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Golub-Kahan-Lanczos based preconditioner for least squares problems in overdetermined and underdetermined cases

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Abstract

We present an effective preconditioner for solving least squares problems in full ranked overdetermined and underdetermined cases. The preconditioner, generated from Golub-Kahan-Lanczos method, can approximately replace a few largest singular values by one without altering the rest. This property accelerates the convergence, thereby improves the efficiency of the algorithm for solving the least squares problems with ill-conditioned system matrix which is caused by large singular values. In this paper we focus on the overdetermined and the underdetermined cases.

Key words: Least squares problems; Preconditioner; Lanczos bidiagonalization process; Krylov subspace method; Golub-Kahan-Lanczos method

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1 Introduction

In this paper, we assume that the least squares problems are in the form as

$$\min \|b - Ax\|_2, \tag{1}$$

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where $A_{m\times n}$ is a full-ranked coefficient matrix which is large and sparse.

In the situation that $m = n$, we can obtain an approximate solution by solving the linear system $Ax = b$ and minimize the residual in the sense of 2-norm. The minimal norm residual method, based on the iterative Krylov methods, is a suitable algorithm to obtain the optimal approximation, and full details can be found in [2]. We have superscript T denoted the transposition of a matrix, and use subscript to indicate the size of matrix. The overdetermined cases

$$\min \|b - Ax\|_2, A \in R_{m\times n}, m > n \tag{2}$$

and the underdetermined cases

$$\min \|b - Ax\|_2, A \in R_{m\times n}, m < n \tag{3}$$

are taken into consideration in the following.

In this paper, we take the preconditioner as a left preconditioner in both overdetermined and underdetermined cases. To the overdetermined system (2) in least squares problems, we generally translate the corresponding linear system

$$Ax = b, A \in R_{m\times n}, m > n, \tag{4}$$

into a normal equation by premultipling A^T on both sides. R is the set of real number here and in the following. Similarly, we translate the underdetermined system (3) into a normal equation in the same way in the corresponding linear system

$$Ax = b, A \in R_{m\times n}, m < n. \tag{5}$$

Thereby we have the normal equation in the following form

$$A^T Ax = A^T b. \tag{6}$$

We notice that the coefficient matrix in (6) is symmetric positive definite, so the normal equation can be solved by the CG method[16]. Thanks to previous researchers, many classic methods, such as CGNE [4] and CGLS[3], can be regarded as an extensions of the CG method and solve least squares problems efficiently. Similarly, the LSQR method[7] is an effective method for solving the least squares problems, so does the LSMR method[15].

For the symmetric positive definition (SPD) matrix, we know the convergence of iterative Krylov methods depends on the condition number κ of the coefficient matrix, in other word, the spectral distribution, where $\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$ with $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denoting the largest and the smallest eigenvalues of A, respectively. To discuss the spectral distribution of $A^T A$ in (6), we give the singular value decomposition of the original coefficient matrix A as follow. Notice that all the matrixes in this paper are full ranked.

We have the singular value decomposition of A in this form

$$A = \hat{U}_{m\times n} D \hat{V}^T_{n\times n}, D = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix}, \tag{7}$$

where $\hat{U}_{m\times n}$ and $\hat{V}_{n\times n}$ are both unitary matrices, σ_i denotes the singular value that $\sigma_1 > \sigma_2 > \cdots > \sigma_n$. From (7), we have

$$A^T A = \hat{V}_{n\times n} D^2 \hat{V}^T_{n\times n}, \tag{8}$$

which can be regarded as the eigenvalue decomposition of the coefficient matrix in the normal equation (6).

If we denote $\Sigma = diag\{\sigma_1^2, \sigma_2^2, \cdots, \sigma_r^2\}$, where $r = \min(m, n)$, it could be easily concluded that the spectral distribution of the coefficient matrix in (6) is Σ. Therefore, the condition numbers of linear systems can be presented as $\kappa(A^T A) = \frac{\sigma_1^2}{\sigma_r^2}$. To accelerate the convergence, thereby improve the algorithm, we expect the condition number to be as small as possible. Therefore, removing the smallest eigenvalue from the spectrum of the coefficient matrix is purpose of the preconditioner. Also, we leave the rest unchanged. Such kind of preconditioners and relevant applications can be located in [8], [9] and [10].

Also, when the property of ill-condition is caused by a few largest eigenvalues, we expect a preconditioner, from the similar point of view, to eliminate the largest eigenvalues from the spectrum in order to accelerate the convergence. A preconditioner formed by Lanczos bidiagonalization is formulated to change the largest singular values to one approximately without altering the others, so that the preconditioner change the corresponding eigenvalues in normal equations. In the ill-conditioned overdetermined case and the ill-conditioned underdetermined case, we utilize the preconditioner to speed up the convergence. To illustrate the effects of the preconditioners proposed in this paper, we utilize two methods to solve a series of the least squares problems. Of course, we divide every experiments into two parts, using preconditioner and not using it.

In the following sections, the process of Lanczos bidiagonalization will be stated in section 2; the preconditioners for solving overdetermined and underdetermined least squares problems (2) (3) will be defined in section 3; numerical examples are demonstrated in section 4; conclusions are presented in section 5 finally.

2 The process of Lanczos bidiagonalization

2.1 Standard Lanczos bidiagonalization

Lanczos biorthogonalization, which can be located in [6] [4], is an important process in methods like LSQR[7], BiCG[11] and BiCGSTAB[12]. A variation of Lanczos biorthogonalization, formed as

$$AV_n = U_{n+1}B, B = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_n & \alpha_n \\ & & & \beta_{n+1} \end{pmatrix}, \tag{9}$$

is denoted as Golub-Kahan-Lanczos method [5] , where V_n and U_{n+1} are both unitary matrices and we assume A is a matrix of size $n \times n$. One characteristic of decomposition (9) is that the lower bidiagonal matrix B shares the same singular values as A's. Furthermore, we have analyzed and concluded in the previous section that the singular values distribution of A directly reflects the spectral distribution of A^TA in problems (6). Hence we expect a preconditioner based on Lanczos bidiagonalization to optimize spectral distributions of system matrices in least squares problems. Some similar preconditioner based on the Golub-Kahan-Lanczos bidiagonalization for square coefficient matrixes has been proposed and applied. For example, inreference[13], the author optimized the spectral distribution of a ill-posed coefficient matrix by a Lanczos-based preconditioner.

However, limited by the dimension of the coefficient matrix in overdetermined and underdetermined cases, the algorithm will break down when maximal number of iteration is greater than both row dimension and column dimension. Therefore, in order to be applied to overdetermined and underdetermined cases, the standard form of Golub-Kahan-Lanczos method requires modification. To extend applications of the Lanczos-based preconditioner, we define variants of the preconditioner which can be utilized in overdetermined cases and underdetermined cases, thereby it is available for least squares problems. At first, we give the standard algorithm for Golub-Kahan-Lanczos method as stated in [5].

Algorithm 1 Standard Golub-Kahan-Lanczos bidiagonalization

1. $\beta_1 = \|b\|_2$, $u_1 = \frac{b}{\beta_1}$, $v_0 = 0$
2. for $i = 1, 2, ..., n$
3. $\quad p_k = A^T u_k - \beta_k v_{k-1}$
4. $\quad \alpha_k = \|p_k\|_2$
5. $\quad v_k = \frac{p_k}{\alpha_k}$
6. $\quad q_k = Av_k - \alpha k u_k$
7. $\quad \beta_{k+1} = \|q_k\|_2$
8. $\quad u_{k+1} = \frac{q_k}{\beta_{k+1}}$

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The $\alpha's$ and $\beta's$ generated in the above algorithm are equal to the ones in (9), also rows of V and U in (9) are obtained through Algorithm 1 as v_k and u_k respectively. Therefore, we could establish the Lanczos bidiagonalization form by a series of iterations performed according to Algorithm 1, when the coefficient matrix A is of size $n \times n$.

To define the Lanczos-based preconditioners in overdetermined cases and underdetermined cases, we have to modify algorithm 1, the standard Lanczos bidiagonalization process, in order to accommodate the situations that the coefficient matrices are m-by-n and $m \neq n$.

2.2 Modified Lanczos bidiagonalization

The main distinction between the overdetermined, or underdetermined, determined and square cases is the dimension of the coefficient matrix A. As stated before, the matrix B, generated by Lanczos bidiagonalization, and A in (9) share the same singular value distribution. We limit the steps of Lanczos bidiagonalizaion process under the minimal number between m and n where A is m-by-n. We utilize iterative Krylov subspace methods to solve the linear systems (6), with symmetric positive definite coefficient matrices. Therefore we conclude easily that the rank of B can not exceed the minimum of m and n. Then, a restrictive condition should be added to the corresponding Lanczos bidiagonalization process to terminate it in appropriate number of steps.

Different from (9), We set a termination rule that the maximal iteration in Golub-Kahan-Lanczos bidiagonalization is less or equal to the minimum between the row dimension and the column dimension toensure that the algorithm will terminate in appropriate number of steps. Following this rule, we have the bidiagonalization decomposition of A in overdetermined situation as

$$AV_{n\times n} = U_{m\times(n+1)}B_n, B_n = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_n & \alpha_n \\ & & & \beta_{n+1} \end{pmatrix}, \tag{10}$$

and the bidiagonalization decomposition of A in underdetermined situation as

$$AV_{n\times m} = U_{m\times(m+1)}B_m, B_m = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_m & \alpha_m \\ & & & \beta_{m+1} \end{pmatrix}. \tag{11}$$

Considering the computational cost of the Lanczos bidiagonalization process, we try to avoid bidiagonalizing A completely. The preconditioner, mentioned in

the previous section and defined in the next section, is structured for the purpose of changing the largest singular values to one, in order to optimize the condition numbers of normal equation (6). Hence, we stop the Lanczos dibiagonalization process when the current smallest singular value σ_k, generated in the kth step of Lanczos dibiagonalization process, is much smaller than the largest one σ_1. We set a scalar number δ to be the threshold of termination, i.e, terminates when $\sigma_k < \delta\sigma_1$. If the bidiagonalization process stops at the kth step, the bidiagonalization composition is of the form below

$$AV_{n\times k} = U_{m\times(k+1)}B_k, B_k = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_k & \alpha_k \\ & & & \beta_{k+1} \end{pmatrix}. \tag{12}$$

M Rezghi set the scalar number δ as the square root of machine precision in [13] while applying it in ill-conditioned systems derived from blurring images. Since δ is a scalar to judge whether we should terminate the Lanczos bidiagonalization process and the Lanczos bidiagonalization process aims to remove the largest singular values, the choice of δ has different effects in different numerical examples. We will present the influence caused the change of δ under different numerical examples and iterative methods in the section of experiments. In general ill-conditioned systems, we need not to set δ so small and some cases will be presented in the 4th section. Here we add the above two restrictive conditions to standard Lanczos bidiagonalization, then we have modified Lanczos bidiagonalization as following.

Algorithm 2 Modified Lanczos bidiagonalization

1. $\beta_1 = \|b\|_2$, $u_1 = \frac{b}{\beta_1}$, $v_0 = 0$, $r = min\{m, n\}$, δ
2. for $i = 1, 2, ..., r$
3. $p_k = A^T u_k - \beta_k v_{k-1}$
4. $\alpha_k = \|p_k\|_2$
5. $v_k = \frac{p_k}{\alpha_k}$
6. $q_k = Av_k - \alpha k u_k$
7. $\beta_{k+1} = \|q_k\|_2$
8. $u_{k+1} = \frac{q_k}{\beta_{k+1}}$
9. get singular values of B: $\sigma_1, \sigma_2, \cdots, \sigma_i$
10. if $\sigma_i < \delta\sigma_1$, break down.
11. end

In this section, we introduced the standard Lanczos bidiagonalization process in Algorithm 1, and defined the modified Lanczos bidiagonalization process in

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Algorithm 2, which is adapted to the overdetermined and the underdetermined situations. A preconditioner based on modified Lanczos bidiagonalization process will be introduced and defined in the next section.

3 Lanczos-based preconditioner for least squares problems

To solve the least squares problems formed as (2) and (3), we solve the corresponding linear systems (4) and (5) instead by translating them into normal equations (6) respectively. If we have the singular value decompositions of A which are structured as (7), and the singular value distributions are scattered and wide, that is the largest singular value is much greater than the smallest one, thereby the condition number of the normal equation (6) will be terribly greater according to analysis of (8). For the purpose of speeding up the convergence, we expect to optimize, or reduce, the condition number of A^TA. Since the condition number of normal equations (6) could be presented as $\kappa(A^TA) = \frac{\sigma_1^2}{\sigma_r^2}$ where σ_1 and σ_r denote the largest and the smallest singular value of A, enlargement or elimination of the smallest singular values, and decrease or elimination of the largest singular values are both effective methods to reduce the condition number. Deflation-based preconditioners, like the deflation preconditioner and the balancing preconditioner[8, 9, 10], have such characteristics and properties to eliminate smallest eigenvalues of system matrix. We do not pay much attention to the preconditioners based on deflation, but the preconditioners functioned for decreasing, or eliminating, the largest ones are what we concern. In the following, all the preconditioners based on Lanczos bidiagonalization are defined for the overdetermined cases (2) and the underdetermined cases (3).

First we shall discuss the situation of the underdetermined case. In linear system (5), the coefficient matrix A has the singular value decomposition illustrated as (7). We assume a diagonal matrix

$$D_k = diag\{\sigma_1, \sigma_2, \cdots, \sigma_k\},$$

where σ_i with $i = 1, 2, \cdots, k$, denotes the first k largest singular values of A. The Lanczos bidiagonalization process for underdetermined cases within k steps have been proposed as (12).

On the premise that B, which is structured by Lanczos bidiagonalization, shares the same singular values with A, we have the following conclusion that: the B_m derived from (11) has singular value decomposition form as

$$B_m = \tilde{U}_{(m+1)\times(m+1)} \begin{pmatrix} D \\ 0 \end{pmatrix}_{(m+1)\times m} \tilde{V}^T_{m\times m},$$

where D in the above equation is equal to the one in (7), with $\tilde{U}_{m+1}$ and $\tilde{V}_{m\times m}$ both unitary matrices. Similarly, the B_k derived from (12) has singular value decomposition form as

$$B_k = \tilde{U}_k \begin{pmatrix} D_k \\ 0 \end{pmatrix} \tilde{V}_k^T, \tag{13}$$

where D_k has been defined at the beginning in this section, with $\tilde{U}_k$ and $\tilde{V}_k$ both unitary matrices.

When we consider the underdetermined case (11), some deductions are stated as follow. We use singular value decomposition of B replacing the one in (11) and we have

$$AV_{n\times m} = U_{m\times(m+1)}\tilde{U}_{(m+1)\times(m+1)} \begin{pmatrix} D \\ 0 \end{pmatrix}_{(m+1)\times m} \tilde{V}_{m\times m}^T.$$

The dimension of matrices are denoted as subscripts in previous sections, and now the subscripts will be omitted for simplification. Then we postmultiply $\tilde{V}$ on both sides and we have

$$AV\tilde{V} = U\tilde{U} \begin{pmatrix} D \\ 0 \end{pmatrix},$$

Here we set $\bar{V} = V\tilde{V} = \{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_m\}$ and $\bar{U} = U\tilde{U} = \{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_{m+1}\}$. As for equation

$$A\bar{V} = \bar{U} \begin{pmatrix} D \\ 0 \end{pmatrix},$$

we regard it as a singular value decomposition of A, similar to (7), approximately. If we set $\bar{U}_m = \{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_m\}$, the first m columns of $U\tilde{U}$, we assume that

$$\begin{aligned} \bar{U}_m &= \hat{U} \\ \bar{V} &= \hat{V} \end{aligned}$$

where $\hat{U}$ and $\hat{V}$ are obtained from (7).

Now we focus on the formulation (8). If a matrix is structured as

$$P = \bar{V}D^{-2}\bar{V}^T,$$

then combining with the previous assumption($\bar{V} = \hat{V}$), it gives that

$$\begin{aligned} PA^TA &= \bar{V}D^{-2}\bar{V}^T\hat{V}D^2\hat{V}^T \\ &= \bar{V}I\bar{V}^T \\ &= I. \end{aligned}$$

It seems that we could have obtained solution directly through the application of such a preconditioner P. In view of computation, however, it is inadvisable for

the following reasons: 1. the preconditioner P is based on a complete Lanczos bidiagonalization, so this process has expensive computational cost even no less than direct methods.; 2. the $\bar{V}$ is approximately equal to $\hat{V}$ in practical implement, but we give the above deduction just in theory, without the consideration of computational errors. Although we can not utilize the preconditioner P in practical computation, a variant of P based on incomplete Lanczos bidiagonalization is defined as follow to solve underdetermined least squares problems.

Here we construct a preconditioner P which is similar with the one mentioned above with merely replacing B_m(from (11)) by B_k(from (12)). After simple deduction, we have

$$P = \bar{V} \begin{pmatrix} D_k^{-2} & 0 \\ 0 & I_{m-k} \end{pmatrix} \bar{V}^T,$$

where D_k is from (13). We set $\bar{V}_k = V\tilde{V}_k$ is the first k columns of $\bar{V}$, where $\tilde{V}_k$ is obviously the first k columns of $\tilde{V}$. Hence we set $\bar{V} = [\bar{V}_k, \bar{V}_{m-k}]$. Based on the definition of $\bar{V}$, we have

$$I = \bar{V}\bar{V}^T = \bar{V}_k\bar{V}_k^T + \bar{V}_{m-k}\bar{V}_{m-k}^T.$$

Analyzing the above information, it gives that

$$\begin{array}{rcl} P & = & \bar{V}_k D_k^{-2} \bar{V}_k^T + \bar{V}_{m-k}\bar{V}_{m-k}^T \\ & = & V\tilde{V}_k D_k^{-2} \tilde{V}_k^T V^T + (I_{m\times m} - \bar{V}_k\bar{V}_k^T) \\ & = & V(B_k^T B_k)^{-1} V^T + (I_{m\times m} - VV^T). \end{array}$$

where V and B_k can both be obtained through Algorithm 2. If we utilize P as a left preconditioner in normal equation (6) for underdetermined cases (5), we have

$$PA^TA = \hat{V} \begin{pmatrix} I_k & 0 \\ 0 & D_{m-k}^2 \end{pmatrix} \hat{V}^T,$$

where $D_{m-k} = diag\{\sigma_{k+1}, \sigma_{k+2}, \cdots, \sigma_m\}$ with σ_i's denoting the $m-k$ smallest singular values.

According to the statement above, we can conclude that the Laczos-based preconditioner has the property to change k largest singular values of coefficient matrix A, or k largest eigenvalues of the system matrix in normal equation (6) in other word, to one without touching the others. The preconditioner is able to optimize the condition number of normal equation (6) when the ill condition is caused by these large singular values. Since $k \ll m$, the computational cost is greatly reduced, so is the computational error. The conclusion, furthermore, is under the premise that the linear system corresponding to least squares problems is underdetermined, so that

$$P_{under} = V(B_k^T B_k)^{-1} V^T + (I_{n\times n} - VV^T) \tag{14}$$

could be used as a left-preconditioner in underdetermined least squares problems. Next we consider the overdetermined cases.

In the overdetermined cases, we construct a Lanczos-based preconditioner that follows the same strategy as stated in the previous subsection. To solve the overdetermined system (4), we solve the normal equation (6) instead to obtain approximate solution. Considering the decomposition form (8) of $A^T A$, we expect to construct a preconditioner, similar to the underdetermined cases, presented as

$$P = \hat{V} \begin{pmatrix} D_k^{-2} & 0 \\ 0 & I_{n-k} \end{pmatrix} \hat{V}^T.$$

Through an analogical deduction to underdetermined cases, a preconditioner formed as

$$P_{over} = V(B_k^T B_k)^{-1} V^T + (I_{n\times n} - VV^T) \tag{15}$$

can be used as a left-preconditioner in overdetermined least squares problems. B_k and $V_{n\times k}$ can be obtained from Algorithm 2. Furthermore it is not computationally costly because of $k \ll n$.

From the above discussion, we can see that the forms of the Lanczos-based preconditioners in over- and under- determined cases are the same, although we deduced them in separate ways. Also, such a preconditioner for the linear system with a square coeffcient matrix has the same form. Therefore, we can conclude that we deduce the preconditioners, proposed in this paper, from the point of overdetermined and underdetermined cases and ultimately get a result similar to the one in square problems, which has been proposed in [13]. Of course, the result of this paper can also be regarded as the expansion of the application of the Lanczos-based preconditioner into the overdetermined and underdetermined least squares problems. Now we unify the preconditioner as follow

$$P = V(B_k^T B_k)^{-1} V^T + (I - VV^T), \tag{16}$$

which can be used as a left preconditioner in ordinary linear systems, overdetermined least squares problems and underdetermined least squares problems. The relevant numerical experiments are presented in the following section, from which we can see the effects of Lanczos-based preconditioners.

4 Numerical experiments

In this section, we will take a series of numerical examples to present the effect of the Lanczos-based preconditioner in the least squares problems. At first, we introduce two iterative methods as the basic algorithm for solving these underdetermined and overdetermined problems. Here, we choose an old and classic method as the first one for solving the least squares problems. It is the CGLS

method[3]. In this method, we first transform the least squares problems into symmetric positive definite(SPD) problems by the normal equations then solve it by the CG method[16]. Integrating the above ideas, we have the CGLS method. Now we present the preconditioned CGLS method algorithm 3, where we just consider the situation of left precondition.

Algorithm 3 Preconditioned CGLS method

1. select x_0 as the initial guess, $r_0 = b - Ax_0$ and P as the preconditioner
2. initialization: we set $\bar{r}_0 = A^T r_0$, $\hat{r}_0 = P\bar{r}_0, f_0 = z_0$
2. for $i = 0, 1, 2, ...$
3. $\quad g_i = Af_i$
4. $\quad \alpha_i = (\hat{r}_i, \bar{r}_i)/\|g_i\|_2^2$
5. $\quad x_{i+1} = x_i + \alpha_i f_i$
6. $\quad r_{i+1} = r_i - \alpha_i g_i$
8. $\quad \bar{r}_{i+1} = A^T r_{i+1}$
9. $\quad \hat{r}_{i+1} = P\bar{r}_{i+1}$
10. $\quad \beta_i = (\hat{r}_{i+1}, \bar{r}_{i+1})/(\hat{r}_i, \bar{r}_i)$
11. $\quad f_{i+1} = \hat{r}_{i+1} + \beta_i f_i$
12. endfor

The second method to solve the least squares problems is the BAGMRES method[14], a variant of the GMRES method[1]. In this method, the least squares problems will be post-multiplied by a matrix B, an arbitrary nonsingular matrix. Now we give the BAGMRES method as Algorithm4.

Ex.	Group and name	id	#rows	#cols	Nonzeros	Problem kind
1	JGD_Forest/TF10	1944	99	107	622	Combinatorial
2	JGD_Forest/TF11	1945	216	236	1607	Combinatorial
3	HB/wm3	277	207	260	2948	Economic
4	Pajek/Sandi_sandi	1520	314	360	613	Bipartite graph
5	Meszaros/refine	1759	29	62	153	Linear programming
6	JGD_margulies/flower_4_1	2155	121	129	386	Combinatorial

Table 1: The structures of six test underdetermined problems

Algorithm 4 BA-GMRES with k restart

1. select x_0 as the initial guess, $r_0 = B(b - Ax_0)$ and $\nu_1 = r_0/\|r_0\|_2$
2. for $i = 1, 2, ..., m$
3. $\quad \omega_i = BA\nu_i$
4. $\quad$ for $j = 1, 2, ..., i$
5. $\qquad h_{j,i} = (\omega_i, \nu_j)$
6. $\qquad \omega_i = \omega_i - h_{j,i}\nu_j$
7. $\quad$ endfor
8. $\quad h_{i+1,i} = \|\omega_i\|_2$
9. $\quad \nu_{i+1} = \omega_i / h_{i+1,i}$
10. $\quad$ Compute y_m to minimize $\|\hat{r_i}\|_2 = \|\|\hat{r_0}\|_2 e_1 - \overline{H}_i y\|_2$
11. $\quad$ if $\|r_i\|_2 < \tau$
12. $\qquad x_i = x_0 + [\nu_1, ..., \nu_i]y_i$
13. $\qquad$ stop
14. $\quad$ endif
15. endfor
16. set $x_0 = x_k$ and return to line 2 until convergence

In the following numerical experiments, the examples all come from practical applications from [17].

All the required information about the underdetermined and overdetermined cases is contained in Table 1 and Table 2 respectively. They both consist of group, number of rows, columns and nonzero elements and the type of problem of each example.

In the next two subsections, we solve the above 12 problems by the PCGLS method and the BAGMRES method combined with the Lanczos-based preconditioners. Then we change the scalar δ, involving the termination rule of the modified Lanczos bidiagonalization, and show its influence on the iterative process. Because the preconditioner is designed to modify the singular values, the distributions of singular values under different scalar δ's will be presented as well.

Ex.	Group and name	id	#rows	#cols	Nonzeros	Problem kind
7	HB/abb313	5	313	176	1557	Least squares
8	JGD_margulies/cat_ears_3_1	2151	204	181	542	Combinatorial
9	JGD_margulies/cat_ears_4_1	2153	377	313	938	Combinatorial
10	JGD_margulies/flower_5_1	2157	211	201	602	Combinatorial
11	JGD_margulies/flower_7_1	2159	463	393	1178	Combinatorial
12	Pajek/Cities	1457	55	46	1342	Weighted bipartite graph

Table 2: The structures of six test overdetermined problems

4.1 The acceleration of iterative processes

To discuss the acceleration of iterative processes, we refer to the PCGLS method and the BAGMRES method in [14, 4]. For the BAGMRES method, we have the following relation between the initial residual and the one from the kth iteration in underdetermined cases,

$$\|Br_k\|_2 = \|CA^T r_k\| \leq 2(\frac{\sigma_1 - \sigma_m}{\sigma_1 + \sigma_m})^k \|Br_0\|_2, \tag{17}$$

where C is a nonsingular matrix, $\kappa(C)$ is the condition number of matrix C and σ's denote the singular values of BA. And we have the relation between r_0 and r_k as

$$\|Br_k\|_2 = \|CA^T r_k\| \leq 2\sqrt{\kappa(C)}(\frac{\sigma_1 - \sigma_n}{\sigma_1 + \sigma_n})^k \|Br_0\|_2, \tag{18}$$

where C is a nonsingular matrix, $\kappa(C)$ is the condition number of matrix C and σ's denote the singular values of BA. More information of the above conclusion can be found in [14]. Now we give the convergence analysis of the PCGLS method, that is

$$\|e_k\|_A \leq 2(\frac{\sigma_1 - \sigma_r}{\sigma_1 + \sigma_r})^k \|e_0\|_A, \tag{19}$$

where $r = \min(m, n)$ and σ's denoting the singular values of PA^TA.

Based on equation (17), (18) and (19), it is obvious that we can accelerate the convergence if the gap between the largest singular value of normal equations and the smallest one is narrowed. In this paper, the Lanczos-based preconditioner is just for resetting the largest singular values to one, which can be regarded as shrink of the singular value distribution. Now, the effect of the Lanczos-based preconditioner in underdetermined cases is shown from Figure 1 to Figure 6.

In the numerical experiments, we set the tolerance $tol = 10^{-12}$, the maximal number of iteration $max_it = 1000$ and the restarted number in the BAGMRES method $restart = 600$. Furthermore, the scalar δ upon which to terminates the

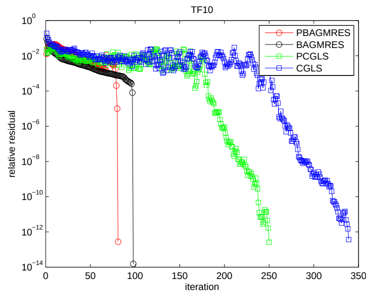

Figure 1: Relative residuals *vs* iterations in `TF10`

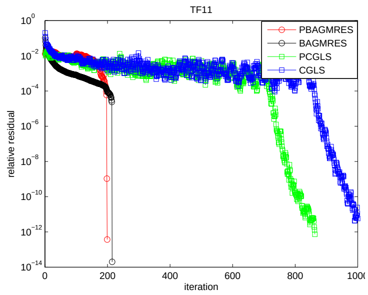

Figure 2: Relative residuals *vs* iterations in `TF11`

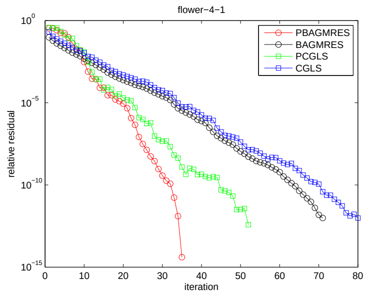

Figure 3: Relative residuals *vs* iterations in `wm3`

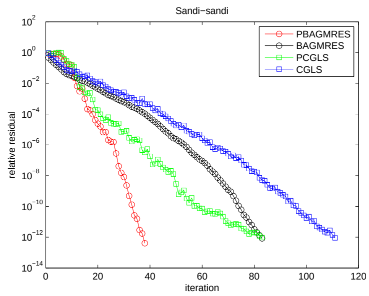

Figure 4: Relative residuals *vs* iterations in `Sandi_sandi`

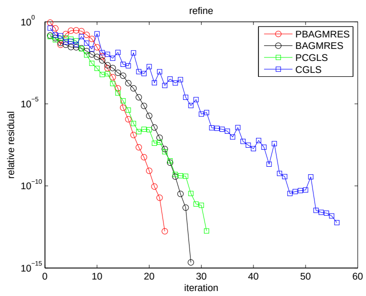

Figure 5: Relative residuals *vs* iterations in `refine`

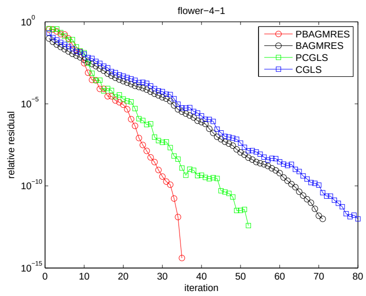

Figure 6: Relative residuals *vs* iterations in `flower_4_1`

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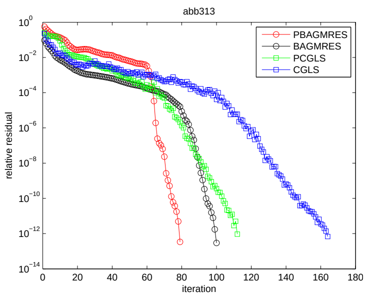

Figure 7: Relative residuals *vs* iterations in `abb313`

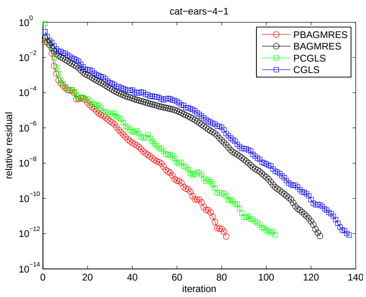

Figure 8: Relative residuals *vs* iterations in `itercat_ears_4_1`

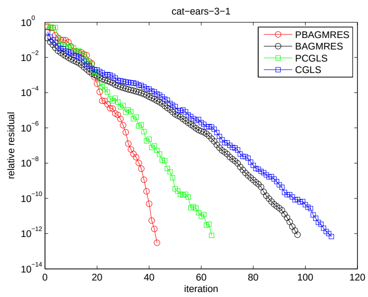

Figure 9: Relative residuals *vs* iterations in `itercat_ears_3_1`

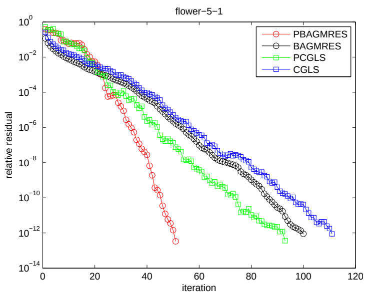

Figure 10: Relative residuals *vs* iterations in `iterflower_5_1`

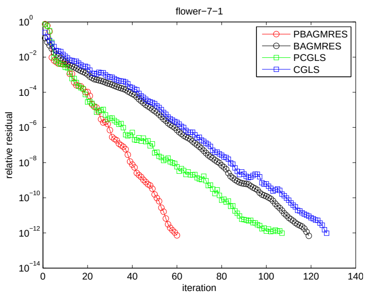

Figure 11: Relative residuals *vs* iterations in `iterflower_7_1`

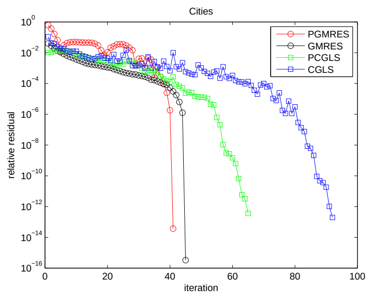

Figure 12: Relative residuals *vs* iterations in `itercities`

Lanczos bidiagonalization process is 0.05 in the test. We set $B = PA^T$ in the preconditioned BAGMRES method and $B = A^T$ in the nonpreconditioned BAGMRES method. From Figure 1 to Figure 6, we can see that both the BAGMRES method and the PCGLS method are accelerated by the Lanczos-based preconditioner as we expected. Next we show the iterative process while solving the overdetermined problems.

Figure 7 to Figure 12 present the results of experiments with the tolerance $tol = 10^{-12}$, the maximal number of iteration $max\_it = 1000$ and the restarted number in the BAGMRES method $restart = 600$. The scalar δ upon which to terminate the Lanczos bidiagonalization process is 0.05 in the test. Similarly, we set $B = PA^T$ in the preconditioned BAGMRES method and $B = A^T$ in the nonpreconditioned BAGMRES method. In Figure 7 to Figure 12, it is obvious that Lanczos-based preconditioners also accelerate the iterative processes in these overdetermined problems, so we think the proconditioner proposed in this paper is helpful to optimize the structure of coefficient matrix thereby accelerate the convergence. Moreover, all the numerical examples here are derived from practical applications. We believe, therefore, the Lanczos preconditioner has the result as

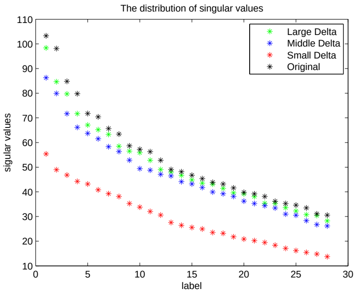

Figure 13: The distribution of singular values in TF10

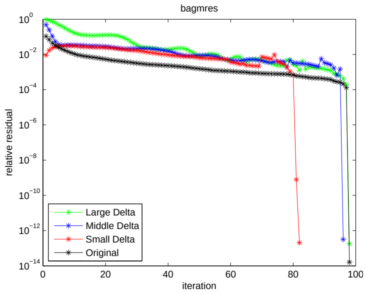

Figure 14: The iterative process of BAGMRE in TF10

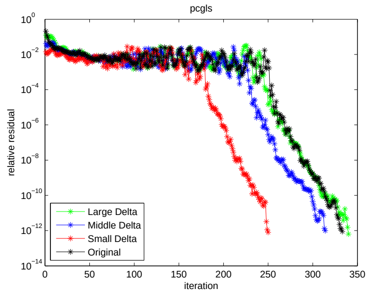

Figure 15: The iterative process of PCGLS in TF10

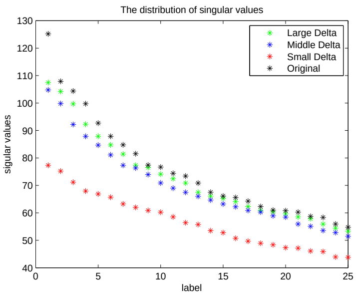

Figure 16: The distribution of singular values in TF11

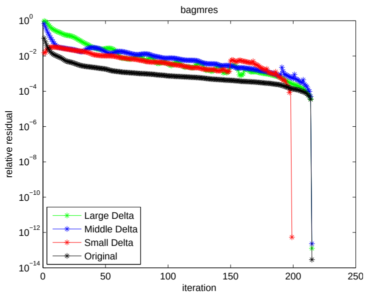

Figure 17: The iterative process of BAGMRE in TF11

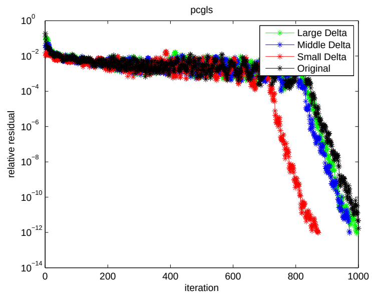

Figure 18: The iterative process of PCGLS in TF11

we expected.

4.2 The influence of the scalar δ

Referring to the illustration above, we have known that the scalar δ is used as a termination rule during the implementation of the Lanczos bidiagonalization process. By the definition of scalar δ, the smaller the δ is, the more large singular values will be replaced by one. It means that we can narrow the distribution of singular values. In the following experiments, we set the scalar δ to three different values and take TF10 and TF11 as the underdetermined examples. We test the distributions of the coefficient matrix of corresponding normal equations, the iterative process of the BAGMRES method and the PCGLS method. The results of TF10 and TF11 with varying scalar δ are presented in Figure13-15 and Figure 16-18 respectively.

As for the overdetermined cases, we take abb313 as the first numerical examples. The singular values distribution and iterative processes of this example are illustrated by Figure 19-21.

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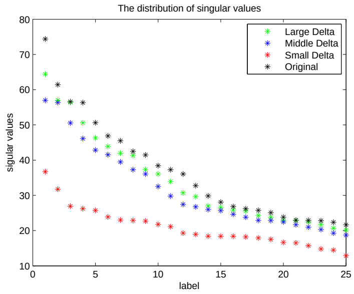

Figure 19: The distribution of singular values in `abb313`

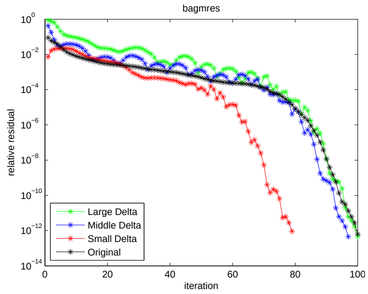

Figure 20: The iterative process of BAGMRE in `abb313`

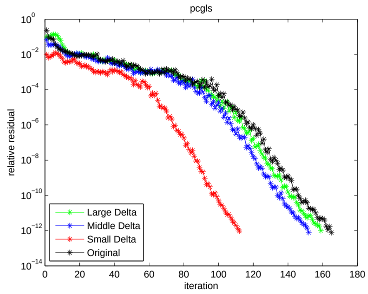

Figure 21: The iterative process of PCGLS in `abb313`

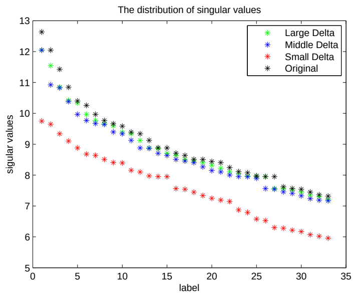

Figure 22: The distribution of singular values in `cat_ears_4_1`

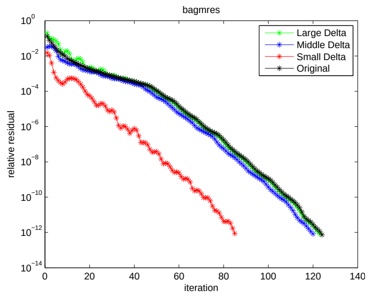

Figure 23: The iterative process of BAGMRE in `cat_ears_4_1`

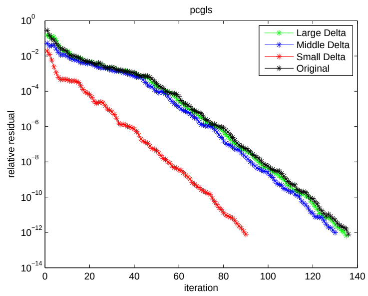

Figure 24: The iterative process of PCGLS in `cat_ears_4_1`

Similarly, the singular value distribution and iterative process regarding to different δ of the example `cat_ears_4_1` are presented in Figure 22-24.

In the above twelve figures, we classify the δ into three classes: the large delta, the middle delta and the small delta. The different δ stand for different preconditioners, upon which we denote the corresponding singular value distribution and iterative process by colorful points and lines. Theoretically, the small delta is able to reset most largest singular values while the large delta reset least largest singular values. Furthermore, required data of the experiments is presented in Table 3 and Table 4, in which k stands for the step of the Lanczos bidiagonalization process, $iter_{BAGMRES}$ and $iter_{PCGLS}$ represent the number of iterations of the BAGMRES method and the PCGLS method, respectively.

From Figure 13, Figure 16, Figure 19 and Figure 22, we can observe that the preconditioner with smaller δ indeed narrows the singular value distribution better than the ones led by larger δ. However, we fail to replace the largest singular values by one, although the improvement has brought us better convergence that is shown in Figure 14-15, Figure 17-18, Figure 20-21 and Figure 23-24. Through Table 3 and Table 4, we can also find that the number of iterations decreases

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Example	TF10		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		99	333
$\delta = 0.8$	2	99	340
$\delta = 0.3$	6	97	314
$\delta = 0.05$	22	83	250
Example	TF11		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		216	1000
$\delta = 0.8$	2	216	995
$\delta = 0.3$	5	216	972
$\delta = 0.05$	24	200	872

Table 3: The information along with the change of scalar δ in underdetermined cases `TF10` and `TF11`

Example	abb313		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		101	165
$\delta = 0.8$	2	101	159
$\delta = 0.3$	5	98	152
$\delta = 0.05$	24	80	112
Example	cat_ears_4_1		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		125	136
$\delta = 0.8$	2	124	135
$\delta = 0.3$	5	121	130
$\delta = 0.05$	40	86	90

Table 4: The information along with the change of scalar δ in underdetermined cases `abb313` and `cat_ears_4_1`

obviously while the δ decreasing. In small-scale problem, the Lanczos-based preconditioner can reset the largest singular values closer to one than in large-scale problems, which is easy to testify by a simple numerical deduction. We suppose that the reason why the preconditioner fails to reset the largest singular values to one, just decreasing them instead, is the accumulation of calculation errors and the assumption

$$\bar{U}_m = \hat{U}$$
$$\bar{V} = \hat{V}.$$

From another experiment, the matrix B constructed in Lanczos bidiagonalization

process has approximately equal singular values with coefficient matrix A. Merely focusing on the numerical value, the gap between the singular values of B and A may be underestimated and even ignored. Nevertheless, the gap will be enlarged when we assume the above equalities without considering the calculation errors. In the above experiments, we can also notice that the different δ influence the iterative process distinctly in different method so the perturbation analysis of the Lanczos-based preconditioner may give us a theoretical explanation of the difference between the theory and the numerical experiment. This supposition is remained to be testified in the future work.

5 Conclusions

To the overdetermined and the underdetermined least squares problems, we choose the BA-GMRES method and the PCGLS method to solve them respectively. Variants of the Lanczos bidiagonalization process are defined in the situation that coefficient matrices are not square, and the algorithm of modified Lanczos bidiagonalization is illustrated as conclusion. When we suffer from the ill-conditioned system matrices, the preconditioners based on modified Lanczos bidiagonalization, P structured for the overdetermined cases and the underdetermined cases respectively, are imposed on iterative Krylov subspace methods to accelerate convergence. Finally we prove our statements with numerical experiments and conclude that the preconditioner defined in this paper is effective to solve least squares problems in overdetermined and underdetermined cases.

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Classical Model of Prandtl's Boundary Layer Theory for Radial Viscous Flow: Application of $(G'/G)-$ Expansion Method

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Abstract

In this paper, the exact closed-form solutions of the Prandtl's boundary layer equation for radial flow models with uniform or vanishing mainstream velocity are derived by using the $(G'/G)-$expansion method. Many new exact solutions are found for the boundary layer equation, which are expressed by the hyperbolic, trigonometric and rational functions. The solutions are valid for all values of the parameter β. It is shown that the $(G'/G)-$expansion method is effective and can be used for many other nonlinear differential equations of mathematical physics.

Keywords: $(G'/G)-$Expansion method; Prandtl's boundary layer equation; Exact solutions

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1 Introduction

Many real world problems in nonlinear science associated with mechanical, structural, aeronautical, ocean, electrical, and control systems can be summarized as solving nonlinear differential equations which arise from mathematically modelling such problems. Therefore, the study of nonlinear differential equations has been an active area of research for the past few years. Investigating integrability and finding exact solutions to such nonlinear differential equations have extensive applications in many scientific fields such as hydrodynamics, fluid dynamics, general relativity, condensed matter physics, solid-state physics, nonlinear optics, neurodynamics, fibre-optic communication and so on. These exact solutions, if reported are helpful for the numerical analyst to verify the complex numerical codes and are also useful in stability analysis for solving special nonlinear models.

In recent years, much attention has been devoted to the development of several powerful and useful methods for finding exact and approximate solutions of nonlinear differential equations. These research methods for solving nonlinear differential equations include the bilinear method and multilinear method [1], classical Lie symmetry method [2], nonclassical Lie group approach [3], Clarkson-Kruskal's direct method [4], deformation mapping method [5], homogenous balance method [6], Weierstrass elliptic function expansion method [7], F-expansion method [8], transformed rational function method [9], auxiliary equation method [10], sine–cosine method [11], tanh-function method [12], Backlund transformation method [13], simplest equation method [14, 15], exponential function rational expansion method [16] and so forth.

Prandtl [17] initiated the concept of a boundary layer in large Reynolds number flows in 1904 and he also showed how the Navier-Stokes equation could be simplified to yield approximate solutions. Prandtl introduced boundary layer theory to understand the flow behavior of a viscous Newtonian fluid near a solid boundary. Prandtl's boundary layer equations arise in various physical models of fluid mechanics. The equations of the boundary layer theory have been the subject of considerable interest, since they represent an important simplification of the original Navier-Stokes equations. These equations arise in the study of steady flows produced by wall jets, free jets and liquid jets, the flow past a stretching plate/surface, flow

induced due to a shrinking sheet and so on. These boundary layer equations are usually solved subject to specific boundary conditions depending upon the physical model investigation. Blasius [18] solved the Prandtl's boundary layer equations for a flat moving plate problem and found a power series solution of the model. Falkner and Skan [19] generalized the Blasius problem by considering the boundary layer flow over an wedge inclined at certain angle. Sakiadis [20] studied the boundary layer flow over a continuously moving rigid surface with a constant speed. Crane [21] was the first one who investigated the boundary layer flow due to a stretching surface and developed the exact solutions of boundary layer equations. Gupta and Gupta [22] extended the Crane's work and for the first time introduced the concept of heat transfer with the stretching sheet boundary layer flow. Schlichting [23] was the first to apply the boundary layer theory to the steady flow produced by a free two-dimensional jet emerging into a fluid at rest and solved the resulting ordinary differential equation numerically. Later, Bickley [24] solved the differential equation analytically. The concept of the boundary layer to laminar jets is discussed fully in standard texts on boundary layer theory such as by Schlichting [25] and Rosenhead [26]. More recently, the similarity solution of axisymmetric non-Newtonian wall jet with swirl effects was obtained by Kolar [27]. Naz et al. [28] and Mason [29] studied the general boundary layer equations for two-dimensional and radial flows by using the classical Lie group approach and recently Naz et al. [30] provided the similarity solutions of the Prandtl's boundary layer equations by implementing the non-classical symmetry method.

The $(G'/G)-$expansion method is a powerful mathematical tool for finding exact solutions of certain nonlinear ordinary differential equations. The $(G'/G)-$expansion method was introduced by Wang in [31] for constructing the exact solutions of some nonlinear evolution equations. To express the applicability and effectiveness of the $(G'/G)-$expansion method, further research has been accomplished by a diverse group of researchers (see, for example, papers $[32-34]$). The importance of our present work is to find some new class of exact closed-form solutions of Prandtl's boundary layer equation for radial flow models with constant or uniform main stream velocity by employing the $(G'/G)-$expansion method.

2 Mathematical model

The Prandtl's boundary layer equation, for the stream function $\phi(r,\theta)$, for radial flow with uniform or vanishing mainstream velocity is [26]

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2-\frac{1}{r}\frac{\partial\phi}{\partial r}\frac{\partial^2\phi}{\partial\theta^2}-\nu\frac{\partial^3\phi}{\partial\theta^3}=0, \tag{1}$$

where (r,θ) denote the cylindrical polar coordinates and ν is the kinematic viscosity. The velocity components $u(r,\theta)$ and $v(r,\theta)$, in the r and θ directions, are related to stream function $\phi(r,\theta)$ as

$$u(r,\theta)=\frac{1}{r}\frac{\partial\phi}{\partial\theta},\quad v(r,\theta)=-\frac{1}{r}\frac{\partial\phi}{\partial r}. \tag{2}$$

By the use of Lie group theoretic method of infinitesimal transformations [2], the general form of similarity solution for equation (1) is

$$\phi(r,\theta)=r^{2-\beta}H(\xi),\quad \xi=\frac{\theta}{r^\beta}, \tag{3}$$

where β is the constant determined from further conditions and $\xi=\theta/r^\beta$ is the similarity variable. By the substitution of Eq. (3) into Eq. (1), we obtain the third-order nonlinear ordinary differential equation in $H(\xi)$, viz.,

$$\nu\frac{d^3H}{d\xi^3}+(2-\beta)H\frac{d^2H}{d\xi^2}+(2\beta-1)\left(\frac{dH}{d\xi}\right)^2=0. \tag{4}$$

Equation (4) is the general form of Prandtl's boundary layer equation for radial flow of a viscous incompressible fluid. The boundary layer equation is usually solved subject to certain boundary conditions depending upon the particular physical model under investigation. Here, we find the exact closed-form solutions of Eq. (4) using the $(G'/G)-$expansion method. The paper is organised as follows. In Section 3, we provide a brief summary of the $(G'/G)-$expansion method. In Sections 4, we apply this method to solve nonlinear Prandtl's boundary layer equation for radial flow. Finally, some concluding remarks are presented in Section 5.

3 A description of the $(G'/G)-$expansion method

In this section, we present a brief summary of the $(G'/G)-$expansion method for solving nonlinear ordinary differential equations. The essence of the $(G'/G)-$expansion method is given in the following steps:

Step 1: We consider a general form of a nonlinear ordinary differential equation

$$P\left[U(z), \frac{dU}{dz}, \frac{d^2U}{dz^2}, \frac{d^3U}{dz^3}, ...\right] = 0, \tag{5}$$

where U is an unknown function of z and P is a polynomial in U and its various derivatives.

Step 2: According to the $(G'/G)-$expansion method, one assumes that the solution of ODE (5) can be written as a polynomial in (G'/G) as follows:

$$U(z) = \sum_{i=0}^{M} \beta_i \left(\frac{G'}{G}\right)^i, \tag{6}$$

where $G = G(z)$ satisfies the second-order linear ODE with constant coefficients, namely

$$\frac{d^2G}{dz^2} + \lambda\frac{dG}{dz} + \mu G = 0, \tag{7}$$

with β_i $(i = 0, 1, 2, ..., M)$, λ and μ being constants to be determined. The integer M is found by considering the homogenous balance between the highest order derivatives and nonlinear terms appearing in ODE (5).

Step 3: The positive integer M can be accomplished by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (5) as follows:

If we define the degree of $U(z)$ as $D[U(z)] = M$, then the degree of other expressions is defined by

$$\begin{aligned} D\left[\frac{d^qU(z)}{dz^q}\right] &= M + q, \\ D\left[U^r\left(\frac{d^qU(z)}{dz^q}\right)^s\right] &= Mr + s(q + M). \end{aligned} \tag{8}$$

Therefore, we can get the value of M in Eq. (6).

Step 4: We substitute Eq. (6) into Eq. (5) and then use ODE (7) to collect all terms with same order of (G'/G) together. The left-hand side of (5) is then converted into polynomial in (G'/G). Now by equating each coefficient of this polynomial to zero, we obtain a system of algebraic equations for β_i, λ and μ.

Step 5: Since the three types of general solutions of Eq. (7) are well known, we substitute the values of β_i and the general solutions of Eq. (7) into Eq. (6) and obtain three types of solutions of the ODE (5).

4 Application of the $(G'/G)-$expansion method

In this section, we employ the $(G'/G)-$expansion method to obtain solutions of Prandtl's boundary layer Eq. (4).

We assume that the solutions of Eq. (4) are of the form

$$H(\xi) = \sum_{i=0}^{M} A_i \left(\frac{G'(\xi)}{G(\xi)} \right)^i, \tag{9}$$

where $G(\xi)$ satisfies the second-order linear ODE with constant coefficients, viz.,

$$\frac{d^2G}{d\xi^2} + \lambda \frac{dG}{d\xi} + \mu G = 0 \tag{10}$$

with λ and μ being constants.

The balancing procedure yields $M = 1$, so the solution of the ODE (4) is of the form

$$H(\xi) = A_0 + A_1 \left(\frac{G'(\xi)}{G(\xi)} \right). \tag{11}$$

Now substituting Eq. (11) into Eq. (4), making use of the ODE (10), collecting all terms with same powers of (G'/G) and equating each coefficient to zero, yields the

following system of algebraic equations:

$$\begin{aligned}
2\beta A_1^2\mu^2 - \beta A_0A_1\lambda\mu - A_1\lambda^2\mu\nu + 2A_0A_1\lambda\mu - 2A_1\mu^2\nu - A_1^2\mu^2 &= 0,\\
3\beta A_1^2\lambda\mu - \beta A_0A_1\lambda^2 - 2\beta A_0A_1\mu - A_1\lambda^3\nu + 2A_0A_1\lambda^2 - 8A_1\lambda\mu\nu + 4A_0A_1\mu &= 0,\\
\beta A_1^2\lambda^2 - 3\beta A_0A_1\lambda + 2\beta A_1^2\mu - 7A_1\lambda^2\nu + A_1^2\lambda^2 + 6A_0A_1\lambda - 8A_1\mu\nu + 2A_1^2\mu &= 0,\\
\beta A_1^2\lambda - 2\beta A_0A_1 - 12A_1\lambda\nu + 4A_1^2\lambda + 4A_0A_1 &= 0,\\
3A_1^2 - 6A_1\nu &= 0.
\end{aligned}$$

Solving this system of algebraic equations, with the aid of Mathematica, we obtain

$$\lambda = 2\sqrt{\mu}, \quad A_0 = \lambda\nu, \quad A_1 = 2\nu. \tag{12}$$

Substituting these values of A_0, A_1 and the corresponding solution of ODE (4) into Eq. (11), we obtain the following three types of solutions of Eq. (1):

Case 1: When $\lambda^2 - 4\mu > 0$

For this case we obtain the hyperbolic function solution given by

$$H(\xi) = \lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \delta\frac{C_1\sinh(\delta\xi) + C_2\cosh(\delta\xi)}{C_1\cosh(\delta\xi) + C_2\sinh(\delta\xi)}\right), \tag{13}$$

where $\delta = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

Reverting back to the original variables (r,θ), the corresponding stream function is given by

$$\phi(r,\theta) = r^{2-\beta}\left[\lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \delta\frac{C_1\sinh\left(\delta\frac{\theta}{r^\beta}\right) + C_2\cosh\left(\delta\frac{\theta}{r^\beta}\right)}{C_1\cosh\left(\delta\frac{\theta}{r^\beta}\right) + C_2\sinh\left(\delta\frac{\theta}{r^\beta}\right)}\right)\right]. \tag{14}$$

Case 2: When $\lambda^2 - 4\mu < 0$

Here we obtain the trigonometric function solution

$$H(\xi) = \lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \epsilon\frac{-C_1\sin(\epsilon\xi) + C_2\cos(\delta\xi)}{C_1\cos(\epsilon\xi) + C_2\sin(\epsilon\xi)}\right), \tag{15}$$

where $\epsilon = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants. The corresponding stream function is given as

$$\phi(r,\theta) = r^{2-\beta}\left[\lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \epsilon\frac{-C_1\sin\left(\epsilon\frac{\theta}{r^\beta}\right) + C_2\cos\left(\epsilon\frac{\theta}{r^\beta}\right)}{C_1\cos\left(\epsilon\frac{\theta}{r^\beta}\right) + C_2\sin\left(\epsilon\frac{\theta}{r^\beta}\right)}\right)\right]. \tag{16}$$

Case 3: When $\lambda^2 - 4\mu = 0$

For this case we obtain the rational function solution

$$H(\xi) = \lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right). \tag{17}$$

In the form of stream function, the solution is expressed as

$$\phi(r,\theta) = r^{2-\beta}\left[\lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\frac{\theta}{r^\beta}}\right)\right], \tag{18}$$

where C_1 and C_2 are arbitrary constants.

5 Concluding remarks

We have employed the (G'/G)-expansion method for obtaining exact closed-form solutions of the well-known Prandtl's boundary layer equation for radial flow models with uniform main stream velocity. The advantage of this method is that in this method, there is no need to apply the initial and boundary conditions at the outset. This method yields a general solution with free parameters which can be identified by the specific conditions. Also the general solutions obtained by (G'/G)-expansion method are not approximate solutions. Prandtl's boundary layer equations arise in various physical models of fluid dynamics and thus the exact solutions obtained maybe very useful and significant for the explanation of some practical physical models dealing with Prandtl's boundary layer theory.

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On properties of meromorphic solutions for a certain q-difference Painlevé equation

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Abstract

The main purpose of this paper is to investigate some properties on transcendental meromorphic solutions of a certain q-difference Painevé equation

$$f(qz)+f(z)+f(\frac{z}{q})=\frac{az+b}{f(z)}+c,$$

where a,b and c are complex constants such that $|a|+|b|\neq 0$. We obtain some results on the value distribution of $f(z)$ and $\Delta_q f(z):=f(qz)-f(z)$, and the non-existence of rational solutions, which extend some earlier results by Qi and Yang, Chen et al.

Key words: q-difference equation; solution; zero order.
Mathematical Subject Classification (2010): 39A 50, 30D 35.

1 Introduction and Main Results

In this paper, we shall assume that readers are familiar with the basic theorems and the standard notations of the Nevanlinna value distribution theory of meromorphic functions such as $m(r,f)$, $N(r,f)$, $T(r,f)$, $\cdots$, (see Hayman [12], Yang [19] and Yi and Yang [20]). We also use $S(r,f)$ to denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r on a set $F\subset[1,+\infty$ of logarithmic density 1, where the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty}\frac{1}{\log r}\int_{[1,r]\cap F}\frac{1}{t}dt.$$

Throughout this paper, the set F of logarithmic density 1 can be not necessarily the same at each occurrence.

A century ago, Painlevé and his colleagues [15] classified all equations of Painlevé type of the form

$$w''(z)=F(z;w;w'),$$

where F is rational in w and w' and (locally) analytic in z. They singled out a list of 50 equations, six of which could not be integrated in terms of known functions. These equations are now known as the differential Painlevé equations. The first two of these equations are P_I and P_{II}:

$$w''=6w^2+z,\quad w''=2w^2+zw+\alpha,$$

where α is a complex constant.

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Differential Painlevé equations have been an important research subject in the field of the Mathematics and the Physics since the beginning of last century. They occur in many physical situations—-plasma physics, statistical mechanics, nonlinear waves, and so on. Therefore, Painlevé equations have attracted much interest as the reduction of solution equations which are solvable by inverse scattering transformations, and so on.

In the past 22 years, the discrete Painlevé equations have become important research problems (see [7]). For example, the discrete P_I equation can be expressed by

$$y_{n+1} + y_{n-1} = \frac{an+b}{y_n} + c,$$

and the discrete P_{II} equation can be expressed by

$$y_{n+1} + y_{n-1} = \frac{(an+b)y_n + c}{1-y_n^2},$$

where a, b, c are real constants, $n \in \mathbb{N}$.

In 2006-2007, Halburd and Korhonen used the analogues of Nevanlinna value distribution theory to single out the difference Painlevé I and II equations from the following form

$$w(z+1) + w(z-1) = R(z, w), \tag{1}$$

where $R(z, w)$ is rational in w and meromorphic in z (see [9, 10, 11]). They obtained that if (1) has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation, or (1) can be transformed by a linear change in w to some difference equations, which include the difference Painlevé I equation

$$w(z+1) + w(z-1) = \frac{az+b}{w(z)} + c, \tag{2}$$

and the difference Painlevé II equation

$$w(z+1) + w(z-1) = \frac{(az+b)w(z) + c}{1-w(z)^2}, \tag{3}$$

where a, b, c are complex constants.

Chen et al [4, 5, 16] studied some properties of finite order transcendental meromorphic solutions of (2)-(3), and obtained a lot of interesting results.

Recently, there were lots of results about q-difference operators, q-difference equations, and so on (see [2, 6, 8, 18, 21, 22]), by applying the analogue of Logarithmic Derivative Lemma on q-difference operators, which was firstly established by Barnett, Halburd, Korhonen and Morgan [1] in 2007. By comparing these results of differences and q-differences, we find that the usual shift $f(z+c)$ of a meromorphic function are replaced by the q-difference $f(qz)$, and the difference $\Delta_c f = f(z+c) - f(z)$ are replaced by $\Delta_q f(z) = f(qz) - f(z)$, $q \in \mathbb{C}\backslash\{0, 1\}$.

In 2015, Qi and Yang [17] investigated the following equations

$$f(qz) + f(\frac{z}{q}) = \frac{az+b}{f(z)} + c, \tag{4}$$

$$f(qz) + f(\frac{z}{q}) = \frac{(az+b)f(z) + c}{1-f(z)^2}, \tag{5}$$

which can be seen as q-difference analogues of (2) and (3), and obtained some theorems as follows.

Theorem 1.1 *[17, Theorem 1.1]. Let $f(z)$ be a transcendental meromorphic solution with zero order of equation (4), and a, b, c be three constants such that a, b cannot vanish simultaneously. Then,*

(i) $f(z)$ has infinitely many poles.

(ii) If $a \neq 0$, then $f(z)$ has infinitely many finite values.

(iii) If $a = 0$ and $f(z)$ takes a finite value A finitely often, then A is a solution of $2z^2 - cz - b = 0$.

Theorem 1.2 *[17, Theorem 1.2]. Let a, b, c and $|q| \neq 1$ be four constants, (i) if $a \neq 0$, then equation (4) has no rational solution;*

(ii) if $a = 0$, then the rational solutions of the equation (4) must satisfy $f(z) = B + \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are relatively prime polynomials and satisfy $\deg P < \deg Q$ and $2z^2 - cz - b = 0$.

Theorem 1.3 *[17, Theorem 1.3]. Let a, b, c be constants with $ac \neq 0$, and $f(z)$ be a transcendental meromorphic solution with zero order of equation (5). Then $f(z)$ has infinitely many poles and infinitely many finite values.*

Inspired by the above results, we further investigate some properties of transcendental meromorphic solutions of the q-difference Painlevé equation

$$f(qz) + f(z) + f(\frac{z}{q}) = \frac{az + b}{f(z)} + c, \tag{6}$$

which is different from (4) and (5) to some extent, and obtain the following theorems.

Theorem 1.4 *Let a, b, c be complex constants such that $|a| + |b| \neq 0$, and $f(z)$ be a zero-order transcendental meromorphic solution of the q-difference Painlevé equation (6).*

(i) If $a \neq 0$, $p(z)$ is a polynomial of degree $k(\geq 0)$ and $|q| \neq 1$, then $f(z) - p(z)$ has infinitely many zeros; if $a = 0$, then the Borel exceptional values of $f(z)$ can only come from the set $E = \{z|\ 3z^2 - cz - b = 0\}$;

(ii) $f(z)$ and $\Delta_q f(z)$ have infinitely many poles, where $\Delta_q f(z) = f(qz) - f(z)$.

Theorem 1.5 *Let a, b, c be complex constants such that $|a| + |b| \neq 0$.*

(i) If $a \neq 0$, then (6) has no rational solution.

(ii) If $a = 0$, then (6) has a nonzero constant solution $f(z) = B$, where B satisfies $3B^2 - cB - b = 0$. Furthermore, if $c^2 + 12b = 0$, then (6) has no nonconstant rational solution.

2 Some Lemmas

To prove our results, we require some lemmas as follows.

Lemma 2.1 *[14, Theorem 2.5] Let $f(z)$ be a transcendental meromorphic solution of order zero of a q-difference equation of the form*

$$U_q(z, f)P_q(z, f) = Q_q(z, f),$$

where $U_q(z, f), P_q(z, f)$ and $Q_q(z, f)$ are q-difference polynomials such that the total degree $\deg U_q(z, f) = n$ in $f(z)$ and its q-shifts, whereas $\deg Q_q(z, f) \leq n$. Moreover, we assume that $U_q(z, f)$ contains just one term of maximal total degree in $f(z)$ and its q-shifts. Then

$$m(r, P_q(z, f)) = o(T(r, f)),$$

on a set of logarithmic density 1.

Remark 2.1 *The above lemma can be called see as a type of a q-difference analogue of Clunie lemma, recently proved by Barnett et al.; see [1, Theorem 2.1].*

Remark 2.2 *Here, a q-difference polynomial of $f(z)$ for $q \in \mathbb{C}\backslash\{0,1\}$ is a polynomial in $f(z)$ and finitely many of its q-shifts $f(qz), \ldots, f(q^n z)$ with meromorphic coefficients in the sense that their Nevanlinna characteristic functions are $o(T(r,f))$ on a set of logarithmic density 1.*

Lemma 2.2 *[1, Theorem 2.5] Let $f(z)$ be a nonconstant zero-order meromorphic solution of $P_q(z,f)=0$, where $P_q(z,f)$ is a q-difference polynomial in $f(z)$. If $P_q(z,a)\not\equiv 0$ for slowly moving target $a(z)$, then*

$$m(r, \frac{1}{f-a}) = o(T(r,f))$$

on a set of logarithmic density 1.

Lemma 2.3 *[21, Theorem 1.1 and 1.3] Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C}\setminus\{0\}$. Then*

$$T(r,f(qz)) = (1+o(1))T(r,f), \qquad N(r,f(qz)) = (1+o(1))N(r,f),$$

on a set of lower logarithmic density 1.

Lemma 2.4 *(Valiron-Mohon'ko) ([13]). Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in $f(z)$,*

$$R(z,f(z)) = \frac{\sum_{i=0}^{m} a_i(z)f(z)^i}{\sum_{j=0}^{n} b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z,f(z))$ satisfies that

$$T(r,R(z,f(z))) = dT(r,f) + O(\Psi(r)),$$

where $d=\max\{m,n\}$ and $\Psi(r)=\max_{i,j}\{T(r,a_i),T(r,b_j)\}$.

3 Proof of Theorem 1.4

Suppose that $f(z)$ is a zero-order transcendental meromorphic solution of (6).

(i) If$a \neq 0$, and $p(z)$ is a polynomial of degree $k(\geq 0)$. Let $p(z) = a_k z^k + \cdots + a_1 z + a_0$. Let $g(z) = f(z) - p(z)$. Substituting $f(z) = g(z) + p(z)$ into equation (6), we have

$$g(qz) + p(qz) + g(z) + p(z) + g(\frac{z}{q}) + p(\frac{z}{q}) = \frac{az+b}{g(z)+p(z)} + c.$$

It follows that

$$\begin{aligned} P_q(z,g) := &\left[g(qz)+p(qz)+g(z)+p(z)+g(\frac{z}{q})+p(\frac{z}{q})\right][g(z)+p(z)] \\ &-(az+b)-c\left[g(z)+p(z)\right]=0. \end{aligned} \tag{7}$$

From (7), we have

$$P_q(z,0) = \left[p(qz)+p(z)+p(\frac{z}{q})\right]p(z) - (az+b) - cp(z). \tag{8}$$

If $p(z) \equiv 0$, then $P_q(z,0) = -(az+b) \not\equiv 0$. If $k=0$ and $p(z) = a_0 \equiv \alpha \in \mathbb{C} \setminus \{0\}$, then $P_q(z,0) = 3\alpha^2 - (az+b) - c\alpha \not\equiv 0$. If $k \geq 1$ and a_k is a nonzero constant, then, we have from (8) that

$$P_q(z,0) = \left[p(qz) + p(z) + p(\frac{z}{q})\right] p(z) - (az+b) - cp(z) = (q^k + 1 + \frac{1}{q^k})a_k^2 z^{2k} + \cdots . \quad (9)$$

Since $|q| \neq 1$, we have $q^k + 1 + \frac{1}{q^k} \neq 0$, then $P_q(z,0) \not\equiv 0$. Thus, we have by Lemma 2.2 that

$$m(r, \frac{1}{g}) = S(r,g).$$

Then, we get

$$N\left(r, \frac{1}{f-p}\right) = N\left(r, \frac{1}{g}\right) = T(r,g) + S(r,g) = T(r,f) + S(r,f). \quad (10)$$

Since $f(z)$ is transcendental, $f(z) - p(z)$ has infinitely many zeros.

If $a = 0$ and $p(z) = \beta \notin E$, then we have

$$P_q(z,0) = 3\beta^2 - c\beta - b \not\equiv 0.$$

Set $g(z) = f(z) - \beta$, by using the same argument as above, we can obtain $N(r, \frac{1}{f-\beta}) = T(r,f) + S(r,f)$.. Therefore, we can obtain that the Borel exceptional values of $f(z)$ can only come from the set $E = \{z | 3z^2 - cz - b = 0\}$.

(ii) From (6), we have

$$f(z)\left[f(qz) + f(z) + f(\frac{z}{q})\right] = az + b + cf(z). \quad (11)$$

It follows from Lemma 2.1 and (11) that

$$m\left(r, f(qz) + f(z) + f(\frac{z}{q})\right) = S(r,f). \quad (12)$$

By applying Lemma 2.4 for (6), we have

$$T\left(r, f(qz) + f(z) + f(\frac{z}{q})\right) = T(r,f) + S(r,f). \quad (13)$$

And by Lemma 2.3 we get

$$\begin{aligned} N\left(r, f(qz) + f(z) + f(\frac{z}{q})\right) &\leq N(r, f(qz)) + N(r, f(z)) + N\left(r, f(\frac{z}{q})\right) \\ &= 3(1+o(1))N(r,f) \quad (14) \end{aligned}$$

on a set of lower logarithmic density 1. Thus, by combining (12)-(14), we have

$$T(r,f) \leq 3(1+o(1))N(r,f) + S(r,f). \quad (15)$$

Since $f(z)$ is transcendental, $f(z)$ has infinitely many poles.

Next, we prove that $\Delta_q f(z)$ has infinitely many poles. Set $z = qw$, then we can rewrite (6) as the form

$$f(q^2w) + f(qw) + f(w) = \frac{aqw + b}{f(qw)} + c. \quad (16)$$

Then it follows from (16) that

$$f(qw)\left[f(q^2w)+f(qw)+f(w)\right]=aqw+b+cf(qw). \tag{17}$$

Since $\Delta_q f(w)=f(qw)-f(w)$, we have $f(qw)=\Delta_q f(w)+f(w)$ and $f(q^2w)=\Delta_q f(qw)+\Delta_q f(w)+f(w)$. Substituting them into (17), we get

$$[\Delta_q f(w)+f(w)]\left[\Delta_q f(qw)+2\Delta_q f(w)+3f(w)\right]=(aqw+b)+c\left[\Delta_q f(w)+f(w)\right],$$

i.e.,

$$\begin{aligned}-3f(w)^2=&\left[\Delta_q f(qw)+5\Delta_q f(w)-c\right]f(w)-(aqw+b)\\&+\left[\Delta_q f(qw)+2\Delta_q f(w)-c\right]\Delta_q f(w).\end{aligned} \tag{18}$$

Since $f(z)$ is a zero-order transcendental meromorphic function and $z=qw$, by Lemma 2.3, we get that $f(w)$ is of zero order. Thus, by Lemma 2.3 again, we have that $f(w),\Delta_q f(w),\Delta_q f(qw)$ are of zero-order. Then by Lemma 2.3 again, we have

$$N(r,\Delta_q f(qw))\le N(r,\Delta_q f(w))+S(r,f). \tag{19}$$

Thus, from (18) and (19) we have

$$\begin{aligned}2N(r,f(w))=&N\left(r,\left[\Delta_q f(qw)+3\Delta_q f(w)-c\right]f(w)-(aqw+b)\right.\\&+\left[\Delta_q f(qw)+\Delta_q f(w)-c\right]\Delta_q f(w)\\\le&N(r,f(w))+5N(r,\Delta_q f(w))+O(\log r)+S(r,f).\end{aligned}$$

That is,

$$N(r,f(w))\le 5N(r,\Delta_q f(w))+S(r,f). \tag{20}$$

Then, it follows from (15) and (20) that

$$T(r,f(w))\le 15N(r,\Delta_q f(w))+S(r,f). \tag{21}$$

Since $f(z)$ is transcendental, that is, $f(w)$ is transcendental, we have from (21) that $\Delta_q f(w)$ has infinitely many poles, that is, $\Delta_q f(z)$ has infinitely many poles.

Therefore, we complete the proof of Theorem 1.4.

4 Proof of Theorem 1.5

Sppose that $f(z)$ is a nonzero rational solution of (6), and has poles $z_1,z_2,\ldots,z_k$. Then, we let

$$\frac{\alpha_{is_i}}{(z-z_i)^{s_i}}+\cdots+\frac{\alpha_{is_1}}{(z-z_i)},\quad i=1,2,\ldots,k$$

be the principal parts of $f(z)$ at z_i respectively, where $\alpha_{is_i}\ne 0,\ldots,\alpha_{is_1}$ are constants, Thus, we can write $f(z)$ as the following form

$$f(z)=\sum_{i=1}^{k}\left(\frac{\alpha_{is_i}}{(z-z_i)^{s_i}}+\cdots+\frac{\alpha_{is_1}}{(z-z_i)}\right)+\beta_0+\beta_1 z+\cdots+\beta_m z^m, \tag{22}$$

where $\beta_0,\beta_1,\ldots,\beta_m$ are constants.

Next, we affirm that $\beta_m = \cdots = \beta_1 = 0$. Suppose that $\beta_m \neq 0 (m \geq 1)$. For sufficiently large z, by (22), we have

$$f(z) = \beta_m z^m (1 + o(1)), \tag{23}$$
$$f(qz) = \beta_m q^m z^m (1 + o(1)), \tag{24}$$
$$f(\frac{z}{q}) = \beta_m q^{-m} z^m (1 + o(1)). \tag{25}$$

By (6), we have

$$\left[f(qz) + f(z) + f(\frac{z}{q}) \right] f(z) = az + b + cf(z). \tag{26}$$

Substituting (23)-(25) into (26), we have

$$(1 + q^m + q^{-m})\beta_m^2 z^{2m}(1 + o(1)) = az + b + c\beta_m z^m (1 + o(1)).$$

Since $|q| \neq 1$, we have $1 + q^m + q^{-m} \neq 0$. And since $\beta_m \neq 0$, we can see the above equation is a contradiction for sufficiently large z. Hence we have $\beta_1 = \cdots = \beta_m = 0$.

(i) Suppose that $a \neq 0$. If $\beta_0 \neq 0$, then for sufficiently large z, by (23)-(25), we have

$$f(qz) = f(z) = f(\frac{z}{q}) = \beta_0 + o(1). \tag{27}$$

Substituting (27) into (26), we conclude that

$$(3\beta_0 + o(1))(\beta_0 + o(1)) = az + b + c(\beta_0 + o(1)),$$

which is a contradiction to the assumption that $a \neq 0$. Thus, $\beta_0 = 0$. Then we have $\beta_0 = \beta_1 = \cdots = \beta_m = 0$. Thus, $f(z)$ can be rewritten by (22) as

$$f(z) = \frac{P(z)}{R(z)}, \tag{28}$$

where

$$P(z) = pz^k + p_{k-1}z^{k-1} + \cdots + p_0, \quad R(z) = rz^t + r_{t-1}z^{t-1} + \cdots + r_0, \tag{29}$$

where $p, p_{k-1}, \ldots, p_0$ and $r, r_{t-1}, \ldots, r_0$ are constants such that $pr \neq 0$ and $k < t$. Then substituting (28) into (6), we have

$$\begin{aligned} &P(qz)P(z)R(z)R(\frac{z}{q}) + P(z)^2 R(qz)R(\frac{z}{q}) + P(\frac{z}{q})P(z)R(qz)R(z) \\ &= (az + b)R(qz)R(z)^2 R(\frac{z}{q}) + cP(z)R(qz)R(z)R(\frac{z}{q}). \end{aligned} \tag{30}$$

Then since $k < t$, we can see that the degree of the left side of (30) does not exceed $2k + 2t$, and the degree of the right side of (30) is equal to $1 + 4t$ by $a \neq 0$. Thus, we can get a contradiction. Therefore, we have that (6) has no nonzero rational solution when $a \neq 0$.

(ii) Suppose that $a = 0$. If $f(z) = B$ is a nonzero constant solution of (6), we can easily get from (6) that B satisfies $3B^2 - cB - b = 0$. Now, we prove that (6) has no rational solution if $a = 0$ and $c^2 + 12b = 0$. Suppose that $f(z)$ is a nonconstant rational solution of (6). Since $\beta_m = 0 (m \geq 1)$, $f(z)$ can be rewritten as the form (28), where

$P(z)$ and $R(z)$ satisfy (29) with $k \leq t$. Suppose that $k < t$. Substituting (28) into (6), we have

$$\begin{aligned} &P(qz)P(z)R(z)R(\frac{z}{q}) + P(z)^2R(qz)R(\frac{z}{q}) + P(\frac{z}{q})P(z)R(qz)R(z) \\ &\quad = bR(qz)R(z)^2R(\frac{z}{q}) + cP(z)R(qz)R(z)R(\frac{z}{q}). \end{aligned} \tag{31}$$

If $k < t$, then it follows from (31) that there exists only one term $bR(qz)R(z)^2R(\frac{z}{q})$ with maximal degree, which is a contradiction. Thus, we have $k = t$. Then, it follows by (29) and (30) that

$$\begin{aligned} &\frac{pq^kz^k + p_{k-1}q^{k-1}z^{k-1} + \cdots + p_0}{rq^tz^t + r_{t-1}q^{t-1}z^{t-1} + \cdots + r_0} + \frac{pz^z + p_{k-1}z^{k-1} + \cdots + p_0}{rz^t + r_{t-1}z^{t-1} + \cdots + r_0} \\ &+ \frac{pq^{-k}z^k + p_{k-1}q^{-(k-1)}z^{k-1} + \cdots + p_0}{rq^{-t}z^t + r_{t-1}q^{-(t-1)}z^{t-1} + \cdots + r_0} \\ &= \frac{b(rz^t + r_{t-1}z^{t-1} + \cdots + r_0)}{pz^k + p_{k-1}z^{k-1} + \cdots + p_0} + c. \end{aligned} \tag{32}$$

Then it follows from (32) that

$$3B^2 - cB - b = 0,$$

as $z \to \infty$, where $B = \frac{p}{r} \neq 0$. Therefore, $f(z)$ can be rewritten as

$$f(z) = B + \frac{G(z)}{H(z)}, \tag{33}$$

where $G(z)$ and $H(z)$ are relatively prime polynomials and satisfy $\deg G(z) = \mu < \deg H(z) = \nu$, B is a constant satisfying $3B^2 - cB - b = 0$. Denote

$$G(z) = \xi z^\mu + \xi_{\mu-1}z^{\mu-1} + \cdots + \xi_0, \quad H(z) = \eta z^\nu + \eta_{\nu-1}z^{\nu-1} + \cdots + \eta_0, \tag{34}$$

where $\xi, \xi_{\mu-1}, \ldots, p_0$ and $\eta, \eta_{\nu-1}, \ldots, \eta_0$ are constants such that $\xi\eta \neq 0$. Substituting (34) into (6) and noting $3B^2 - cB - b = 0$, we have

$$\begin{aligned} &(4B - c)G(z)H(qz)H(z)H(\frac{z}{q}) + BG(qz)H(z)^2H(\frac{z}{q}) + BG(\frac{z}{q})H(z)^2H(qz) \\ &= -G(qz)G(z)H(z)H(\frac{z}{q}) - G(z)^2H(qz)H(\frac{z}{q}) - G(\frac{z}{q})G(z)H(z)H(qz). \end{aligned} \tag{35}$$

By observing the coefficients and degrees of all terms of the above equation, and combining with $\nu > \mu$, we have that the term with maximal degree of (35) is

$$\left[(4B - c) + Bq^{\mu-\nu} + Bq^{\nu-\mu}\right]\xi\eta^3z^{\mu+3\nu}.$$

Since $3B^2 - cB - b = 0$ and $c^2 + 12b = 0$, we have $B = \frac{c}{6}$. And by $|q| \neq 1$, we can get that $(4B - c) + Bq^{\mu-\nu} + Bq^{\nu-\mu} \neq 0$. In fact, if $(4B - c) + Bq^{\mu-\nu} + Bq^{\nu-\mu} = 0$, *i.e.*

$$B = \frac{c}{4 + q^{\mu-\nu} + q^{\nu-\mu}}.$$

Then, we have

$$\frac{c}{4 + q^{\mu-\nu} + q^{\nu-\mu}} = \frac{c}{6}.$$

By solving the above equation, we get $|q| = 1$, a contradiction. Thus, (35) is a contradiction for sufficiently large z. Therefore, if $a = 0$ and $c^2 + 12b = 0$, then (6) has no nonconstant rational solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors drafted the manuscript, read and approved the final manuscript.

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New approximation of fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces

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Abstract

We prove necessary and sufficient conditions for the strong convergence of the modified two-step iteration process to the fixed point of asymptotically demicontractive mappings in real Banach spaces.

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1 Introduction

Let K be a nonempty subset of a real Banach space X and X^* be its dual space. We denote by J the normalized duality mapping from X into 2^{X^*} defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If X is strictly convex, then J is single-valued. In the sequel, we shall denote the single-valued duality mapping by j.

Let $T : K \to K$ be a mapping.

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Definition 1.1. T is called a *k-strictly asymptotically pseudo-contractive mapping* with sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ if for all $x, y \in K$ there exists $j(x-y) \in J(x-y)$ and a constant $k \in [0, 1)$ such that

$$\begin{aligned} &\langle (I-T^n)x - (I-T^n)y, j(x-y)\rangle \\ &\geq \frac{1}{2}(1-k)\left\|(I-T^n)x-(I-T^n)y\right\|^2 - \frac{1}{2}(k_n^2-1)\left\|x-y\right\|^2 \end{aligned} \tag{1.1}$$

for all $n \in \mathbb{N}$.

Definition 1.2. T is called an *asymptotically demicontractive mapping* with sequence $\{k_n\} \subset [0, \infty)$, $\lim_{n\to\infty} k_n = 1$ if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and for all $x \in K$ and $x^* \in F(T)$, there exists $k \in [0, 1)$ and $j(x-x^*) \in J(x-x^*)$ such that

$$\langle x - T^n x, j(x-x^*)\rangle \geq \frac{1}{2}(1-k)\left\|x-T^n x\right\|^2 - \frac{1}{2}(k_n^2-1)\left\|x-x^*\right\|^2 \tag{1.2}$$

for all $n \in \mathbb{N}$.

Definition 1.3. $T : K \to K$ is called *uniformly L-Lipschitizian* if there exists a constant $L > 0$ such that

$$\left\|T^n x - T^n y\right\| \leq L\left\|x-y\right\|, \tag{1.3}$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

The classes of k-strictly asymptotically pseudo-contractive and asymptotically demicontractive mappings are introduced by Liu [3]. It is easy to see that a k-strictly asymptotically pseudo-contrative mapping with a non-empty fixed point set $F(T)$ is asymptotically demicontractive.

In Hilbert spaces, it is shown in [3] that (1.1) and (1.2) are equivalent to the following inequalities:

$$\left\|T^n x - T^n y\right\| \leq k_n^2\left\|x-y\right\|^2 + k\left\|(I-T^n)x-(I-T^n)y\right\|^2$$

and

$$\left\|T^n x - T^n y\right\|^2 \leq k_n^2\left\|x-y\right\|^2 + \left\|x-T^n x\right\|^2,$$

respectively.

By using the modified Mann iteration method [4] introduced by Schu [7], Liu [3] proved a convergence theorem for the iterative approximation of fixed points of k-strictly asymptotically pseudo-contractive mappings and asymptotically demicontractive mappings in Hilbert spaces.

Osilike [6] extended the results of Liu [3] about the iterative approximation of fixed points of k-strictly asymptotically demicontractive mappings from Hilbert spaces to much more general real q-uniformly smooth Banach spaces, $1 < q < \infty$ and specifically proved the following results.

Theorem 1.4. *Let $q > 1$ and X be a real q-uniformly smooth Banach space. Let K be a closed convex and bounded subset of X and $T : K \to K$ a completely continuous uniformly L-Lipschitizian asymptotically demicontractive mapping with a sequence $k_n \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the conditions*

(i) $0 \leq \alpha_n, \beta_n \leq 1$, $n \geq 1$;

(ii) $0 < \epsilon \leq c_q \alpha_n^{q-1}(1 + L\beta_n)^q \leq \frac{1}{2}q(1-k)(1+L)^{-(q-2)} - \epsilon$ *for all* $n \geq 1$ *and for some* $\epsilon > 0$; *and*

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$.

Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \quad n \geq 1 \end{cases}$$

converges strongly to a fixed point of T.

Remark 1.5. For Hilbert spaces, in Theorem 1.4, if we put $q = 2$, $c_q = 1$ and $\beta_n = 0$, then Theorems 1 and 2 of Liu [3] follow.

Recently Chidume and Măruşter [1] made a comprehensive and very useful survey on the main convergence properties of the modified Mann iteration method for the demicontractive mappings.

The purpose of this work is to prove necessary and sufficient conditions for the strong convergence of the modified two-step iteration process to the fixed point of asymptotically demicontractive mappings in real Banach spaces. Our results extend and improve the results of Igbokwe [2], Liu [3], Moore and Nnoli [5].

2 Main results

The following results are useful:

Lemma 2.1. ([8]) *For all $\varrho, \varsigma \in X$ and $j(\varrho + \varsigma) \in J(\varrho + \varsigma)$,*

$$\|\varrho + \varsigma\|^2 \leq \|\varrho\|^2 + 2\mathrm{Re}\,\langle \varsigma, j(\varrho + \varsigma)\rangle .$$

Lemma 2.2. ([2]) *Let X be a normed space and K be a nonempty convex subset of X. Let $T : K \to K$ be uniformly L-Lipschitzian mapping and let $\{t_n\}$ and $\{\beta_n\}$ be the sequences in $[0, 1]$. For arbitrary $\varrho_1 \in K$, generate the sequence $\{\varrho_n\}$ by*

$$\begin{cases} \varrho_{n+1} = (1-t_n)\varrho_n + t_n T^n \varsigma_n, \\ \varsigma_n = (1-\beta_n)\varrho_n + \beta_n T^n \varrho_n, \quad n \geq 1. \end{cases}$$

Then

$$\|\varrho_n - T\varrho_n\| \leq \|\varrho_n - T^n \varrho_n\| + L(1+L)^2 \left\|\varrho_{n-1} - T^{n-1}\varrho_{n-1}\right\|. \tag{2.1}$$

We now prove our main results.

Lemma 2.3. *Let X be a real Banach space and K be a nonempty convex subset of X. Let $T : K \to K$ be an uniformly $L-$Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $\varrho_1 \in K$, generate the sequence $\{\varrho_n\}$ by*

$$\begin{cases} \varrho_{n+1} = (1-t_n)\varrho_n + t_n T^n \varsigma_n, \\ \varsigma_n = (1-\beta_n)\varrho_n + \beta_n T^n \varrho_n, \quad n \geq 1, \end{cases} \tag{2.2}$$

where $\{t_n\}$ and $\{\beta_n\}$ are the sequences in $[0, 1]$ satisfying

(i) $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $\lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} \beta_n$.

Then (a) the sequence $\{\varrho_n\}$ is bounded,

(b) $\liminf_{n\to\infty} \|\varrho_{n+1} - T^n \varrho_{n+1}\| = 0$,

(c) $\liminf_{n\to\infty} \|\varrho_n - T^n \varrho_n\| = 0$,

(d) $\liminf_{n\to\infty} \|\varrho_n - T\varrho_n\| = 0$.

Proof. Since T is asymptotically demicontractive, then

$$\langle \varrho - T^n\varrho, j(\varrho - \varrho^*)\rangle \geq \frac{1}{2}(1-k)\|\varrho - T^n\varrho\|^2 - \frac{1}{2}(k_n^2 - 1)\|\varrho - \varrho^*\|^2$$

and hence

$$\|\varrho - T^n\varrho\| \leq \sqrt{\frac{(2\|\varrho - T^n\varrho\| + (k_n^2 - 1)\|\varrho - \varrho^*\|)\|\varrho - \varrho^*\|}{1-k}}.$$

Therefore, by the triangle inequality,

$$\|\varrho - \varrho^*\| \leqslant \|T^n\varrho - \varrho^*\| + \sqrt{\frac{(2\|\varrho - T^n\varrho\| + (k_n^2 - 1)\|\varrho - \varrho^*\|)\|\varrho - \varrho^*\|}{1-k}}. \tag{2.3}$$

Now we shall prove that

$$\liminf_{n\to\infty} \|\varrho_{n+1} - T^n\varrho_{n+1}\| = 0.$$

If $\varrho_n = T\varrho_n$ for all $n \geqslant m$ for some $m \in \mathbb{N}$, then (2.3) trivially holds, as we have

$$\begin{aligned} \|\varrho_{n+1} - T^n\varrho_{n+1}\| &= \|\varrho_{n+1} - T^nT\varrho_{n+1}\| = \left\|\varrho_{n+1} - T^{n+1}\varrho_{n+1}\right\| \\ &= 0 \end{aligned}$$

for all $n \geq m$.

Suppose now that there exists the smallest positive integer n_0 such that $\varrho_{n_0} \neq T\varrho_{n_0}$. Put

$$\begin{aligned} a_0 := {} & \|T^{n_0}\varrho_{n_0} - \varrho^*\| \\ & + \sqrt{\frac{\left(2\|\varrho_{n_0} - T^{n_0}\varrho_{n_0}\| + (k_{n_0}^2 - 1)\|\varrho_{n_0} - \varrho^*\|\right)\|\varrho_{n_0} - \varrho^*\|}{1-k}} + 1. \end{aligned}$$

Then clearly

$$\|\varrho_{n_0} - \varrho^*\| \leq a_0. \tag{2.4}$$

To prove that $\liminf_{n\to\infty} \|\varrho_{n+1} - T^n\varrho_{n+1}\| = 0$, we shall assume, to the contrary, that $\liminf_{n\to\infty} \|\varrho_{n+1} - T^n\varrho_{n+1}\| = 2\delta > 0$. Then there exists $n_0' \in \mathbb{N}$ such that $\|\varrho_{n+1} - T^n\varrho_{n+1}\| \geqslant \delta$ for all $n \geq n_0'$.

Also, by $\lim_{n\to\infty} k_n = 1$ and (ii), we may suppose that

$$\begin{aligned} t_n &\leq \min\left\{\frac{1}{1+2L}, \frac{(1-k)\delta^2}{24(1+L)(1+2L)a_0^2}\right\}, \\ \beta_n &\leq \min\left\{\frac{1}{1+L}, \frac{(1-k)\delta^2}{24L(1+L)a_0^2}\right\}, \\ k_n^2 - 1 &\leq \frac{(1-k)\delta^2}{24a_0^2} \end{aligned} \tag{2.5}$$

for all $n \geq n_0'$.

We now show that the sequence $\{\varrho_n\}$ is bounded. By induction we shall show that

$$\|\varrho_n - \varrho^*\| \leqslant a_0 \tag{2.6}$$

for all $n \geq n_0'$.

It is clear that (2.6) holds for $n = n_0$. Assume it is true for some $n > N := \max\{n_0, n_0'\}$, that is, $\|\varrho_n - \varrho^*\| \leq a_0$ for some $n \geq N$. Then

$$\begin{aligned} \|\varrho_n - T^n\varrho_n\| &\leq \|\varrho_n - \varrho^*\| + \|T^n\varrho_n - \varrho^*\| \\ &\leq (1+L)\|\varrho_n - \varrho^*\| \\ &\leq (1+L)a_0, \end{aligned}$$

$$\begin{aligned} \|\varsigma_n - \varrho^*\| &= \|(1-\beta_n)\varrho_n + \beta_n T^n\varrho_n - \varrho^*\| \\ &= \|\varrho_n - \varrho^* - \beta_n(\varrho_n - T^n\varrho_n)\| \\ &\leq \|\varrho_n - \varrho^*\| + \beta_n\|\varrho_n - T^n\varrho_n\| \\ &\leq a_0 + (1+L)a_0\beta_n \\ &\leq 2a_0, \end{aligned}$$

$$\begin{aligned} \|\varrho_n - T^n\varsigma_n\| &\leq \|\varrho_n - \varrho^*\| + \|T^n\varsigma_n - \varrho^*\| \\ &\leq \|\varrho_n - \varrho^*\| + L\|\varsigma_n - \varrho^*\| \\ &\leq (1+2L)a_0, \end{aligned}$$

and

$$\begin{aligned} \|\varrho_{n+1} - \varrho^*\| &= \|(1-t_n)\varrho_n + t_n T^n\varsigma_n - \varrho^*\| \\ &= \|\varrho_n - \varrho^* - t_n(\varrho_n - T^n\varsigma_n)\| \\ &\leq \|\varrho_n - \varrho^*\| + t_n\|\varrho_n - T^n\varsigma_n\| \\ &\leq a_0 + (1+2L)a_0 t_n \\ &\leq 2a_0. \end{aligned} \tag{2.7}$$

On the other hand, by Lemma 2.1,

$$\begin{aligned}
\|\varrho_{n+1}-\varrho^*\|^2 &= \|(1-t_n)\varrho_n + t_n T^n \varsigma_n - \varrho^*\|^2 \\
&= \|\varrho_n - \varrho^* - t_n(\varrho_n - T^n\varsigma_n)\|^2 \\
&\le \|\varrho_n-\varrho^*\|^2 - 2t_n \left\langle \varrho_n - T^n\varsigma_n, j(\varrho_{n+1}-\varrho^*)\right\rangle \\
&= \|\varrho_n-\varrho^*\|^2 - 2t_n \left\langle \varrho_{n+1} - T^n\varrho_{n+1}, j(\varrho_{n+1}-\varrho^*)\right\rangle \\
&\quad + 2t_n \left\langle T^n\varsigma_n - T^n\varrho_{n+1}, j(\varrho_{n+1}-\varrho^*)\right\rangle + 2t_n \left\langle \varrho_{n+1}-\varrho_n, j(\varrho_{n+1}-\varrho^*)\right\rangle .
\end{aligned}$$

Since T is asymptotically demicontractive mapping, we obtain

$$\begin{aligned}
\|\varrho_{n+1}-\varrho^*\|^2 \le \|\varrho_n-\varrho^*\|^2 &- (1-k)t_n \|\varrho_{n+1} - T^n\varrho_{n+1}\|^2 \\
&+ \left(k_n^2-1\right) t_n \|\varrho_{n+1}-\varrho^*\|^2 \\
&+ 2(1+L)t_n \|\varrho_{n+1}-\varrho_n\| \|\varrho_{n+1}-\varrho^*\| \\
&+ 2Lt_n \|\varsigma_n - \varrho_n\| \|\varrho_{n+1}-\varrho^*\| .
\end{aligned} \tag{2.8}$$

Consider the following estimates,

$$\begin{aligned}
\|\varsigma_n - \varrho_n\| &= \|(1-\beta_n)\varrho_n + \beta_n T^n\varrho_n - \varrho_n\| \\
&= \beta_n \|\varrho_n - T^n\varrho_n\| \\
&\le (1+L)a_0 t_n,
\end{aligned}$$

and

$$\begin{aligned}
\|\varrho_{n+1} - \varrho_n\| &= \|(1-t_n)\varrho_n + t_n T^n\varsigma_n - \varrho_n\| \\
&= t_n \|\varrho_n - T^n\varsigma_n\| \\
&\le (1+2L)a_0 t_n,
\end{aligned}$$

so that (2.8), takes the form

$$\begin{aligned}
\|\varrho_{n+1}-\varrho^*\|^2 \le \|\varrho_n-\varrho^*\|^2 &- (1-k)t_n \|\varrho_{n+1} - T^n\varrho_{n+1}\|^2 \\
&+ \left(k_n^2-1\right) t_n \|\varrho_{n+1}-\varrho^*\|^2 \\
&+ 2(1+L)(1+2L)a_0 t_n^2 \|\varrho_{n+1}-\varrho^*\| \\
&+ 2L(1+L)a_0 t_n \beta_n \|\varrho_{n+1}-\varrho^*\| .
\end{aligned}$$

Then, by (2.5),

$$\begin{aligned}
\|\varrho_{n+1}-\varrho^*\|^2 &\le \|\varrho_n-\varrho^*\|^2 - (1-k)\delta^2 t_n \\
&\quad + 4a_0^2 \left[\left(k_n^2-1\right) + (1+L)(1+2L)t_n + L(1+L)\beta_n\right] t_n \\
&\le \|\varrho_n-\varrho^*\|^2 - (1-k)\delta^2 t_n + \frac{1}{2}(1-k)\delta^2 t_n
\end{aligned}$$

and hence

$$\|\varrho_{n+1}-\varrho^*\|^2 \leqslant \|\varrho_n-\varrho^*\|^2 - \frac{1}{2}(1-k)\delta^2 t_n. \tag{2.9}$$

Thus $\|\varrho_{n+1}-\varrho^*\| \le \|\varrho_n-\varrho^*\| \le a_0$ and so we proved (2.6). Therefore, we proved (a).

From (2.9) we have that for every $r > N$,

$$\begin{aligned}\frac{1}{2}(1-k)\delta^2 \sum_{n=N}^{r} t_n &\le \sum_{n=N}^{r} (\|\varrho_n - \varrho^*\|^2 - \|\varrho_{n+1} - \varrho^*\|^2) \\ &\le \|\varrho_N - \varrho^*\|^2 .\end{aligned}$$

Hence we have $\sum_{n=1}^{\infty} t_n < \infty$, a contradiction with the condition (i). Therefore, our assumption $\delta > 0$ was wrong. Thus

$$\liminf_{n\to\infty} \|\varrho_{n+1} - T^n \varrho_{n+1}\| = 0. \tag{2.10}$$

Therefore, we proved (b).

Now according to Lemma 2.1, substituting $\varrho = u + v$ and $\varsigma = -v$, we obtain

$$\|u+v\|^2 \ge \|u\|^2 + 2\langle v, j(u)\rangle ,$$

which is mainly due to Igbokwe [2].

By (2.2) we have

$$\begin{aligned}\|\varrho_{n+1} - T^n\varrho_{n+1}\|^2 &= \|(1-t_n)\varrho_n + t_n T^n \varsigma_n - T^n \varrho_{n+1}\|^2 \\ &= \|\varrho_n - T^n\varrho_n - t_n(\varrho_n - T^n\varsigma_n) - (T^n\varrho_{n+1} - T^n\varrho_n)\|^2 .\end{aligned} \tag{2.11}$$

Then by (2.11) we get

$$\begin{aligned}\|\varrho_{n+1} - T^n\varrho_{n+1}\|^2 \ge{}& \|\varrho_n - T^n\varrho_n\|^2 \\ &- 2\langle t_n(\varrho_n - T^n\varsigma_n) + (T^n\varrho_{n+1} - T^n\varrho_n), j(\varrho_n - T^n\varrho_n)\rangle .\end{aligned}$$

Thus

$$\begin{aligned}\|\varrho_n - T^n\varrho_n\|^2 &\le \|\varrho_{n+1} - T^n\varrho_{n+1}\|^2 \\ &\quad + 2\langle t_n(\varrho_n - T^n\varsigma_n) + (T^n\varrho_{n+1} - T^n\varrho_n), j(\varrho_n - T^n\varrho_n)\rangle \\ &\le \|\varrho_{n+1} - T^n\varrho_{n+1}\|^2 \\ &\quad + 2\|t_n(\varrho_n - T^n\varsigma_n) + (T^n\varrho_{n+1} - T^n\varrho_n)\| \|\varrho_n - T^n\varrho_n\| .\end{aligned} \tag{2.12}$$

Further,

$$\begin{aligned}\|t_n(\varrho_n - T^n\varsigma_n) + (T^n\varrho_{n+1} - T^n\varrho_n)\| &\le t_n\|\varrho_n - T^n\varsigma_n\| + \|T^n\varrho_{n+1} - T^n\varrho_n\| \\ &\le (1+2L)a_0 t_n + L\|\varrho_{n+1} - \varrho_n\| \\ &\le (1+2L)a_0 t_n + L(1+2L)a_0 t_n \\ &= (1+L)(1+2L)a_0 t_n.\end{aligned}$$

Therefore, from (2.12), we get

$$\|\varrho_n - T^n\varrho_n\|^2 \le \|\varrho_{n+1} - T^n\varrho_{n+1}\|^2 + 2(1+L)^2(1+2L)a_0^2 t_n. \tag{2.13}$$

From (2.13), (ii) and (b),

$$\liminf_{n\to\infty} \|\varrho_n - T^n\varrho_n\| = 0. \tag{2.14}$$

Thus we proved (c).

At last, from (2.14) and Lemma 2.2, we obtain (d). This completes the proof. $\square$

Theorem 2.4. *Let X be a real Banach space and K be a nonempty convex subset of X. Let $T : K \to K$ be an uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $\varrho_1 \in K$, generate the sequence $\{\varrho_n\}$ by*

$$\begin{cases} \varrho_{n+1} = (1-t_n)\varrho_n + t_n T^n \varsigma_n, \\ \varsigma_n = (1-\beta_n)\varrho_n + \beta_n T^n \varrho_n, \quad n \geq 1, \end{cases}$$

where $\{t_n\}$ and $\{\beta_n\}$ are the sequences in $[0,1]$ satisfying

(i) $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $\lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} \beta_n$.

If T is completely continuos, then $\{\varrho_n\}$ converges strongly to some fixed point of T in K.

Proof. From Lemma 2.3, $\liminf_{n\to\infty} \|\varrho_n - T\varrho_n\| = 0$. Therefore, there exists a subsequence $\{\varrho_{n_j}\}$ of $\{\varrho_n\}$ such that $\lim_{j\to\infty} \left\|\varrho_{n_j} - T\varrho_{n_j}\right\| = 0$. Since $\{\varrho_{n_j}\}$ is bounded and T is completely continuous, then $\{T\varrho_{n_j}\}$ has a subsequence $\{T\varrho_{n_{j_k}}\}$, which converges strongly. Hence $\{\varrho_{n_{j_k}}\}$ converges strongly. Let $\lim_{k\to\infty} \varrho_{n_{j_k}} = p$. Then $\lim_{k\to\infty} T\varrho_{n_{j_k}} = Tp$. Thus we have $\lim_{k\to\infty} \left\|\varrho_{n_{j_k}} - T\varrho_{n_{j_k}}\right\| = \|p - Tp\| = 0$. Hence $p \in F(T)$. From (2.9) and Lemma 2.3 it follows that $\lim_{n\to\infty} \|\varrho_n - p\| = 0$. This completes the proof. □

Remark 2.5. 1. We generalize the results of Liu [3] from Hilbert spaces to more general Banach spaces. Moreover the boundedness assumption on the subset K is removed.

2. One can see that, with $\sum_{n=1}^{\infty} t_n = \infty$, the condition $\sum_{n=1}^{\infty} t_n^2 < \infty$ is not always true. Let us take $t_n = \frac{1}{\sqrt{n}}$. Then obviously $\sum_{n=1}^{\infty} t_n = \infty$, but $\sum_{n=1}^{\infty} t_n^2 = \infty$. Hence the results of Igbokwe [2] are need to be improve.

3. We improve the results of Moore and Nnoli [5] by removing the conditions like $\liminf_{n\to\infty} d(\varrho_n, F(T)) = 0$.

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VISCOSITY APPROXIMATION OF SOLUTIONS OF FIXED POINT AND VARIATIONAL INCLUSION PROBLEMS

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Abstract. In this paper, fixed point and variational inclusion problems are investigated based on a viscosity approximation method. Strong convergence theorems are established without the aid of metric projections in the framework of Hilbert spaces.

Keywords: maximal monotone operator; fixed point; proximal point algorithm; zero point.

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1. Introduction

A very common problem in diverse areas of mathematics and physical sciences consist of finding a solution which satisfies certain constraints. This problem is referred to as the convex feasibility problem. It can be described as follows: Suppose $C_1, C_2, \cdots, C_r$, where r is some positive integer, are finitely many nonempty convex closed subset of a Hilbert space H with $C = \cap_{i=1}^{r} \neq \emptyset$. The convex feasibility problem is to find a point in C. In the real world, many important problems have reformulations which require finding fixed points of some nonlinear operators, for instance, evolution equations, complementarity problems, mini-max problems, variational inequalities and zero point problems; see [1-13] and the references therein.

In this paper, we are concerned with the problem of finding a common solution of fixed point and inclusion problems. Many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as this problem. One of the most popular methods for solving inclusion problems goes back to the work of Browder [14]. The basic ideas is to reduce inclusion problems to fixed point problems of nonlinear operators. In this paper, we study a regularization method for two monotone and a nonexpansive mappings. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a viscosity approximation method is introduced. A strong convergence theorem of common solutions is established. In Section 4, applications of the main results are discussed.

2. Preliminaries

In what follows, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty, convex and closed subset of H. Let $S : C \to C$ be a mapping. $Fix(S)$ stands for the fixed point set of S; that is, $Fix(S) := \{x \in C : x = Sx\}$. Recall that S is said to be *κ-contractive* iff there exists a constant $\kappa \in (0, 1)$ such that

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$\|Sx - Sy\| \le \kappa\|x - y\|$, $\forall x, y \in C$. It is well known that every contractive mapping has a unique fixed point in metric spaces. The Picard iterative algorithm $x_{n+1} = Sx_n$ converge to the fixed point of S. S is said to be *nonexpansive* iff $\|Sx - Sy\| \le \|x - y\|$, $\forall x, y \in C$. If C is a bounded, closed, and convex subset of H, then $F(S)$ is not empty; see [15] and the references therein. Since the nonexpansivity of S, the Picard iterative algorithm may not converge to fixed points of S. The Mann iterative algorithm is powerful and efficient to study fixed points of nonexpansive mappings. However, in infinite dimensional spaces, the Mann iterative algorithm is only weak convergence. To obtain strong convergence of the Mann iterative algorithm, different regularization methods have been investigated recently; see [16]-[29] and the references therein.

Let $A : C \to H$ be a mapping. Recall that A is said to be *monotone* iff $\langle Ax-Ay, x-y\rangle \ge 0$, $\forall x, y \in C$. Recall that A is said to be *inverse-strongly monotone* iff there exists a constant $\alpha > 0$ such that $\langle Ax - Ay, x - y\rangle \ge \alpha\|Ax - Ay\|^2$, $\forall x, y \in C$. For such a case, A is also said to be *α-inverse-strongly monotone*. It is not hard to see that every inverse-strongly monotone mapping is monotone and continuous. Recall that a set-valued mapping $B : H \rightrightarrows H$ is said to be *monotone* iff, for all $x, y \in H$, $f \in Bx$ and $g \in By$ imply $\langle x - y, f - g\rangle \ge 0$. In this paper, we use $B^{-1}(0)$ to stand for the zero point of B. A monotone mapping $B : H \rightrightarrows H$ is *maximal* iff the graph $Graph(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$, for all $(y, g) \in Graph(B)$ implies $f \in Bx$. For a maximal monotone operator B on H, and $r > 0$, we may define the single-valued resolvent $J_r : H \to Dom(B)$, where $Dom(B)$ denote the domain of B. It is known that J_r is firmly nonexpansive, and $B^{-1}(0) = F(J_r)$.

In this paper, we study fixed points of nonexpansive mappings and zero points of two monotone mappings based on a viscosity approximation method. Strong convergence theorems are established in the framework of Hilbert spaces. The results obtained in this paper mainly improve the corresponding results in [23]-[29]. In order to prove our main results, we also need the following lemmas.

Lemma 2.1 [30] *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition $a_{n+1} \le (1 - t_n)a_n + t_nb_n + c_n$, $\forall n \ge 0$, where $\{t_n\}$ is a number sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$, $\{b_n\}$ is a number sequence such that $\limsup_{n\to\infty} b_n \le 0$, and $\{c_n\}$ is a positive number sequence such that $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.2. [31] *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal monotone operator on H. Then $(A + B)^{-1}(0) = F(J_r(I - rA))$.*

Lemma 2.3. [32] *Let H be a Hilbert space, and A an maximal monotone operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have $J_\lambda x = J_\mu\left(\left(1 - \frac{\mu}{\lambda}\right)J_\lambda x + \frac{\mu}{\lambda}x\right)$, where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.*

Lemma 2.4. [14] *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T be a nonexpansive mapping on C. Then $I - T$ is demiclosed at origin.*

3. Main results

Theorem 3.1. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal*

monotone operator on H. Let S be a fixed κ-contraction and let T be a nonexpansive mapping on C. Assume $Dom(B) \subset C$ and $(A+B)^{-1}(0) \cap Fix(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$ and let $\{r_n\}$ be a positive real number sequence in $(0, 2\alpha)$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$, $y_n = \alpha_n Sx_n + (1-\alpha_n)Tx_n$, $x_{n+1} \approx (I+r_nB)^{-1}(y_n - r_nAy_n)$, $\forall n \geq 0$. Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - (I+r_nB)^{-1}(y_n - r_nAy_n)\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 2\alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in (A+B)^{-1}(0) \cap Fix(T)$, where $\bar{x} = Proj_{(A+B)^{-1}(0)\cap Fix(T)} S\bar{x}$.

Proof. First, we show that $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Using the restrictions imposed on $\{r_n\}$, one see that $I - r_nA$ is nonexpansive. Indeed, we have

$$\begin{aligned}
&\|(I-r_nA)x - (I-r_nA)y\|^2 \\
&\leq \|x-y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

That is, $\|(I-r_nA)x - (I-r_nA)y\| \leq \|x-y\|$. Fixing $p \in (A+B)^{-1}(0) \cap Fix(T)$, we find that

$$\begin{aligned}
\|y_n - p\| &\leq \alpha_n\|Sx_n - p\| + (1-\alpha_n)\|Tx_n - p\| \\
&\leq \alpha_n\|Sx_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - p\| + \alpha_n\|Sp - p\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|e_n\| + \|(I+r_nB)^{-1}(y_n - r_nAy_n) - p\| \\
&\leq e_n + \|(y_n - r_nAy_n) - (I - r_nA)p\| \\
&\leq e_n + \big(1-\alpha_n(1-\kappa)\big)\|x_n - p\| + \alpha_n(1-\kappa)\frac{\|Sp-p\|}{1-\kappa} \\
&\leq \max\{\|x_n - p\|, \frac{\|Sp-p\|}{1-\kappa}\} + e_n \\
&\ \vdots \\
&\leq \max\{\|x_0 - p\|, \frac{\|Sp-p\|}{1-\kappa}\} + \sum_{i=0}^{\infty} e_i < \infty.
\end{aligned}$$

This proves that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$. Notice that

$$\|y_n - y_{n-1}\| \leq \big(1-\alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|.$$

Setting $z_n = y_n - r_nAy_n$, one further has

$$\begin{aligned}
\|z_n - z_{n-1}\| &\leq \|y_n - y_{n-1}\| + \|r_n - r_{n-1}\|\|Ay_{n-1}\| \\
&\leq \big(1-\alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + |r_n - r_{n-1}|\|Ay_{n-1}\| \qquad (3.1)\\
&\quad + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|.
\end{aligned}$$

Putting $J_{r_n} = (I + r_n B)^{-1}$, it follows from Lemma 2.3 that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\le e_n + e_{n-1} + \|J_{r_{n-1}} z_{n-1} - J_{r_{n-1}}\big(\frac{r_{n-1}}{r_n} z_n + (1 - \frac{r_{n-1}}{r_n}) J_{r_n} z_n\big)\| \\
&\le e_n + e_{n-1} + \|(1 - \frac{r_{n-1}}{r_n})(J_{r_n} z_n - z_{n-1}) + \frac{r_{n-1}}{r_n}(z_n - z_{n-1})\| \\
&\le e_n + e_{n-1} + \frac{|r_n - r_{n-1}|}{r_n}\|z_n - J_{r_n} z_n\| + \|z_n - z_{n-1}\|,
\end{aligned}$$

which implies from (3.1) that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\le e_n + e_{n-1} + \frac{|r_n - r_{n-1}|}{r_n}\|z_n - J_{r_n} z_n\| + \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| \\
&\quad + |r_n - r_{n-1}|\|Ay_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\| \\
&\le \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + e_n + e_{n-1} \\
&\quad + |r_n - r_{n-1}|\Big(\|Ay_{n-1}\| + \frac{\|J_{r_n} z_n - z_n\|}{r_n}\Big) + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|.
\end{aligned}$$

From the restrictions imposed on the control sequences, we have

$$\sum_{n=1}^{\infty}\Big(e_n + e_{n-1} + |r_n - r_{n-1}|\Big(\|Ay_{n-1}\| + \frac{\|J_{r_n} z_n - z_n\|}{r_n}\Big) + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|\Big) < \infty.$$

Using Lemma 2.1, we find $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $\|\cdot\|^2$ is convex, we have $\|y_n - p\|^2 \le \alpha_n\|Sx_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2$, from which it follows that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \|(y_n - r_n Ay_n) - (p - r_n Ap)\|^2 + 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2 \\
&\le \|y_n - p\|^2 - r_n(2\alpha - r_n)\|Ay_n - Ap\|^2 + 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2 \\
&\le \alpha_n\|Sx_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ay_n - Ap\|^2 \\
&\quad + 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
r_n(2\alpha - r_n)\|Ay_n - Ap\|^2 \le\ & \alpha_n\|Sx_n - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
& + 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2.
\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\|Ay_n - Ap\| = 0. \tag{3.2}$$

Put $\lambda_n = (I + r_n B)^{-1}(y_n - r_n Ay_n)$. Since $(I + r_n B)^{-1}$ is firmly nonexpansive, one has

$$\begin{aligned}
\|\lambda_n - p\|^2 &\le \langle (y_n - r_n Ax_n) - (p - r_n Ap), \lambda_n - p\rangle \\
&\le \frac{1}{2}\big(\|y_n - p\|^2 + \|\lambda_n - p\|^2 - \|y_n - \lambda_n - r_n(Ay_n - Ap)\|^2\big) \\
&\le \frac{1}{2}\big(\|y_n - p\|^2 + \|\lambda_n - p\|^2 - \|y_n - \lambda_n\|^2 + 2r_n\|\lambda_n - y_n\|\|Ay_n - Ap\|\big).
\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le e_n^2+\alpha_n\|Sx_n-p\|^2+\|x_n-p\|^2-\|y_n-\lambda_n\|^2\\ &\quad+2r_n\|\lambda_n-y_n\|\|Ay_n-Ap\|+2e_n\|\lambda_n-p\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|y_n-\lambda_n\|^2 &\le e_n^2+\alpha_n\|Sx_n-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad+2r_n\|\lambda_n-y_n\|\|Ay_n-Ap\|+2e_n\|\lambda_n-p\|.\end{aligned}$$

Using the restrictions imposed on the control sequences and (3.2), we arrive at

$$\lim_{n\to\infty}\|y_n-\lambda_n\|=0. \tag{3.3}$$

Note that $\|x_n-Tx_n\|\le\|x_n-x_{n+1}\|+\|\lambda_n-y_n\|+\|y_n-Tx_n\|+e_n$. This finds from (3.3) $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Next, we show that

$$\limsup_{n\to\infty}\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle\le 0, \tag{3.4}$$

where $\bar{x}$ is the unique fixed point of the mapping $Proj_{(A+B)^{-1}(0)\cap Fix(T)}S$. To show this inequality, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $\limsup_{n\to\infty}\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle=\lim_{i\to\infty}\langle S\bar{x}-\bar{x},y_{n_i}-\bar{x}\rangle\le 0$, Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to $\hat{x}$. Without loss of generality, we assume that $y_{n_i}\rightharpoonup\hat{x}$. Since $\|x_n-y_n\|\le\|x_n-x_{n+1}\|+\|\lambda_n-y_n\|+e_n$, one has $x_{n_i}\rightharpoonup\hat{x}$. Using Lemma 2.4, one has $\hat{x}\in Fix(T)$. Since $y_n-r_nAy_n\in\lambda_n+r_nB\lambda_n$, that is, $\frac{y_n-\lambda_n-r_nAy_n}{r_n}\in B\lambda_n$. Let $\mu\in B\nu$. Since B is monotone, we find that $\langle\frac{y_n-\lambda_n}{r_n}-\mu-Ay_n,\lambda_n-\nu\rangle\ge 0$. Hence, one has $0\le\langle -A\hat{x}-\mu,\hat{x}-\nu\rangle$. This implies that $-A\hat{x}\in B\hat{x}$, that is, $\hat{x}\in(A+B)^{-1}(0)$. This shows (3.4) holds. Notice that

$$\begin{aligned}\|y_n-\bar{x}\|^2 &\le\alpha_n\langle Sx_n-S\bar{x},y_n-\bar{x}\rangle+\alpha_n\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle+(1-\alpha_n)\|Tx_n-p\|\|y_n-\bar{x}\|\\ &\le\big(1-\alpha_n(1-\kappa)\big)\|x_n-\bar{x}\|\|y_n-\bar{x}\|+\alpha_n\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle.\end{aligned}$$

It follows that $\|y_n-\bar{x}\|^2\le(1-\alpha_n(1-\kappa))\|x_n-\bar{x}\|^2+2\alpha_n\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle$.
Hence, we have

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &\le\|(y_n-r_nAy_n)-(I-r_nA)\bar{x}\|^2+2e_n\|\lambda_n-\bar{x}\|+e_n^2\\ &\le(1-\alpha_n(1-\kappa))\|x_n-\bar{x}\|^2+2\alpha_n\langle S\bar{x}-\bar{x},y_n-\bar{x}\rangle+2e_n\|\lambda_n-\bar{x}\|+e_n^2.\end{aligned}$$

An application of Lemma 2.1 to the above inequality yields that $\lim_{n\to\infty}\|x_n-\bar{x}\|=0$. This completes the proof.

4. Applications

Let C be a nonempty closed and convex subset of a Hilbert space H. Let i_C be the indicator function of C, that is, $i_C(x)=\infty, x\notin C$, $i_C(x)=0, x\in C$. Since i_C is a proper lower and semicontinuous convex function on H, the subdifferential ∂i_C of i_C is maximal monotone. So, we can define the resolvent J_r of ∂i_C for $r>0$, i.e., $J_r:=(I+r\partial i_C)^{-1}$. Letting $x=J_ry$, we find that

$$y\in x+r\partial i_Cx\Longleftrightarrow y\in x+rN_Cx\Longleftrightarrow x=Proj_Cy,$$

where $Proj_C$ is the metric projection from H onto C and $N_Cx:=\{e\in H:\langle e,v-x\rangle,\forall v\in C\}$.

Theorem 4.1. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $T : C \to C$ be a nonexpansive mapping. Assume that $VI(C, A) \cap Fix(T)$ is not empty. Let $S : C \to C$ be a fixed κ-contraction. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$, $y_n = \alpha_n Sx_n + (1-\alpha_n)Tx_n$, $x_{n+1} \approx Proj_C(y_n - r_n Ay_n)$, $\forall n \geq 0$. Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - Proj_C(y_n - r_n Ay_n)\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 2\alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in VI(C, A) \cap Fix(T)$, where $\bar{x} = Proj_{VI(C,A)\cap Fix(T)} S\bar{x}$.*

Proof. Putting $B = \partial i_C$ in Theorem 3.1, we find that $J_{r_n} = Proj_C$. This finds from Theorem 3.1 the desired conclusion immediately.

Next, we consider the problem of finding a solution of a Ky Fan inequality [7], which is known as an equilibrium problem in the terminology of Blum and Oettli; see [33] and the references therein.

Let B be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } B(x, y) \geq 0, \quad \forall y \in C. \tag{4.1}$$

To study equilibrium problem (4.1), we may assume that B satisfies the following restrictions:

(R-a) $B(y, x) + B(x, y) \leq 0$, $\forall x, y \in C$;
(R-b) $B(x, x) = 0$, $\forall x \in C$;
(R-c) $B(x, y) \geq \limsup_{t\downarrow 0} B(tz + (1-t)x, y)$, $\forall x, y, z \in C$,
(R-d) $y \mapsto B(x, y)$, $\forall x \in C$, is lower semi-continuous and convex.

The following lemmas can be found in [22] and [33].

Lemma 4.2. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $B : C \times C \to \mathbb{R}$ be a bifunction with (R-a), (R-b), (R-c) and (R-d). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that $rB(z, y) + \langle y - z, z - x\rangle \geq 0$, $\forall y \in C$. Further, define*

$$T_r x = \Big\{ z \in C : rB(z, y) + \langle y - z, z - x\rangle \geq 0, \ \forall y \in C \Big\} \tag{4.2}$$

for all $r > 0$ and $x \in H$. Then T_r is single-valued and firmly nonexpansive and $F(T_r) = EP(F)$ is closed convex.

Lemma 4.3. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let B be a bifunction from $C \times C$ to $\mathbb{R}$ with (R-a), (R-b), (R-c) and (R-d). Let A_B be a multivalued mapping of H into itself defined by*

$$A_B x = \begin{cases} \{z \in H : \langle y - x, z\rangle \leq B(x, y), \ \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{4.3}$$

Then A_B is a maximal monotone operator with domain $D(A_B) \subset C$, $EP(B) = A_B^{-1}(0)$, where $FP(B)$ stands for the solution set of (4.1), and $T_r x = (I + rA_B)^{-1}x$, $\forall x \in H$, $r > 0$, where T_r is defined as in (4.2).

Theorem 4.4. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $B : C \times C \to \mathbb{R}$ be a bifunction with (R-a), (R-b), (R-c) and (R-d). Let $T : C \to C$ be a nonexpansive mapping. Assume that $EP(B) \cap Fix(T)$ is not empty. Let $S : C \to C$ be a fixed κ-contraction and let $T_{r_n} = (I+r_nA_B)^{-1}$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$ and $x_{n+1} \approx T_{r_n}(\alpha_n Sx_n + (1-\alpha_n)Tx_n)$, $\forall n \geq 0$, Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - T_{r_n}(\alpha_n Sx_n + (1-\alpha_n)Tx_n)\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 2\alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in EP(B) \cap Fix(T)$, where $\bar{x} = Proj_{EP(B)\cap Fix(T)} S\bar{x}$.*

Proof. Putting $A = 0$ in Theorem 3.1, we find that $J_{r_n} = T_{r_n}$. From Theorem 3.1, we draw the desired conclusion immediately.

Recall that a mapping $T : C \to T$ is said to be α-strictly pseudocontractive iff there exits a constant $\alpha \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \alpha\|(I-T)x - (I-T)y\|^2 + \|x-y\|^2, \quad \forall x, y \in C.$$

The class of strictly pseudocontractive mappings was first introduced by Browder and Petryshyn [28]. It is known if T is α-strictly pseudocontractive, then $I - T$ is $\frac{1-\alpha}{2}$-inverse strongly monotone.

Finally, we consider the problem of common fixed point problems of nonlinear mappings.

Theorem 4.5. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T_1 be a nonexpansive mapping and let T_2 be a α-strictly pseudocontractive mapping on C. Let S be a fixed κ-contraction on C. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$, $y_n = \alpha_n Sx_n + (1-\alpha_n)T_1x_n$, $x_{n+1} \approx (1-r_n)y_n + r_nT_2y_n$, $\forall n \geq 0$, Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - (1-r_n)y_n - r_nT_2y_n\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 1-\alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(T_1) \cap Fix(T_2)$, where $\bar{x} = Proj_{Fix(T_1)\cap Fix(T_2)} S\bar{x}$.*

Proof. Putting $A = I - T_2$, we find A is $\frac{1-\alpha}{2}$-inverse strongly monotone. We also have $VI(C, A) = Fix(T_2)$ and $r_nT_2y_n + (1-r_n)y_n = Proj_C(y_n - r_nAy_n)$. In view of Theorem 3.1, we obtain the desired result immediately.

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ON THE STABILITY OF ADDITIVE ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

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Abstract. In this paper, we solve the following additive ρ-functional inequalities

$$N\left(f(x+y)-f(x)-f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right),t\right)\geq\frac{t}{t+\varphi(x,y)} \tag{0.1}$$

and

$$N\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)-f(x)-f(y)\right),t\right)\geq\frac{t}{t+\varphi(x,y)} \tag{0.2}$$

in fuzzy normed spaces, where ρ is a fixed real number with $\rho\neq 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [10] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [6, 12, 27]. In particular, Bag and Samanta [2], following Cheng and Mordeson [5], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [11]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 16, 17] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 16, 17, 18] Let X be a real vector space. A function $N : X\times\mathbb{R}\to[0,1]$ is called a *fuzzy norm* on X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

(N_1) $N(x,t)=0$ for $t\leq 0$;
(N_2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
(N_3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;
(N_4) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$.
(N_6) for $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [15, 16].

Definition 1.2. [2, 16, 17, 18] Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x\in X$ such that $\lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty}x_n=x$.

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Definition 1.3. [2, 16, 17, 18] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [24] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x)+\frac{1}{2}f(y)$ is called the *Jensen equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 9, 13, 14, 19, 22, 23, 25]).

Park [20, 21] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we solve the additive ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in fuzzy Banach spaces by using the direct method.

In Section 3, we solve the additive ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in fuzzy Banach spaces by using the direct method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

2. Additive ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $\rho \neq 1$. We need the following lemma to prove the main results.

Lemma 2.1. *Let $f : X \to Y$ be a mapping satisfying*

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \tag{2.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is additive.

Proof. Letting $x = y = 0$ in (2.1), we get $-f(0) = 0$ and so $f(0) = 0$.

Replacing y by x in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) = \rho(f(x+y) - f(x) - f(y))$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. $\square$

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$N\left(f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \varphi(x, y)} \tag{2.3}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t + \frac{1}{2}\Phi(x, x)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.3), we get

$$N\left(f(2x) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x, x)} \tag{2.5}$$

and so

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)}$$

for all $x \in X$. Hence

$$\begin{aligned}
&N\left(2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right) \qquad (2.6)\\
&\geq \min\left\{N\left(2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, N\left(2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right)\right\}\\
&= \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right), \frac{t}{2^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right), \frac{t}{2^{m-1}}\right)\right\}\\
&\geq \min\left\{\frac{\frac{t}{2^l}}{\frac{t}{2^l} + \varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{\frac{t}{2^{m-1}}}{\frac{t}{2^{m-1}} + \varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\}\\
&= \min\left\{\frac{t}{t + 2^l \varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{t}{t + 2^{m-1}\varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\}\\
&\geq \frac{t}{t + \frac{1}{2}\sum_{j=l+1}^{m} 2^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}\right)}
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (2.2) and (2.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

By (2.3),

$$\begin{aligned} N\Big(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right) \\ -\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right), 2^nt\Big) \geq \frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} N\Big(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right) \\ -\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right), t\Big) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)} = \frac{t}{t+2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{t}{t+2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$A(x+y)-A(x)-A(y) = \rho\left(2A\left(\frac{x+y}{2}\right)-A(x)-A(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive, as desired. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned} N\left(f(x+y)-f(x)-f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right), t\right) \\ \geq \frac{t}{t+\theta(\|x\|^p+\|y\|^p)} \end{aligned} \tag{2.7}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x)-A(x), t\right) \geq \frac{(2^p-2)t}{(2^p-2)t+2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^jx, 2^jy\right) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.3). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^nx)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x)-A(x), t\right) \geq \frac{1}{t+\frac{1}{2}\Phi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (2.5) that

$$N\left(f(x)-\frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t+\varphi(x,x)}$$

and so

$$N\left(f(x) - \frac{1}{2}f(2x), t\right) \geq \frac{2t}{2t + \varphi(x,x)} = \frac{t}{t + \frac{1}{2}\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying (2.7). Then $A(x) := N$-$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2^p)t}{(2-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

3. Additive ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a fuzzy number with $\rho \neq 1$.

Lemma 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho\left(f(x+y) - f(x) - f(y)\right) \tag{3.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is additive.

Proof. Letting $y = 0$ in (3.1), we get $2f\left(\frac{x}{2}\right) - f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f(x+y) - f(x) - f(y) = 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho(f(x+y) - f(x) - f(y))$$

and so $f(x + y) = f(x) + f(y)$ for all $x, y \in X$. □

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \varphi(x,y)} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N$-$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t + \Phi(x, 0)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = 0$ in (3.3), we get

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x,0)} \tag{3.5}$$

for all $x \in X$. Hence

$$\begin{aligned}
&N\left(2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right) \qquad (3.6)\\
&\geq \min\left\{N\left(2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, N\left(2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right)\right\}\\
&= \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right), \frac{t}{2^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right), \frac{t}{2^{m-1}}\right)\right\}\\
&\geq \min\left\{\frac{\frac{t}{2^l}}{\frac{t}{2^l} + \varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{\frac{t}{2^{m-1}}}{\frac{t}{2^{m-1}} + \varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\}\\
&= \min\left\{\frac{t}{t + 2^l\varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{t}{t + 2^{m-1}\varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\}\\
&\geq \frac{t}{t + \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^j}, 0\right)}
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (3.2) and (3.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

By (3.3),

$$\begin{aligned}
&N\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right.\\
&\quad \left. - \rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 2^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned}
&N\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right.\\
&\left. - \rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), t\right) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = \frac{t}{t + 2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{t}{t+2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$2A\left(\frac{x+y}{2}\right) - A(x) - A(y) = \rho\left(A(x+y) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 3.1, the mapping $A : X \to Y$ is Cauchy additive, as desired. $\square$

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \quad (3.7)$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi\left(2^j x, 2^j y\right) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t + \Phi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (3.5) that

$$N\left(f(x) - \frac{1}{2} f(2x), \frac{t}{2}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{2} f(2x), t\right) \geq \frac{2t}{2t + \varphi(2x, 0)} = \frac{t}{t + \frac{1}{2}\varphi(2x, 0)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

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On the Difference equation $x_{n+1} = Ax_n + \frac{B\sum\limits_{i=0}^{k} x_{n-i}}{C+D\prod\limits_{i=0}^{k} x_{n-i}}$

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Abstract

The main objective of this paper is to study the global stability of the positive solutions and the periodic character of the difference equation

$$x_{n+1} = \frac{A\sum\limits_{i=0}^{k} x_{n-i}}{B + C\prod\limits_{i=0}^{k} x_{n-i}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $A,\ B$ and C are positive real numbers and the initial conditions $x_{-k},\ x_{-k+1}, ...,\ x_{-},\ x_0$ are nonnegative real numbers.

Keywords: difference equations, stability, global stability, periodic solutions.
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1 Introduction

Difference equations have always played an important role in the construction and analysis of mathematical models of biology, ecology, probability theory, genetics, number theory, physics, economic process, and so forth.

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

Ahmed [1] investigated the global asymptotic stability and the periodic character for the rational difference equation,

$$x_{n+1} = \frac{\alpha x_{n-l}}{\beta+\gamma \prod_{i=l}^{k} x_{n-2i}^{p_i}}, \qquad n = 0, 1, ...,$$

where the parameters $\alpha,\ \beta,\ \gamma,\ p_1,\ p_2,\ ...,\ p_k$ are nonnegative real numbers, and $l,\ k$ are nonnegative integers such that $l \leq k$ and the initial conditions x_{-2k}, x_{-2k+1}, ..., x_{-1}, x_0 are arbitrary nonnegative real numbers.

Wang et al. [2] studied the asymptotic behavior of the solutions of the nonlinear difference equation

$$x_{n+1} = \frac{\sum_{i=0}^{l} A_{s_i}\ x_{n-s_i}}{B+C\sum_{j=0}^{k} x_{n-t_j}}, \qquad n = 0, 1, ...,$$

where the initial conditions $x_{-m},\ x_{-m+1},\ ...,\ x_{-1},\ x_0$ are positive real numbers, $m = max\{s_1,\ ...,\ s_l,\ t_1,\ ...,\ t_k\}$, $s_1,\ ...,\ s_l,\ t_1,\ ...,\ t_k$ are nonnegative integers, and A_{s_i}, B, C are arbitrary positive real numbers.

Zayed et al. [3] investigated the boundedness character, the periodic character, the convergence and the global stability of positive solutions of the difference equation

$$x_{n+1} = \frac{A+\sum_{i=0}^{k}\alpha_i\ x_{n-i}}{\sum_{i=0}^{k}\beta_i\ x_{n-i}}, \qquad n = 0, 1, ...,$$

where the coefficients $A,\ \alpha_i,\ ,\beta_i$ and the initial conditions $x_{-k},\ x_{-k+1},\ ...,\ x_{-1},\ x_0$ are positive real numbers, while k is a positive integer number.

In [4] Ibrahim et al. studied the global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-m}}{\beta+\gamma\prod_{j=0}^{k} x_{n-i_j}}, \qquad n = 0, 1, ...,$$

where the parameters $\alpha,\ \beta,\ \gamma$ and initial conditions are non-negative real numbers, $\{i_0 < i_1 < ... < i_k\}$ is a set of nonnegative even integers and m is an odd positive integer

Hamza et al. [5] studied the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{A\prod_{i=l}^{k} x_{n-2i-1}}{B+C\prod_{j=0}^{k-1} x_{n-2i}}, \qquad n = 0, 1, ...,$$

where $A,\ B,\ C$ are nonnegative parameters and $l,\ k$ are nonnegative integers for $l < k$. They discussed the existence of unbounded solutions under certain conditions for $l = 0$.

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In [6] El-Metwally investigated the global stability character and the oscillatory of the solutions of the following difference equation

$$y_{n+1} = \frac{\alpha y_n \prod_{i=l}^{k} x_{n-2i-1}}{\beta + \gamma \sum_{i=0}^{k} y_{n-2i-1}^{p} \prod_{i=0}^{k} y_{n-2i-1}}, \qquad n = 0, 1, ...,$$

where $\alpha,\ \beta,\ \gamma,\ p \in (0,\ \infty)$ with the initial conditions $y_0,\ y_{-1},\ ...,\ y_{-2k},\ y_{-2k-1} \in (0,\ \infty)$. For more results in the direction of this study, see, for example, [1–27] and the papers therein.

The aim of this paper to study some qualitative behavior of the positive solutions of a higher order difference equation

$$x_{n+1} = Ax_n + \frac{B \sum_{i=0}^{k} x_{n-i}}{C + D \prod_{i=0}^{k} x_{n-i}}, \qquad n = 0,\ 1,\ ..., \tag{1}$$

where the parameters $A,\ B,\ C$ and D are positive real numbers and the initial conditions $x_{-k},\ x_{-k+1}, ...,\ x_{-},\ x_0$ are nonnegative real numbers.

2 Preliminaries

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 *(Equilibrium Point)*
A point $\overline{x} \in I$ is called an equilibrium point of the difference equation (2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of the difference equation (2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 2 *(Stability)*
Let $\overline{x} \in (0,\ \infty)$ be an equilibrium point of the difference equation (2). Then, we have

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(i) The equilibrium point $\overline{x}$ of the difference equation (2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of the difference equation (2) is called locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of the difference equation (2) is called global attractor if for all $x_{-k}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of the difference equation (2) is called globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of the difference equation (2).
(v) The equilibrium point $\overline{x}$ of the difference equation (2) is called unstable if $\overline{x}$ is not locally stable.

Definition 3 *(Periodicity)*
A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Definition 4 *The linearized equation of the difference equation (2) about the equilibrium $\overline{x}$ is the linear difference equation*

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0, \tag{4}$$

where

$$p_i = \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}.$$

Theorem 1 *[1]: Assume that $p_i \in R$, $i = 1, 2, ..., k$ and k is non-negative integer. Then*

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ... \ .$$

3 Change of variables

By using the change of variables $x_n = \left(\frac{C}{D}\right)^{\frac{1}{k+1}} y_n$, the equation (1) reduces to the following difference equation

$$y_{n+1} = A y_n + \frac{r \sum_{i=0}^{k} y_{n-i}}{1 + \prod_{i=0}^{k} y_{n-i}}, \qquad n = 0,\ 1,\ ..., \tag{5}$$

where $r = \frac{B}{C}$ and the initial conditions $y_n,\ y_{n-1}, ...,\ y_{n-k+1},\ y_{n-k}$ are positive real numbers.

4 Local Stability of the Equilibrium Point

In this section, we study the local stability character of the equilibrium point of Eq.(5).

Eq.(5) has equilibrium point and is given by

$$\overline{y} = A\overline{y} + \frac{r \sum_{i=0}^{k} \overline{y}_{n-i}}{1 + \prod_{i=0}^{k} \overline{y}_{n-i}},$$

or

$$\overline{y}\,(1 - A)\left(1 + \overline{y}^{k+1}\right) = r(k+1)\overline{y}.$$

Thus $\overline{y}_1 = 0$ is always an equilibrium point of Eq. (5). If $A < 1$ and $\frac{r(k+1)}{1-A} > 1$; then the only positive equilibrium point $\overline{y}_2$ of Eq. (5) is given by

$$\overline{y}_2 = \left(\frac{r(k+1)}{1-A} - 1\right)^{\frac{1}{k+1}}.$$

Theorem 2 *The equilibrium $\overline{y}_1$ of Eq. (5) is locally asymptotically stable if*

$$A + r(k+1) < 1.$$

Proof: Let $f:(0,\ \infty)^{k+1}\longrightarrow(0,\ \infty)$ be a continuous function defined by

$$f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})=Au_n+\frac{r\sum\limits_{i=0}^{k}u_{n-i}}{1+\prod\limits_{i=0}^{k}u_{n-i}}. \tag{7}$$

Therefore, it follows that

$$\begin{aligned}
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_n} &= A+\frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-r\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=1}^{k}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2},\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-1}} &= \frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-ru_n\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=2}^{k}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2},\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-2}} &= \frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-ru_nu_{n-1}\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=3}^{k}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2},\\
&\ \ \vdots\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-k}} &= \frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-r\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=0}^{k-1}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2}.
\end{aligned}$$

At $\overline{y}_1=0$, we have

$$\begin{aligned}
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_n} &= A+r\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-1}} &= ...=\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-k}}=r,
\end{aligned}$$

and the linearized equation of Eq. (5) about $\overline{y}_1=0$, is the equation

$$z_{n+1}-(A+r)\,z_n-rz_{n-l}-...-ry_{n-k}=0,$$

It follows by Theorem 1 that, Eq. (5) is asymptotically stable if and only if

$$|A+r|+|r|+...+|r|<1,$$

and so

$$A+r\,(k+1)<1.$$

The proof is complete.

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Theorem 3 *The equilibrium $\overline{y}_1$ of Eq. (5) is unstable if $A + r(k+1) > 1$.*

Theorem 4 *The equilibrium $\overline{y}_2$ of Eq. (5) is stable if*

$$Ar + (1-A)(1-rk-A) < r.$$

Proof: At $\overline{y}_2 = \left(\frac{r(k+1)}{1-A} - 1\right)^{\frac{1}{k+1}}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial u_n} &= A + \frac{r\left(1+\frac{r(k+1)}{1-A}-1\right)-r(k+1)\left(\frac{r(k+1)}{1-A}-1\right)}{\left(1+\frac{r(k+1)}{1-A}-1\right)^2} \\
&= A + \frac{r\left(\frac{r(k+1)}{1-A}\right)-r(k+1)\left(\frac{r(k+1)-1+A}{1-A}\right)}{\left(\frac{r(k+1)}{1-A}\right)^2} = A + \frac{\left(\frac{r(k+1)}{1-A}\right)(r-r(k+1)+1-A)}{\left(\frac{r(k+1)}{1-A}\right)^2} \\
&= A + \frac{(r-rk-r+1-A)}{\left(\frac{r(k+1)}{1-A}\right)} = A + \frac{(1-A)(1-rk-A)}{r(k+1)} \\
\frac{\partial f}{\partial u_{n-1}} &= \ldots = \frac{\partial f}{\partial u_{n-k}} = \frac{(1-A)(1-rk-A)}{r(k+1)},
\end{aligned}$$

and the linearized equation of Eq. (5) about $\overline{y}_2 = \left(\frac{r(k+1)}{1-A} - 1\right)^{\frac{1}{k+1}}$, is the equation

$$z_{n+1} - \left(A + \frac{(1-A)(1-rk-A)}{r(k+1)}\right) z_n - \frac{(1-A)(1-rk-A)}{r(k+1)} z_{n-l} - \ldots - \frac{(1-A)(1-rk-A)}{r(k+1)} y_{n-k} = 0,$$

It follows by Theorem A that, Eq.(5) is stable if and only if

$$\left|A + \frac{(1-A)(1-rk-A)}{r(k+1)}\right| + \left|\frac{(1-A)(1-rk-A)}{r(k+1)}\right| + \ldots + \left|\frac{(1-A)(1-rk-A)}{r(k+1)}\right| < 1,$$

for $rk + A < 1$ we get

$$A + \frac{(1-A)(1-rk-A)}{r} < 1.$$

The proof is complete.

5 Existence of Boundedness Solutions

Here we look at the boundedness nature of solutions of Eq.(5).

Theorem 5 *Every solution of Eq.(5) is bounded if $A + r(k+1) < 1$.*

Proof: Let $\{y_n\}_{n=0}^{\infty}$ be a solution of Eq.(5). It follows from Eq.(5) that

$$0 \leq y_{n+1} = Ay_n + \frac{r\sum\limits_{i=0}^{k} y_{n-i}}{1+\prod\limits_{i=0}^{k} y_{n-i}} < Ay_n + r\sum_{i=0}^{k} y_{n-i} < (A + r(k+1))\,\overline{y}.$$

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this equation is locally asymptotically stable if $A + r(k+1) < 1,$ and converges to the equilibrium point $\overline{y}$. Therefore

$$\limsup_{n\to\infty} y_n \leq (A + r(k+1))\,\overline{y}.$$

Hence, the solution is bounded.

Theorem 6 *Every solution of Eq.(5) is unbounded if* $A > 1$.

Proof: Let $\{y_n\}_{n=0}^{\infty}$ be a solution of Eq.(5). Then from Eq.(5) we see that

$$y_{n+1} = Ay_n + \frac{r\sum_{i=0}^{k} y_{n-i}}{1+\prod_{i=0}^{k} y_{n-i}} > Ay_n + r\sum_{i=0}^{k} y_{n-i} > A\overline{y}.$$

This equation is unbounded because $A > 1$, and $\lim_{n\to\infty} y_n = \infty$. Then by using ratio test $\{y_n\}_{n=0}^{\infty}$ is unbounded from above.

6 Global Stability of the Equilibrium Point

In this section we study the global stability of the positive solutions of Equation (1).

Theorem 7 *The following statements are true*

(a) If $A + r(k+1) < 1$ then the equilibrium point $\overline{y}_1 = 0$ is a global attractor of equation (1).

(b) If $rk + A < 1$ then the equilibrium point $\overline{y}_2 = \left(\frac{r(k+1)}{1-A} - 1\right)^{\frac{1}{k+1}}$ is a global attractor of equation (1).

Proof. (a) From Eq. (7) we can see that the function is increasing of all arguments. Now, we can see that the function $F(y_n,\ y_{n-1},\ ...,\ y_{n-k})$ increasing in $y_n,\ y_{n-1},\ ...,\ y_{n-k+1}$ and x_{n-k}. Then

$$\begin{aligned} & \left[Ay + \frac{r(k+1)y}{1+y^{k+1}} - y\right](y - \overline{y}_1) \\ \leq\ & [Ay + r(k+1)y - y]\,(y - 0) \\ \leq\ & -(1 - A - r(k+1))\,y^2 < 0 \end{aligned}$$

If $A + r(k+1) < 1$, then $F(y,\ y,\ ...,\ y)$ satisfies the inequality

$$[F(y,\ y,\ ...,\ y) - y]\,(y - \overline{y}_1) < 0, \quad \text{for } \overline{y}_1 = 0.$$

According to Theorem 1.10 page 15 in [1], then $\overline{x}_1$ is a global attractor of Eq. (1). This completes the proof.

(b) If $rk + A < 1$, then we can see that the function $f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})$ defined by Eq. (7) increasing of all arguments. Suppose that (m, M) is a solution of the system

$$M = f(M,\ M,\ ...,\ M) \quad \text{and} \quad m = f(m,\ m,\ ...,\ m).$$

Then from Equation (1), we see that

$$M = AM + \frac{r(k+1)M}{1+M^{k+1}}, \text{ and } m = Am + \frac{r(k+1)m}{1+m^{k+1}},$$

then

$$\begin{aligned} (1-A) + (1-A)M^{k+1} &= r(k+1), \\ (1-A) + (1-A)m^{k+1} &= r(k+1), \end{aligned}$$

Subtracting this two equations, we obtain

$$(1-A)\left(M^{k+1} - m^{k+1}\right) = 0$$

under the condition $A \neq 1$,we see that $M = m$. According to Theorem 1.15 page 18 in [1], we see that $\overline{y}_2$ is a global attractor of Equation (1).

7 Existence of Periodic Solutions

In this section we investigate the existence of periodic solutions of Eq.(5).

Theorem 8 *If k is even, then equation (5) has not prime period two solution.*

Proof: Equation (5) can be expressed that

$$y_{n+1} = Ay_n + \frac{r\left(y_n + y_{n-1} + y_{n-2} + ... + y_{n-k}\right)}{1 + y_n y_{n-1} y_{n-2} ... y_{n-k}},$$

For $k = 2m$ is even, then $y_n,\ y_{n-2},\ y_{n-4},\ ...,\ y_{n-k-2},\ y_{n-k}$ are even and $y_{n-1},\ y_{n-3},\ y_{n-5},\ ...,\ y_{n-k-3},\ y_{n-k-1}$ are odd. Suppose that exists there distinct positive solutions

$$...p,\ q,\ p,\ q,\ ...,$$

of Equation (5). Then

$$p = Aq + \frac{r\left((m+1)\,q + mp\right)}{1 + q^{m+1}p^m} \text{ and } q = Ap + \frac{r\left((m+1)\,p + mq\right)}{1 + p^{m+1}q^m}.$$

Therefore,

$$\begin{aligned} p - Aq + q^{m+1}p^{m+1} - Aq^{m+1}p^{m+1} &= r\,(m+1)\,q + rmp, &(7) \\ q - Ap + p^{m+1}q^{m+1} - Ap^{m+1}q^{m+1} &= r\,(m+1)\,p + rmq, &(8) \end{aligned}$$

By subtracting (8) from (7), we have

$$(1 + A + r)(p - q) = 0$$

Since $r + A + 1 \neq 0$, then $p = q$. This is a contradiction. Thus, the proof is completed.

Theorem 9 *If k is odd, then equation (5) has not prime period two solution.*

Proof: When $k = 2m + 1$ is odd, then $y_n,\ y_{n-2},\ y_{n-4},\ ...,\ y_{n-k-3},\ y_{n-k-1}$ are even and $y_{n-1},\ y_{n-3},\ y_{n-5},\ ...,\ y_{n-k-2},\ y_{n-k}$ are odd.

First suppose that there exists distinct positive solutions

$$...p,\ q,\ p,\ q,\ ...,$$

of Equation (5). Then

$$p = Aq + \frac{r\left((m+1)\,q + (m+1)\,p\right)}{1 + q^{m+1}p^{m+1}},$$

and

$$q = Ap + \frac{r\left((m+1)\,p + (m+1)\,q\right)}{1 + p^{m+1}q^{m+1}}.$$

Therefore,

$$p - Aq + q^{m+1}p^{m+2} - Aq^{m+2}p^{m+1} = r\,(m+1)\,q + r\,(m+1)\,p, \tag{9}$$

$$q - Ap + p^{m+1}q^{m+2} - Ap^{m+2}q^{m+1} = r\,(m+1)\,p + r\,(m+1)\,q, \tag{10}$$

Subtracting (10) from (9), we get

$$(p - q)\left((A+1)p^{m+1}q^{m+1} + 1 + A\right) = 0$$

Since $A+1 \neq 0$, then $p = q$. This is a contradiction. Thus, the proof is completed.

8 Numerical Examples

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (5).

Example 1. The zero solution of the difference equation (5) is local stability if $k = 3$, $A = 0.2$, $r = 0.1$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$ (See Fig. 1).

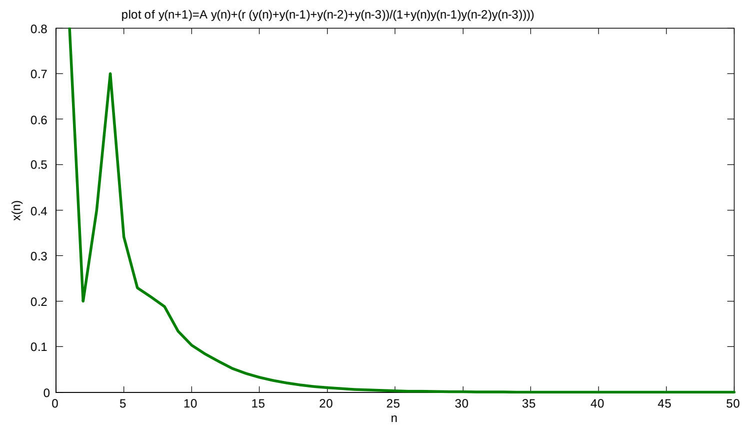

Figure 1. Plot the behavior of the zero solution of equation (5).

Example 2. The positive solution of the difference equation (5) is local stability if $k = 3$, $A = 0.6$, $r = 0.2$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$ (See Fig. 2).

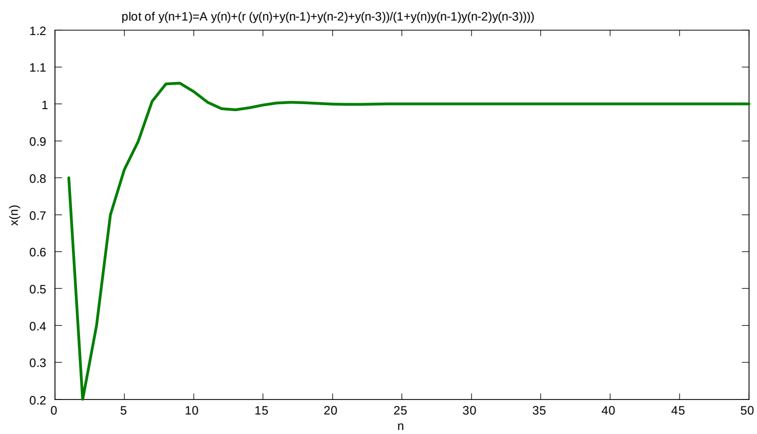

Figure 2. Plot the behavior of the positive solution of equation (5).

Example 3. The solution of the difference equation (5) is global stability if $k = 3$, $A = 0.02$, $r = 0.33$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$ (See Fig. 3).

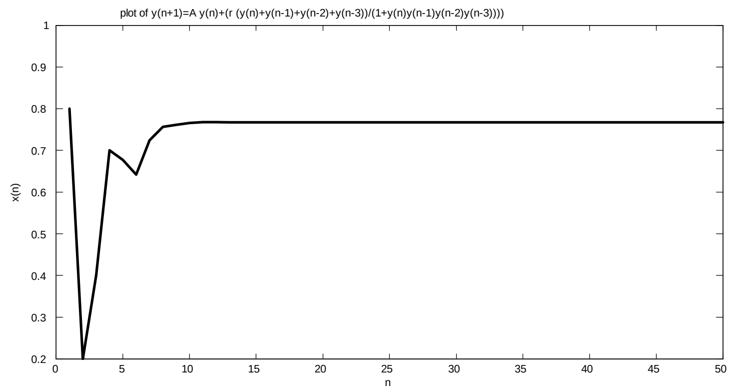

Figure 3. Plot the behavior of the positive solution of equation (5).

Example 4. Figure (4) shows the equation (5) is unbounded when $k = 3$, $A = 1.1$, $r = 0.1$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$.

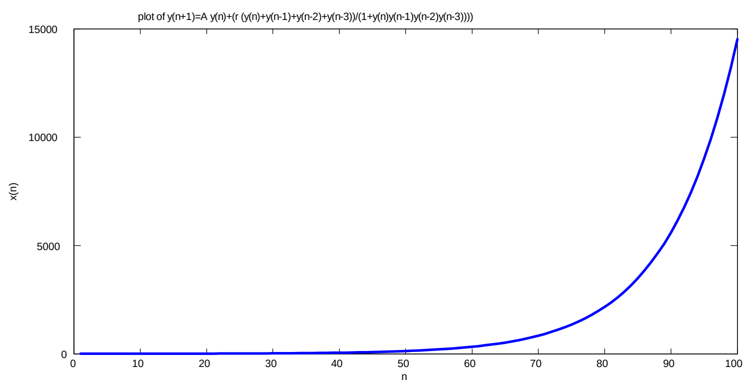

Figure 4. Plot the behavior of the solution of equation (5).

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A Kind of Generalized Fuzzy Integro-differential Equations of Mixed Type and Strong Fuzzy Henstock Integrals*

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Abstract

In this paper, we prove the existence theorem of solutions for a kind of discontinuous fuzzy integro-differential equation of mixed type by using the definition of the $\omega - ACG^*$ for a fuzzy-number-valued function and a generalized controlled convergence theorem of strong fuzzy Henstock integrals.

Keywords: Fuzzy number; $\omega - ACG^*$; Discontinuous fuzzy Integro-differential equation; Controlled convergence theorem; Strong fuzzy Henstock integrals.

1 INTRODUCTION

The Cauchy problems for fuzzy differential equations have been studied by several authors [11, 9, 12, 16, 17, 18] on the metric space (E^n, D) of normal fuzzy convex set with the distance D given by the maximum of the Hausdorff distance between the corresponding level sets. In [16], the author has been proved the Cauchy problem has a uniqueness result if f was continuous and bounded. In [11, 12], the authors presented a uniqueness result when f satisfies a Lipschitz condition. For a general reference to fuzzy differential equations, see a recent

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book by Lakshmikantham and Mohapatra [13] and references therein. In 2002, Xue and Fu [26] established solutions to fuzzy differential equations with right-hand side functions satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets.

However, there are discontinuous systems in which the right-hand side functions $\tilde{f}:[a,b]\times E^n \to E^n$ are not integrable in the sense of Kaleva [11] on certain intervals and their solutions are not absolute continuous functions. Recently, Wu and Gong [24, 25] have combined the fuzzy set theory [27] and nonabsolute integration theory [10], and discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva[11] integration. In order to complete the theory of fuzzy calculus and to meet the solving need of transferring a fuzzy differential equation into a fuzzy integral equation, Gong and Shao [7, 8] have defined the strong fuzzy Henstock integrals and discussed some of their properties and the controlled convergence theorem. So, in [19, 20, 21, 22, 23], the authors used the strong fuzzy Henstock integrals [8], and deal with the Cauchy problem of discontinuous fuzzy systems. In this paper, according to the idea of [4] and using the concept of generalized differentiability [2], the operator j which is the isometric embedding from (E^n, D) onto its range in the Banach space X and the generalized controlled convergence theorems for the strong fuzzy Henstock integrals, we will deal with the Cauchy problem of discontinuous fuzzy integro-differential equations of mixed type as following:

$$\begin{cases} x'(t) = \tilde{f}(t, x(t), \int_0^t k_1(t,s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(t,s)\tilde{h}(s,x(s))\mathrm{d}s), \\ x(0) = x_0, \quad x_0 \in E^n, t \in I_a = [0,a], a \in R^+ \end{cases} \tag{1}$$

where $\tilde{f}, \tilde{g}, \tilde{h}, x$ will be assumed strong fuzzy Henstock integrable and k_1, k_2 are real-valued functions.

To make our analysis possible, in section 2, we will first recall some basic results of fuzzy numbers. In section 3, we give some definitions of $\omega - ACG^*$ of fuzzy-number-valued function. In addition, we present the concept of strong fuzzy Henstock integral and a generalized controlled convergence theorem for the strong fuzzy Henstock integrals. In section 4, we deal with the Cauchy problem of discontinuous fuzzy integro-differential equation of mixed type. And in section 5, we present some concluding remarks.

2 PRELIMINARIES

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two nonempty bounded subset of R^n. The distance between A and B is defined by the Hausdorff metric [6]:

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} \parallel a-b \parallel, \sup_{b\in B}\inf_{a\in A} \parallel b-a \parallel\}.$$

Denote $E^n = \{u : R^n \to [0,1] | u$ satisfies (1)-(4) below$\}$ is a fuzzy number space. where

(1)u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2)u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

(3)u is upper semi-continuous,

(4)$[u]^0 = cl\{x \in R^n | u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n | u(x) \geq \alpha\}$. Then from above (1)-(4), it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha < 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows [6]:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $u, v \in E^n$ and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0, \infty)$

$$D(u, v) = \sup\{d_H([u]^\alpha, [v]^\alpha) : \alpha \in [0, 1]\},$$

where d is the Hausdorff metric defined in $P_k(R^n)$. Then it is easy see that D is a metric in E^n. Using the results [5], we know that

(1) (E^n, D) is a complete metric space,

(2) $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$,

(3)$D(\lambda u, \lambda v) = |\lambda| D(u, v)$ for all $u, v, w \in E^n$ and $\lambda \in R$.

The metric space (E^n, D) has a linear structure, it can be imbedded isomorphically as a cone in a Banach space of function $u^* : I \times S^{n-1} \longrightarrow R$, where S^{n-1} is the unit sphere in R^n, with an imbedding function $u^* = j(u)$ defined by

$$u^*(r, x) = \sup_{\alpha \in [u]^\alpha} < \alpha, x >$$

for all $< r, x > \in I \times S^{n-1}$. (see [5])

Theorem 1 *There exist a real Banach space X such that E^n can be imbedding as a convex cone C with vertex 0 into X. Furthermore the following conclusions hold:*

(1) the imbedding j is isometric,

(2) addition in X induces addition in E^n,

(3) multiplication by nonnegative real number in X induces the corresponding operation in E^n,

(4) $C - C$ is dense in X,

(5) C is closed.

A fuzzy-number-valued function $f : [a, b] \to E^n$ is said to satisfy the condition (H) on $[a, b]$, if for any $x_1 < x_2 \in [a, b]$ there exists $u \in E^n$ such that $f(x_2) = f(x_1) + u$. We call u is the H-difference of $f(x_2)$ and $f(x_1)$, denoted $f(x_2) -_H f(x_1)$ ([11]).

3

For brevity, we always assume that it satisfies the condition (H) when dealing with the operation of subtraction of fuzzy numbers throughout this paper.

It is well-known that the H-derivative for fuzzy-number-functions was initially introduced by Puri and Ralescu [17] and it is based in the condition (H) of sets. We note that this definition is fairly strong, because the family of fuzzy-number-valued functions H-differentiable is very restrictive. For example, the fuzzy-number-valued function $f : [a, b] \to E^n$ defined by $f(x) = C \cdot g(x)$, where C is a fuzzy number, $\cdot$ is the scalar multiplication (in the fuzzy context) and $g : [a, b] \to R^+$, with $g'(t_0) < 0$, is not H-differentiable in t_0 (see [2]). To avoid the above difficulty, in this paper we consider a more general definition of a derivative for fuzzy-number-valued functions enlarging the class of differentiable fuzzy-number-valued functions, which has been introduced in [2] and [3].

Definition 1 ([2]) *Let $\tilde{f} : (a, b) \to E^n$ and $x_0 \in (a, b)$. We say that $\tilde{f}$ is differentiable at x_0, if there exists an element $\tilde{f}'(t_0) \in E^n$, such that*

(1) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0+h) -_H \tilde{f}(x_0), \tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)}{h} = \tilde{f}'(x_0)$$

or

(2) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0 + h), \tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

or

(3) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0 + h) -_H \tilde{f}(x_0), \tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

or

(4) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0+h), \tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)}{h} = \tilde{f}'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}\cdot$ and $-\frac{1}{h}\cdot$, respectively).

4

3 THE STRONG FUZZY HENSTOCK INTEGRAL AND ITS CONTROLLED CONVERGENCE THEOREM

In this section we shall give the definition of the strong Henstock integral for fuzzy-number-valued functions [7, 8] on a finite interval, which is an extension of the usual fuzzy Kaleva integral in [11]. In addition, we define the properties of $\omega - AC^*$ and $\omega - ACG^*$ for fuzzy-number-valued functions. In particular, we shall prove a controlled convergence theorems for the strong fuzzy Henstock integrals.

Definition 2 ([10, 14]) *Let $\delta(x)$ be a positive function defined on the interval $[a, b]$. A division $P = \{[x_{i-1}, x_i] : \xi_i\}$ is said to be $\delta-$fine if the following conditions are satisfied:*

(1) $a - x_0 < x_1 < \cdots < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$.

For brevity, we write $P = \{[u, v]; \xi\}$

Definition 3 ([7, 8]) *A fuzzy-number-valued function $\tilde{f}$ is said to be strong Henstock integrable on $[a, b]$ if there exists a additive fuzzy-number-valued function $\tilde{F}$ on $[a, b]$ such that for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ and for any δ-fine division $P = \{([u, v], \xi)\}$ of $[a, b]$, we have*

$$\begin{aligned} & \sum_{i\in K_n} D(\tilde{f}(\xi_i)(v_i - u_i), \tilde{F}([u_i, v_i])) \\ & + \sum_{j\in I_n} D(\tilde{f}(\xi_j)(v_j - u_j), (-1) \cdot \tilde{F}([u_j, v_{j-1}])) \\ < \ & \varepsilon. \end{aligned}$$

where $K_n = \{i \in \{1, 2, \cdot\cdot, n\}$ such that $\tilde{F}([x_{i-1}, x_i])$ is a fuzzy number and $I_n = \{j \in \{1, 2, \cdot\cdot, n\}$ such that $\tilde{F}([x_j, x_{j-1}])$ is a fuzzy number. We write $\tilde{f} \in SFH[a, b]$.

Definition 4 ([10, 14]) *A real-valued function F is strong absolute continuous ($F \in AC^*$) on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a $\eta > 0$ such that for every finite or infinite sequence of non-overlapping interval $\{[a_i, b_i]\}$, satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \mathcal{O}(F; [a_i, b_i]) < \varepsilon$, where where $\mathcal{O}$ denotes the oscillation of f over $[a_i, b_i]$, i.e.,*

$$\mathcal{O}(f, [a_i, b_i]) = \sup\{|F(x) - F(y)|; x, y \in [a_i, b_i]\}.$$

A real-valued function F is said to be ACG^* on X if X is the union of a sequence of sets $\{X_i\}$ such that on each X_i the function F is $AC^*(X_i)$.

Definition 5 *A fuzzy-number-valued function f defined on $X \subset [a, b]$ is said to be weak generalized absolute continuous ($\tilde{f} \in \omega - ACG^*(X)$) if for every $\lambda \in [0, 1]$, the real-valued function $f^-_\lambda(x)$ and $f^+_\lambda(x)$ are ACG^*.*

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Theorem 2 *If $\tilde{f}$ is strong fuzzy Henstock integrable on $[a,b]$, then its primitive F is $\omega - ACG^*$ on $[a,b]$.*

Proof. For every $\varepsilon > 0$, there is a function $\delta(\xi) > 0$ such that for any δ-fine partial division $P = \{[u,v], \xi\}$ in $[a,b]$, we have

$$\sum D(F([u,v]), f(\xi)(v-u)) < \varepsilon.$$

We assume that $\delta(\xi) \leq 1$. Let

$$X_{n,i} = \{x \in [a,b] : D(f(x), \tilde{0}) \leq n, \frac{1}{n} < \delta(x) \leq \frac{1}{n-1}, x \in [a + \frac{i-1}{n}, a + \frac{i}{n})\}$$

for $n = 2, 3, \cdots, i = 1, 2, \cdots$. Fixed $X_{n,i}$ and let $\{[a_k, b_k]\}$ be any finite sequence of non-overlapping intervals with $a_k, b_k \in X_{n,i}$ for all k. Then $\{([a_k, b_k], a_k)\}$ is a δ-fine partial division of $[a,b]$. Furthermore, if $a_k \leq u_k \leq v_k \leq b_k$, then $\{([a_k, u_k], a_k)\}, \{([a_k, v_k], a_k)\}$ are δ-fine partial division of $[a,b]$. Thus

$$\begin{aligned}
\sum D(F(u_k), F(v_k)) &\leq \sum D(F(a_k), F(u_k)) + \sum D(F(b_k), F(v_k)) \\
&+ \sum D(F(a_k), F(b_k)) \\
&\leq 3\varepsilon + \sum D(f(a_k)(u_k - a_k), \tilde{0}) + \sum D(f(b_k)(b_k - v_k), \tilde{0}) \\
&+ \sum D(f(a_k)(b_k - a_k), \tilde{0}) \leq 3\varepsilon + 3n \sum (b_k - a_k).
\end{aligned}$$

Choose $\eta \leq \frac{\varepsilon}{3n}$ and $\sum (b_k - a_k) < \eta$. Then

$$\sum \mathcal{O}(F, [a_k, b_k]) \leq 3\varepsilon + \varepsilon.$$

Therefore, F is $\omega - AC^*(X_{n,i})$. Consequently, F is $\omega - ACG^*$ on $[a,b]$.

Theorem 3 *If there exists a fuzzy-number-valued function F is continuous and $\omega - ACG^*$ on $[a,b]$ such that $F'(x) = f(x)$ a.e. in $[a,b]$, then f is strong fuzzy Henstock integrable on $[a,b]$ with primitive F.*

Proof. Let F be the primitive of f and $F'(x) = f(x)$ for $x \in [a,b] \setminus S$ where S is of measure zero. For $\xi \in [a,b] \setminus S$, given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that whenever $\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ we have

$$D(F([u,v]), f(\xi)(v-u)) \leq \varepsilon |v-u|.$$

Since F is continuous and $\omega - ACG^*$ on $[a,b]$, there is a sequence of closed sets $\{X_i\}$ such that $\cup_i X_i = [a,b]$ and F is $\omega - AC^*(X_i)$ for each i. Let $Y_1 = X_1, Y_i = X_i \setminus (X_1 \cup X_2 \cdots \cup X_{i-1})$ for $i = 1, 2, \cdots$ and S_{ij} denote the set of points $x \in S \cap Y_i$ such that $j - 1 \leq D(f, \tilde{0}) < j$. Obviously, S_{ij} are pairwise disjointed and their union is the set S. Since F is also $\omega - AC^*(S_{ij})$, there is a $\eta_{ij} < \varepsilon 2^{-i-j} j^{-1}$ such that for any sequence of non-overlapping intervals $\{I_k\}$ with at least one endpoint of I_k belonging to S_{ij} and satisfying $\sum_k |I_k| < \eta_{ij}$

we have $\sum_k D(F(I_k), \tilde{0}) < \varepsilon 2^{-i-j}$. Again, $F(I)$ denotes $F(v) -_H F(u)$ where $I = [u, v]$. Choose G_{ij} to be the union of a sequence of open intervals such that $|G_{ij}| < \eta_{ij}$ and $G_{ij} \supset S_{ij}$ where $|G_{ij}|$ denotes the total length of G_{ij}. Now for $\xi \in S_{ij}$, put $(\xi - \delta(\xi), \xi + \delta(\xi)) \subset G_{ij}$. Hence we have defined a positive function $\delta(\xi)$.

Take any $\delta-$fine division $P = \{[u, v]; \xi\}$. Split the $\sum$ over P into partial sums $\sum_1$ and $\sum_2$ in which $\xi \bar{\in} S$ and $\xi \in S$ respectively and we obtain

$$\begin{aligned} D(f(\xi)(v-u), F([a,b])) &\leq \sum_1 D(f(\xi)(v-u), F([a,b])) \\ &+ \sum_2 D(F([a,b]), \tilde{0}) + \sum_2 D(f(\xi)(v-u), \tilde{0}) \\ &< \varepsilon(b-a) + \sum_{i,j} \varepsilon 2^{-i-j} + \sum_2 j\eta_{ij} \\ &< \varepsilon(b-a) + 2\varepsilon. \end{aligned}$$

That is to say, f is strong fuzzy Henstock integrable to F on $[a, b]$.

Definition 6 *A sequence of fuzzy-number-valued functions $\{G_n(x)\}$ is said to be weak uniformly $ACG^*(U\omega - ACG^*)$ if for every $\lambda \in [0, 1]$, the real-valued functions $\{G_n(x)\}_\lambda^-$ and $\{G_n(x)\}_\lambda^+$ are $UACG^*$.*

Theorem 4 (Controlled Convergence theorem) *If a sequence of strong fuzzy Henstock integrable $\{f_n\}$ satisfies the following conditions:*

(1) $f_n(x) \to f(x)$ almost everywhere in $[a, b]$ as $n \to \infty$;

(2) the primitives $F_n(x) = (SFH)\int_a^x f_n(s)\mathrm{d}x$ of f_n are $\omega - ACG^*$ uniformly in n;

(3) the primitives $F_n(x)$ are equicontinuous on $[a, b]$,
then $f(x)$ is strong fuzzy Henstock integrable on $[a, b]$ and we have

$$\lim_{n\to\infty} (SFH)\int_a^b f_n(x)\mathrm{d}x = (SFH)\int_a^b f(x)\mathrm{d}x.$$

If condition (1) and (2) are replaced by condition (4):

(4) $g(x) \leq f(x) \leq h(x)$ almost everywhere on $[a, b]$, where $g(x)$ and $h(x)$ are steong fuzzy Henstock integrable.

Proof. In view of condition (3), $F(x)$ exist as the limit of $F_n(x)$ and is continuous. In fact, for $\forall\lambda \in [0, 1]$, $(F_n(x))_\lambda^-$ and $(F_n(x))_\lambda^+$ is uniformly ACG^* on $[a, b]$. By the Controlled Convergence theorem of real valued strong Henstock integral([14] Theorem 7.6), $F(x)$ is continuous. Because $F_\lambda^-(x)$ and $F_\lambda^+(x)$ is Henstock integrable on $[a, b]$, it follows condition (2) that F is $\omega - ACG^*$. From theorem 3.2, it remains to show that $F'(x) = f(x)$ almost everywhere. Hence we obtain $f(x)$ is strong fuzzy Henstock integrable on $[a, b]$.

Next, we put $G(x) = (SFH)\int_a^x F(t)\mathrm{d}t$, in view of condition (3), for $\forall\lambda \in [0, 1]$, we have

$$\lim_{n\to\infty} (F_n(x))_\lambda^- = G_\lambda^-(x) = F_\lambda^-(x)$$

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and

$$\lim_{n\to\infty}(F_n(x))^+_\lambda = G^+_\lambda(x) = F^+_\lambda(x).$$

So, let $x = b$, we have

$$\lim_{n\to\infty}(SFH)\int_a^b f_n(x)\mathrm{d}x = (SFH)\int_a^b f(x)\mathrm{d}x.$$

This completes the proof.

4 AN EXISTENCE RESULT OF GENERALIZED FUZZY INTEGRO-DIFFERENTIAL EQUATIONS

By using the Controlled Convergence theorem of strong fuzzy Henstock integral, in this section, we prove a theorem for the existence of solution to the Cauchy problem (1). For any bounded subset A of the Banach space X we denote $\alpha(A)$ the Kuratowski measure of non-compactness of A, i.e the infimum of all $\varepsilon > 0$ such that there exist a finite covering of A by sets of diameter less than ε. For the properties of α we refer to [1] for example.

Lemma 1 ([1]) *Let $H \subset C(I_\gamma, X)$ be a family of strong equicontinuous functions. Then*

$$\alpha(H) = \sup_{t\in I_\gamma}\alpha(H(t)) = \alpha(H(I_\gamma))$$

where $\alpha(H)$ denote the Kuratowski measure of non-compactness in $C(I_\gamma, X)$ and the function $t \to \alpha(H(t))$ is continuous.

Theorem 5 ([1]) *Let D be a closed convex subset of X, and let F be a continuous function from D into itself. If for $x \in D$ the implication*

$$\bar{V} = c\bar{o}n(\{x\} \cup F(V)) \Rightarrow V$$

is relatively compact, then F has a fixed point.

Theorem 6 *If the fuzzy-number-valued function $\tilde{f} : I_a \longrightarrow E^n$ is (SFH) integrable, then*

$$\int_I \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv}\tilde{f}(I),$$

where $\overline{conv}\tilde{f}(I)$ is the closure of the convex of $\tilde{f}(I)$, I is an arbitrary subinterval of I_a, and $|I|$ is the length of I..

Proof. Because of $j \circ \tilde{f}$ is abstract (SH) integrable in a Banach Space, by using the mean valued theorem of (SH) integrals, we have

$$(SH)\int_I j \circ \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv} j \circ \tilde{f}(I) = |I| \cdot j \circ \overline{conv}\tilde{f}(t).$$

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In additional, there exists $(SH)\int_I j\circ\tilde{f}(t)\mathrm{d}t = j\circ\int_I \tilde{f}(t)\mathrm{d}t$.

So, we have $j\circ\int_I \tilde{f}(t)\mathrm{d}t \in |I|\cdot\overline{conv}j\circ\tilde{f}(I)$. And the set $\{|I|\cdot\overline{conv}\tilde{f}(I)\}$ is a closed convex set, we have

$$\int_I \tilde{f}(t)\mathrm{d}t \in |I|\cdot\overline{conv}\tilde{f}(I).$$

Definition 7 *A fuzzy-number-valued function $\tilde{f}: I_a\times E^n \longrightarrow E^n$ is L^1-Carathéodory if the following conditions hold:*

(1) the fuzzy mapping $(x,y)\in E^n\times E^n$ is measurable for all $t\longrightarrow\tilde{f}(t,x,y)$;

(2) the fuzzy mapping $t\in I_a$ is continuous for all $(x,y)\longrightarrow\tilde{f}(t,x,y)$.

We observer that the problem (1) is equivalent to the integral eqution:

$$x(t)=x_0+\int_0^t \tilde{f}(z,x(z),\int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z$$

or

$$x(t)=x_0+(-1)\cdot\int_0^t \tilde{f}(z,x(z),\int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z. \tag{2}$$

Now, we define a notion of a solution.

Definition 8 *A $\omega-ACG^*$ function $x: I_a\to E^n$ is said to be the generalized solutions of the problem (1) if it satisfies the following conditions:*

(1) $x(0)=x_0$;

(2)

$$x'(t)=\tilde{f}(t,x(t),\int_0^t k_1(t,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(t,s)\tilde{h}(s,x(s))\mathrm{d}s).$$

for a. e. $t\in I_a$.

Definition 9 *A continuous function $x: I_a\to E^n$ is said to be the solutions of problem (2) if*

$$x(t)=x_0+\int_0^t \tilde{f}(z,x(z),\int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z$$

or

$$x(t)=x_0+(-1)\cdot\int_0^t \tilde{f}(z,x(z),\int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z.$$

for every $t\in I_a$

For every fuzzy number $x\in C(I_a,E^n)$, we define the norm of x by:

$$H(x,\tilde{0})=\sup_{t\in I_a} D(x,\tilde{0}).$$

Let

$$B(p) = \{x \in C(I_a, E^n) | H(x, \tilde{0}) \leq H(x, \tilde{0}) + p, p > 0\}.$$

Obviously, $B(p)$ is closed and convex in E^n. Define the operator $F : C(I_a, E^n) \to C(I_a, E^n)$ by:

$$F(x)(t) = x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z, s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z, s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z$$

where integrals are in the sense of strong fuzzy Henstock integral.

Let

$$\Gamma(p) = \{F(x) \in C(I_a, E^n) | x \in B(p)\}$$

for each $p > 0$. Let $r(K)$ be the spectral radius of the integral operator K defined by

$$K(u)(t) = \int_0^c k(t, s)u(s)\mathrm{d}s,$$

where the kernel $k \in C(I_a \times I_a, R), u \in C(I_a, E^n)$ and c denotes any fixed valued in I_a.

Next, we give the main result in this section.

Theorem 7 *Suppose that for each $\omega - ACG^*$ function $x : I_a \to E^n$, the functions $\tilde{g}(\cdot, x(\cdot)), \tilde{f}(\cdot, x(\cdot)), \int_0^{(\cdot)} k_1(\cdot, s)\tilde{g}(s, x(s))\mathrm{d}s$, and $\int_0^a k_2(z, s)\tilde{h}(s, x(s))\mathrm{d}s$ are (SFH) integrable, $\tilde{g}, \tilde{f}$, and $\tilde{h}$ are fuzzy L^1-Caratheodory functions. Let $k_1, k_2 : I_a \times I_a \to R^+$ be measurable functions such that $k_1(t, \cdot), k_2(t, \cdot)$ are continuous.*

Assume that there exists $p_0 > 0$ and positive constants L, L_1 and d_1, such that

$$\alpha(j \circ \tilde{g}(I, X)) \leq L\alpha(j \circ X), \quad I \subset I_a, X \subset B(p_0),$$
$$\alpha(j \circ \tilde{h}(I, X)) \leq L_1\alpha(j \circ X), \quad I \subset I_a, X \subset B(p_0),$$
$$\alpha(j \circ \tilde{f}(t, A, C, D)) \leq d_1 \cdot \max\{\alpha(j \circ A), \alpha(j \circ C), \alpha(j \circ D)\} \quad A, C, D \subset B(p_0),$$

where $\tilde{g}(I, X) = \{\tilde{g}(t, x(t)) | t \in I, x \in X\}, \tilde{h}(I, X) = \{\tilde{h}(t, x(t)) | t \in I, x \in X\}$ and

$$\tilde{f}(t, A, C, D) = \{\tilde{f}(t, x_1, x_2, x_3) | (x_1, x_2, x_3) \in A \times C \times D\}$$

where α denotes the Kuratowski measure of non-compactness.

Moreover, let $\Gamma(p_0)$ be equicontinuous, equibounded, and uniformly $\omega-ACG^*$ on I_a. Then, there exists at least on solution of problem (1) on I_c, for some $0 < c \leq a$, such that $d_1 \cdot c < 1$ and $d_1 \cdot c \cdot L \cdot r(K)$.

Proof. By equicontinuity and equiboundedness of $\Gamma(p_0)$ there exists a number $c, 0 < c \leq a$ such that

$$\begin{aligned}
&H(\int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z, s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z, s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
=&\sup_{t \in I_c} D(\int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z, s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z, s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
\leq& p_0,
\end{aligned}$$

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where $p_0 > 0, x \in B(p_0)$. By the definition of F, we have

$$
\begin{aligned}
&H(F(x)(t), \tilde{0})\\
=&H(x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})\\
\leq&H(x_0, \tilde{0}) + H(\int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})\\
\leq&H(x_0, \tilde{0}) + p_0, \quad t \in I_c, x_0 \in E^n.
\end{aligned}
$$

Using Theorem 4, we deduce that the fuzzy-number-valued function F is continuous.

Obviously, there exists $V \subset B$ such that $\overline{V} = \overline{conv}(\{x\} \cup F(V))$ for every $x \in B(p_0)$. Next, we will prove that V is relatively compact.

In fact, let $V(t) = \{v(t) \in E^n | v \in V\}$ for $t \in I_c$. Since $V \subset B(p_0)$ and $F(V) \subset \Gamma(p_0)$, then $V \subset \overline{V}$ is equicontinuous. By Lemma 1, we get that $t \to v(t) = \alpha(j \circ V(t))$ is continuous on I_c. For fixed $t \in I_c$, we divide the interval $[0, t]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = it/m, i = 0, 1, 2 \cdots, m$. Let $V([t_i, t_{i+1}]) = \{u(s) : u \in V, t_i \leq s \leq t_{i+1}, i = 1, 2, \cdots, m-1\}$ By Lemma 1 and the continuity of v, there exists $s_i \in I_i = [t_i, t_{i+1}]$ such that

$$\alpha(j \circ V([t_i, t_{i+1}])) = \sup_{t \in I_c}\{\alpha(j \circ V(s)) | t_i \leq s \leq t_{i+1}\} := v(s_i).$$

For fixed $z \in [0, t]$, we divide the interval $[0, z]$ into m parts: $0 = z_0 < z_1 < \cdots < z_m = z$, where $z_j = jz/m, j = 0, 1, 2 \cdots, m$. Let $V([z_j, z_{j+1}]) = \{u(s) | u \in V, z_j \leq s \leq z_{j+1}\}, j = 0, 1, 2, \cdots, m-1$. By Lemma 1 and the continuity of v, there exists $s_j \in I_j = [z_j, z_{j+1}]$ such that

$$\alpha(j \circ V([z_j, z_{j+1}])) = \sup_{t \in I_c}\{\alpha(j \circ V(s)) | z_j \leq s \leq z_{j+1}\} := v(s_j).$$

Furthermore, we divide the interval $[0, c]$ into m parts: $0 = r_0 < r_1 < \cdots < r_m = c$, where $r_k = kc/m, k = 0, 1, 2 \cdots, m$. Let $V([r_k, r_{k+1}]) = \{u(s) | u \in V, r_k \leq s \leq r_{k+1}\}, j = 0, 1, 2, \cdots, m-1$. By Lemma 1 and the continuity of v, there exists $s_k \in I_k = [r_k, r_{k+1}]$ such that

$$\alpha(j \circ V([r_k, r_{k+1}])) = \sup_{t \in I_c}\{\alpha(j \circ V(s)) | r_k \leq s \leq r_{k+1}\} := v(s_k).$$

By Theorem 3 and Theorem 4, we have

$$F(x)(t) = x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \tilde{f}(z, x(z), \sum_{j=0}^{m-1} \int_{z_j}^{z_{j+1}} k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s,$$
$$\sum_{k=0}^{m-1} \int_{r_k}^{r_{k+1}} k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z \in x_0$$
$$+ \sum_{i=0}^{m-1}(t_{i+1} - t_i)\overline{conv}\tilde{f}(I_i, V(I_i), \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}(k_1(I_i, I_j)\tilde{g}(I_j, V(I_j))),$$
$$\sum_{k=0}^{m-1}(r_{j+1} - r_j)\overline{conv}(k_2(I_i, I_j)\tilde{h}(I_k, V(I_k))),$$

where $k(I, J) = \{k(t, s)|t \in I, s \in J\}$ and $\tilde{g}(I, V(I)) = \{\tilde{g}(t, x(t))|t \in I, x \in V\}$.

Using the condition in assumption and the properties of noncompactness α ([1]), we have

$$\alpha(j \circ F(V)(t))$$
$$\leq \sum_{i=0}^{m-1}(t_{i+1} - t_i)\overline{conv}\alpha(j \circ \tilde{f}(I_i, V(I_i), \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}(k_1(I_i, I_j)\tilde{g}(I_j, V(I_j))),$$
$$\sum_{k=0}^{m-1}(r_{j+1} - r_j)\overline{conv}(k_2(I_i, I_j)\tilde{h}(I_k, V(I_k))))$$
$$\leq \sum_{i=0}^{m-1}(t_{i+1} - t_i)d_1 \max\{(\alpha(j \circ V(I_i)), \alpha j \circ (\sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}(k_1(I_i, I_j)\tilde{g}(I_j, V(I_j)))),$$
$$\alpha j \circ (\sum_{k=0}^{m-1}(r_{j+1} - r_j)\overline{conv}(k_2(I_i, I_j)\tilde{h}(I_k, V(I_k)))).$$

We observe that if

$$\alpha(j \circ V(I_i)) = \max\{(\alpha(j \circ V(I_i)), \alpha j \circ (\sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}(k_1(I_i, I_j)\tilde{g}(I_j, V(I_j)))),$$
$$\alpha(j \circ (\sum_{k=0}^{m-1}(r_{j+1} - r_j)\overline{conv}(k_2(I_i, I_j)\tilde{h}(I_k, V(I_k))))),$$

then

$$\alpha(j \circ V(t)) = \alpha j \circ (\overline{conv}(\{x(t)\} \cup F(V(t))))\alpha(j \circ F(V(t))) \leq d_1 \cdot c \cdot \alpha(j \circ V(t))$$

for every $t \in I_c$. Because $d_1 \cdot c < 1$, we have $\alpha(j \circ V) < \alpha(j \circ V)$. This is a contradiction.

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If

$$\begin{aligned}
&\alpha(j\circ(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j)))))\\
&=\max\{\alpha(j\circ V(I_i)),\alpha j\circ(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j)))),\\
&\alpha(j\circ(\sum_{k=0}^{m-1}(r_{j+1}-r_j)\overline{conv}(k_2(I_i,I_j)\tilde{h}(I_k,V(I_k)))))\},
\end{aligned}$$

we have

$$\begin{aligned}
&\alpha(j\circ F(V)(t))\\
&\le\sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_i,I_j)\alpha(j\circ\tilde{g}(I_j,V(I_j)))\\
&\le\sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1\cdot L\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_i,I_j)\alpha(j\circ V(I_j))\\
&\le\frac{c}{m}\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)\alpha(j\circ V(I_j))\sum_{i=0}^{m-1}k_1(I_i,I_j).
\end{aligned}$$

For $j=0,1,2,\dots,m-1$, there exists $q_j=0,1,2,\dots,m-1$ such that $k_1(I_i,I_j)\le k_1(I_{q_j},I_j)$. So,

$$\alpha(j\circ F(V)(t))\le d_1\cdot c\cdot L\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)v(s_j),\quad s_j\in I_j.$$

Hence

$$\begin{aligned}
&\alpha(j\circ F(V)(t))\\
&\le d_1\cdot c\cdot L\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)(v(s_j)-v(p_j))\\
&+d_1\cdot c\cdot L\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)v(p_j).
\end{aligned}$$

By the continuity of v, we have $j\circ v(s_j)-j\circ v(p_j)<\varepsilon$. Therefore, we have

$$\alpha(j\circ F(V)(t))\le d_1\cdot c\cdot L\cdot\int_0^c k_1(t,s)v(s)\mathrm{d}s$$

for $t\in I_c$. Since $V=\overline{conv}(\{x\}\cup F(V))$, we have $\alpha(j\circ V(t))\le\alpha(j\circ F(V)(t))$, so, $v(t)\le d_1\cdot c\cdot L\cdot\int_0^c k_1(t,s)v(s)\mathrm{d}s$. By Gronwalls inequality, we have $\alpha(j\circ V(t))=0$ for $t\in I_c$. By Arzelá−Ascoli's theorem, we have V is relatively. Consequently,

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by Theorem 5, F has a fixed point. That is to say that problem (1) have at least solutions.

Similary, if

$$\alpha(j \circ (\sum_{k=0}^{m-1}(r_{j+1} - r_j)\overline{conv}(k_2(I_i, I_j)\tilde{h}(I_k, V(I_k)))))$$
$$= \max\{\alpha(j \circ V(I_i)), \alpha j \circ (\sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}(k_1(I_i, I_j)\tilde{g}(I_j, V(I_j)))),$$
$$\alpha(j \circ (\sum_{k=0}^{m-1}(r_{j+1} - r_j)\overline{conv}(k_2(I_i, I_j)\tilde{h}(I_k, V(I_k)))))\},$$

then we have $\alpha(j \circ V(t)) \leq \alpha(j \circ F(V)(t))$. By Arzel$\acute{a}-$Ascoli's theorem, the set V is relatively. By Theorem 5, F has a fixed point which is a solution of the problem (1).

5 CONCLUSIONS

In this paper, we give the definition of the $\omega - ACG^*$ for a fuzzy-number-valued function and a generalized controlled convergence theorem. In addition, we deal with the Cauchy problem of discontinuous fuzzy integro-differential equations of mixed type involving the strong fuzzy Henstock integral in fuzzy number space. The function governing the equations is supposed to be discontinuous with respect to some variables and satisfy nonabsolute fuzzy integrablility. Our result improves the result given in Ref. [11, 2] and [26] (where uniform continuity was required), as well as those referred therein.

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On the Generalized Stieltjes Transform of Fox's Kernel Function and its Properties in the Space of Generalized Functions

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Abstract

In this paper, a Stieltjes transform enfolding some Fox's H-function has been investigated on certain class of generalized functions named as Boehmians. By developing two spaces of Boehmians, the extended transform has been inspected and some general properties are also obtained. An inverse problem is also discussed in some detail.

Keywords: Fox's H-function; Stieltjes transform; Laplace transform; Boehmian space; Distribution space.

1 Introduction

The Fox's H-function is a generalization of the Meijer G-function introduced by Charles Fox [15]. It is defined by the compact notation adopted for

$$H_{p,q}^{m,n}(\omega) = H_{p,q}^{m,n}\left[\omega \left| \begin{array}{l} (a_j, \alpha_j)_{j=1,2,\dots,p} \\ (b_j, \beta_j)_{j=1,2,\dots,q} \end{array} \right. \right]$$

and has an exemplification in terms of the Barnes-type integral [2]

$$H_{p,q}^{m,n}(\omega) = \frac{1}{2\pi i} \int_{\mathcal{L}} \jmath_{p,q}^{m,n}(\varsigma)\, \omega^{\varsigma} d\varsigma,$$

where $\mathcal{L}$ is a path in the complex plane, $\omega^{\varsigma} = \exp\{\varsigma(\log|\omega| + i\arg\omega)\}$, and

$$\jmath_{p,q}^{m,n}(\varsigma) = \frac{\boldsymbol{a}(\varsigma)\,\boldsymbol{b}(\varsigma)}{\boldsymbol{c}(\varsigma)\,\boldsymbol{d}(\varsigma)},$$

where

$$\begin{aligned} \boldsymbol{a}(\varsigma) &:= \prod_{1}^{m} \Gamma\left(b_j - \beta_j\varsigma\right), \boldsymbol{b}(\varsigma) := \prod_{1}^{n} \Gamma\left(1 - a_j + \alpha_j\varsigma\right) \\ \boldsymbol{c}(\varsigma) &:= \prod_{m+1}^{q} \Gamma\left(1 - b_j - \beta_j\varsigma\right) \text{ and } \boldsymbol{d}(\varsigma) := \prod_{n+1}^{p} \Gamma\left(a_j + \alpha_j\varsigma\right), \end{aligned}$$

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with $m,p,q \in \mathbb{N}$, $a_j, b_j \in \mathbb{C}, \alpha_j, \beta_j \in \mathbb{R}^+, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ satisfying $0 < n < p$ and $0 < m < q$, and $\mathbb{C}, \mathbb{R}^+$ and $\mathbb{N}$ denote, respectively, the sets of complex numbers, positive real numbers and positive integers.

We refer to the survey article by Braaksma [2] and the book of Charles Fox [15] for asymptotic behaviour of Fox's H-functions.

Fox's H-functions being an extreme generalization of the generalized hypergeometric functions ${}_pF_q$ are utilized for applications in a large variety of problems connected with statistical distribution theory, structures of random variables, generalized distributions, Mathai's pathway models, versatile integrals, reaction, diffusion, reaction diffusion, engineering, communication, fractional differential and integral equations and many areas of theoretical physics and statistical distribution theory as well.

Recently, utility and importance of H-functions are realized due to their occurrence as kernels of certain integral transforms.

The generalized Stieltjes transform of a function $\varphi(t)$ of one variable with kernel involving Fox's H-function is defined by [5, (1.3)]

$$\chi_g^s(\varphi)(\omega) = \int_0^\infty \omega^{\square 1} H_{2,2}^{1,2}\left[\left(\frac{\xi}{\omega}\right)^\lambda \middle| \begin{array}{l} (a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2) \end{array}\right]\varphi(\xi)\,\mathtt{d}\xi, \tag{1}$$

where $H_{p,q}^{m,n}[\omega]$ is the usual notation of the Fox H-function.

An interesting fact that we find it worthwhile to be mentioned here is that the transform under consideration is a modulation of the Laplace transform

$$\chi_\ell(\varphi)(\omega) = \int_0^\infty H_{2,2}^{1,2}\left[(\xi\omega)^\lambda \middle| \begin{array}{l} (a_1,\alpha_1) \\ (e_1,\gamma_1),(e_2,\gamma_2) \end{array}\right]\varphi(\xi)\,\mathtt{d}\xi \tag{2}$$

that rectified after some iterations and an appropriate choice on its parameter.

Denote by $\mathcal{J}_{c,d}$ the Fréchet space of smooth functions φ defined for all ξ $(0<\xi<\infty)$ by the set $\{\boldsymbol{\delta}_{c,d,k}\}$ of seminorms where

$$\boldsymbol{\delta}_{c,d,k}(\varphi) = \sup_{0<\xi<\infty}\left|\varrho_{c,d}(\log\xi)(\xi D_\xi)^k\sqrt{\xi}\varphi(\xi)\right| < \infty \tag{3}$$

for every choice of $k\,(k\in\mathbb{N}_0)$,

$$\varrho_{c,d}(\log\xi) = \begin{cases} \xi^c, & 1 \le \xi < \infty \\ \xi^d, & 0 < \xi < 1 \end{cases},$$

c and d are being real numbers.

The strong dual of continuous linear forms on $\mathcal{J}_{c,d}$ is denoted by $\mathcal{J}'_{c,d}$.

Let p_1 and q_1 be real numbers defined by $p_1 = \min\left(\operatorname{Re}\frac{b_j}{\beta_j}\right)$ $(j=1,2,...,m)$, $q_1 = \max\left(\operatorname{Re}\frac{a_j \square 1}{\alpha_j}\right)$ $(j=1,2,...,n)$ and related by the pair of inequalities $c+\dfrac{1}{2}+\lambda q_1 < 0$ and $d+\dfrac{1}{2}+\lambda p_1 > 0$.

Then, the extended transform of a distribution $f \in \mathcal{J}'_{c,d}$ is defined as the application of $f(t) \in \mathcal{J}'_{c,d}$ to its kernel (see [5, Theorem 3.1])

$$\omega^{\square 1} H_{2,2}^{1,2}\left[\left(\frac{\xi}{\omega}\right)^\lambda \middle| \begin{array}{l} (a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2) \end{array}\right]$$

giving, by kernel method,

$$\chi_g^s(f)(\omega) = \left\langle f(\xi), \omega^{\square 1} H_{2,2}^{1,2}\left[\left(\frac{\xi}{\omega}\right)^\lambda \middle| \begin{array}{l} (a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2) \end{array}\right]\right\rangle, \tag{4}$$

where ω is a complex number not lying on the negative real axis.

For our consecutive investigation, we denote by $\mathcal{I}_{c,d}$ the subset of those integrable functions of $\mathcal{J}_{c,d}$ assigned by the set $\{\boldsymbol{\delta}_{c,d,k}\}$ and its strong dual $\mathcal{I}'_{c,d}$ of distributions. Then, indeed, $\mathcal{I}_{c,d} \subseteq \mathcal{J}_{c,d}$ and, hence, $\mathcal{J}'_{c,d} \subseteq \mathcal{I}'_{c,d}$. Denote by $\mathcal{D}$ the standard notation of the space of smooth functions whose supports are compact subset of $(0,\infty)$. Then, it is easy to check that $\mathcal{D} \subset \mathcal{I}_{c,d}$ and that topology of $\mathcal{D}$ is stronger than the topology induced on it by $\mathcal{I}_{c,d}$. Hence, the restriction to any $f \in \mathcal{I}'_{c,d}$ to $\mathcal{D}$ is in $\mathcal{D}'$, where $\mathcal{D}'$ is the space of disributions.

We need to establish the following theorem.

THEOREM 1 Given $\varphi \in \mathcal{I}_{c,d}$. Then, $\chi_g^s(\varphi) \in \mathcal{I}_{c,d}$.
PROOF Let $\varphi \in \mathcal{I}_{c,d}$ be given. For the convenience of the reader, we write

$$H_{2,2}^{1,2}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right] = H_{2,2}^{1,2}\left[\left(\frac{y}{\xi}\right)^{\lambda} \middle| \begin{array}{l} (a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2) \end{array}\right].$$

By aid of (3) and (1) and simple computation we write

$$\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\sqrt{\xi}\left(\chi_g^s\right)(\varphi)(\xi)\right| \le \int_0^\infty \left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\sqrt{\xi}\xi^{\square 1}H_{2,2}^{1,2}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| \times |\varphi(y)|\,\mathrm{d}y.$$

This can also be revised to give

$$\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\sqrt{\xi}\left(\chi_g^s\right)(\varphi)(\xi)\right| \le \int_0^\infty \left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\left(\xi^{\square 1}\right)^{\frac{1}{2}}H_{2,2}^{1,2}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| \times |\varphi(y)|\,\mathrm{d}y.$$

By utilizing the Property 2.8

$$\mathcal{D}_z^k\left\{z^w H_{p,q}^{m,n}\left[cz^\sigma \middle| \begin{array}{l} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array}\right]\right\} = z^{w\square k}H_{p+1,q+1}^{m,n+1}\left[cz^\sigma \middle| \begin{array}{l} (-w,\sigma),(a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q},(k-w,\sigma) \end{array}\right]$$

of Kilbas and Saigo [1, p.33] we get

$$\left|\varrho_{c,d}(\log\xi)(\xi D_\xi)^k\sqrt{\xi}\left(\chi_g^s\right)(\varphi)(\xi)\right| \le \int_0^\infty \left|\varrho_{c,d}(\log\xi)\xi^{\square\frac{1}{2}}\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right||\varphi(y)|\,\mathrm{d}y,$$

where

$$\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right] = H_{3,3}^{1,3}\left[\left(\frac{y}{\xi}\right)^{\lambda} \middle| \begin{array}{l} \left(\frac{1}{2},\lambda\right),(a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2),\left(k-\frac{1}{2},\lambda\right) \end{array}\right].$$

Therefore, the asymptotic properties of H-functions, for large ξ, imply

$$\sup_{0<\xi<\infty}\left|\varrho_{c,d}(\log\xi)\xi^{\square\frac{1}{2}}\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| = \sup_{0<\xi<\infty}\left|\xi^c\xi^{\square\frac{1}{2}}\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| < M_1,$$

where M_1 is some positive constant. Similarly, for small ξ, it implies

$$\sup_{0<\xi<\infty}\left|\varrho_{c,d}(\log\xi)\xi^{\square\frac{1}{2}}\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| = \sup_{0<\xi<\infty}\left|\xi^d\xi^{\square\frac{1}{2}}\widehat{H_{3,3}^{1,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| < M_2,$$

where M_2 is a postive constant.
Let $M = \max\{M_1, M_2\}$. Then, by the preceding two formulas, we have

$$\sup_{0<\xi<\infty} \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} \left(\chi_g^s \right) (\varphi) (\xi) \right| \leq M \int_0^\infty |\varphi (y)| \, \mathtt{d}y < \infty,$$

since φ is integrable.

The proof of this theorem is finished.

Definition 2 Let $f \in \acute{\mathcal{I}}_{c,d}$. Then, the Stieltjes transform χ_g^s of $f \in \acute{\mathcal{I}}_{c,d}$ is defined by the inner product

$$\left\langle \chi_g^s (f) (\omega), \varphi (\omega) \right\rangle = \left\langle f (\omega), \chi_g^s (\varphi) (\omega) \right\rangle, \tag{6}$$

where $\varphi \in \mathcal{I}_{c,d}$ is aritrary.

The inner product on the left hand side of (6) is well-defined by Theorem 1. Hence, it may be noted from Equation 6 that the Stieltjes transform of $f \in \acute{\mathcal{I}}_{c,d}$ is a distribution in $\acute{\mathcal{I}}_{c,d}$.

2 Generalized Distributions; Boehmian Spaces

We always assume that readers are aquainted with the concept of Boehmian spaces, if it were otherwise, we would refer to [4], [6 − 14] and [16, 17].

Let us now prove the following Theorems that legitimate the existence of our Boehmian spaces.

The following definition is important for our next investigation.

Definition 3 Given $\varphi, \psi \in \mathcal{I}_{c,d}$, then, for φ and ψ, the product $\otimes$ is defined by

$$(\varphi \otimes \psi) (\omega) = \int_0^\infty \varphi \left(\xi^{\square 1} \omega \right) \frac{\psi (\xi)}{\xi} \mathtt{d}\xi, \tag{7}$$

provided the integral exists.

Theorem 4 Given $\varphi \in \mathcal{I}_{c,d}$, then $\varphi \otimes \psi \in \mathcal{I}_{c,d}$, for every $\psi \in \mathcal{D}$.

Proof On account of (3), we write

$$\begin{aligned} \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} (\varphi \otimes \psi) (\xi) \right| &\leq \int_0^\infty \left| y^{\square 1} \psi (y) \right| \\ &\quad \times \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} \varphi \left(y^{\square 1} \xi \right) \right| \mathtt{d}y \\ &\leq A^* \int_0^\infty \left| y^{\square 1} \psi (y) \right| \mathtt{d}y. \end{aligned}$$

Let $[a_1, a_2]$ be a closed interval containing the support of ψ. Since $\varphi \in \mathcal{I}_{c,d}$, it by considering supremum over all $\xi \, (0 < \xi < \infty)$ follows that

$$\boldsymbol{\delta}_{c,d,k} (\varphi \otimes \psi) \leq A^* \int_{a_1}^{a_2} \left| y^{\square 1} \psi (y) \right| \mathtt{d}y < \infty,$$

for some constant A^*.

Hence, the proof of this theorem is finished.

Let $\curlyvee$ be the product of Mellin type given by

$$(\varphi \curlyvee \psi) (y) = \int_0^\infty \xi^{\square 1} \varphi \left(\xi^{\square 1} y \right) \psi (\xi) \, \mathtt{d}\xi. \tag{8}$$

We generate the space $\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ where Δ is the subset of $\mathcal{D}$ of sequences (δ_n) such that

$$\left.\begin{array}{l}\text{(i)}\ \displaystyle\int_0^\infty \delta_n(\xi)\,\mathtt{d}\xi = 1;\\ \text{(ii)}\ \ |\delta_n(\xi)| < A,\ \ A\in\mathbb{R},\ \ A>0;\\ \text{(iii)}\ \ \operatorname{supp}\delta_n(\xi)\subseteq(a_n,b_n),\ \ a_n,b_n\to 0 \text{ as } n\to\infty,\end{array}\right\}\tag{9}$$

$n\in\mathbb{N},\ \xi\in(0,\infty)$.

In what follows we shall make a free use of the properties of the product $\curlyvee$ that we briefly describe them as follows :

(i) $\varphi_1\curlyvee\varphi_2=\varphi_2\curlyvee\varphi_1$;
(ii) $(\varphi_1\curlyvee\varphi_2)\curlyvee\varphi_3=\varphi_1\curlyvee(\varphi_2\curlyvee\varphi_3)$;
(iii) $(\alpha\varphi_1)\curlyvee\varphi_2=\alpha(\varphi_1\curlyvee\varphi_2)$;
(iv) $\varphi_1\curlyvee(\varphi_2+\varphi_3)=\varphi_1\curlyvee\varphi_2+\varphi_1\curlyvee\varphi_3$.

Following theorem follows from elementary rules of integral calculus. Hence, its proof is deleted.

Theorem 5 Given $\varphi_n,\varphi,\varphi_1,\varphi_2\in\mathcal{I}_{c,d}$, $\alpha\in\mathbb{C}$, and $\psi,\psi_1,\psi_2\in\mathcal{D}$ such that $\varphi_n\to\varphi$ as $n\to\infty$, then the following are true :

(i) $\varphi_n\otimes\psi\to\varphi\otimes\psi$ as $n\to\infty$.
(ii) $\varphi_1\otimes(\psi_1+\psi_2)=\varphi_1\otimes\psi_1+\varphi_2\otimes\psi_2$.
(iii) $\alpha(\varphi\otimes\psi)=\alpha\varphi\otimes\psi=\varphi\otimes(\alpha\psi)$.

Theorem 6 Given $\varphi\in\mathcal{I}_{c,d}$ and $\psi_1,\psi_2\in\mathcal{D}$, then $\varphi\otimes(\psi_1\curlyvee\psi_2)=(\varphi\otimes\psi_1)\otimes\psi_2$.

Proof Let $\varphi\in\mathcal{I}_{c,d}$ and $\psi_1,\psi_2\in\mathcal{D}$. Then, by aid of the integrals (7) and (8), we write

$$\begin{aligned}(\varphi\otimes(\psi_1\curlyvee\psi_2))(\omega) &= \int_0^\infty \varphi\left(\xi^{-1}\omega\right)\frac{(\psi_1\curlyvee\psi_2)(\xi)}{\xi}\mathtt{d}\xi\\ &= \int_0^\infty \psi_2(y)\,y^{-1}\int_0^\infty \varphi\left(\xi^{-1}\omega\right)\frac{\psi_1\left(\xi y^{-1}\right)}{\xi}\mathtt{d}y\mathtt{d}\xi\\ &= \int_0^\infty \psi_2(y)\,\frac{\int_0^\infty \varphi\left(y^{-1}z^{-1}\omega\right)z^{-1}\psi(z)\,\mathtt{d}z}{y}\mathtt{d}y\\ &= \int_0^\infty \psi_2(y)\,\frac{(\varphi\otimes\psi_1)\left(y^{-1}\omega\right)}{y}\mathtt{d}y.\end{aligned}$$

The proof of this theorem is finished.

Theorem 7 Given $(\delta_n)\in\Delta$ and $\varphi\in\mathcal{I}_{c,d}$, then $\varphi\otimes\delta_n\in\mathcal{I}_{c,d}$.

Proof Let $\varphi\in\mathcal{I}_{c,d}$ and $(\delta_n)\in\Delta$ be given. Then, by (3) and the Identity (i) of (9) we have

$$\begin{aligned}\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\sqrt{\xi}(\varphi\otimes\delta_n-\varphi)(\xi)\right| &= \int_0^\infty\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\sqrt{\xi}\varphi_y(\xi)\right|\\ &\quad\times|\delta_n(y)|\,\mathtt{d}y,\end{aligned}\tag{10}$$

where $\varphi_y(\xi)=\varphi\left(\xi y^{-1}\right)y^{-1}-\varphi(\xi)$. Since $\varphi_y(\xi)\in\mathcal{I}_{c,d}$, we from (10), get that

$$\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\sqrt{\xi}(\varphi\otimes\delta_n-\varphi)(\xi)\right|\le A\int_0^\infty|\delta_n(y)|\,\mathtt{d}y,\tag{11}$$

where A is some positive constant.

Hence, by the identities (ii) and (iii) of (9), Equation (11) can be expressed as

$$\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_\xi)^k\sqrt{\xi}(\varphi\otimes\delta_n-\varphi)(\xi)\right|\le AA_1(b_n-a_n)\to 0 \text{ as } n\to\infty.\tag{12}$$

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Hence, the proof of this theorem is finished.

THEOREM 8 Given $\varphi \in \mathcal{I}_{c,d}$, then, for every $(\delta_n) \in \Delta$, we have $\varphi \otimes \delta_n \to \varphi$ in $\mathcal{I}_{c,d}$ as $n \to \infty$.
PROOF Let F_n be a compact subset of $(0, \infty)$ containing $\operatorname{supp} \delta_n$, for all n. Then, on account of (i) of (9), we get

$$\begin{aligned} \left|\varrho_{c,d}(\log \xi)(\xi \mathcal{D}_\xi)^k \sqrt{\xi}(\varphi \otimes \delta_n - \varphi)(\xi)\right| \leq & \int_{F_n} \left|\varrho_{c,d}(\log \xi)(\xi \mathcal{D}_\xi)^k \sqrt{\xi}\varphi\left(x^{\square 1}\xi\right)\right| \\ & \times \frac{|\delta_n(x)|}{x} \mathrm{d}x \\ & + \int_{F_n} \left|\varrho_{c,d}(\log \xi)(\xi \mathcal{D}_\xi)^k \sqrt{\xi}\varphi(\xi)\right| \\ & \times |\delta_n(x)| \, \mathrm{d}x. \end{aligned} \tag{13}$$

Therefore, (13) gives

$$\left|\varrho_{c,d}(\log \xi)(\xi \mathcal{D}_\xi)^k \sqrt{\xi}(\varphi \otimes \delta_n - \varphi)(\xi)\right| \leq A_1 \int_{F_n} \frac{|\delta_n(x)|}{x} \mathrm{d}x + A_2 \int_{F_n} |\delta_n(x)| \, \mathrm{d}x.$$

Considering the supremum over all ξ, $0 < \xi < \infty$, implies

$$\boldsymbol{\delta}_{c,d,k}(\varphi \otimes \delta_n - \varphi) < \infty,$$

for any choice of the real numbers c, d and $k \in \mathbb{N}_0$. Thus, we find that

$$\varphi \otimes \delta_n \to \varphi \text{ in } \mathcal{I}_{c,d} \text{ as } n \to \infty.$$

The proof has been completed .

COROLLARY 9 Given $(\delta_n) \in \Delta$ and $\varphi_1 \otimes \delta_n = \varphi_2 \otimes \delta_n$, then $\varphi_1 = \varphi_2$ for all $\varphi_1, \varphi_2 \in \mathcal{I}_{c,d}$.

The space $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ is constructed.

Addition and multiplication by a scalar in $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ are defined by

$$\left[\frac{\varphi_n}{\delta_n}\right] + \left[\frac{\psi_n}{\varepsilon_n}\right] =: \left[\frac{\varphi_n \otimes \delta_n + \psi_n \otimes \delta_n}{\delta_n \curlyvee \varepsilon_n}\right] \text{ and } \mu \left[\frac{\varphi_n}{\delta_n}\right] =: \left[\frac{\mu \varphi_n}{\delta_n}\right] \quad (\mu \in \mathbb{C}).$$

An extension of $\otimes$ and differentiation to $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ is given as follows

$$\left[\frac{\varphi_n}{\delta_n}\right] \otimes \left[\frac{\psi_n}{\varepsilon_n}\right] =: \left[\frac{\varphi_n \otimes \psi_n}{\delta_n \curlyvee \varepsilon_n}\right] \text{ and } \mathcal{D}^\alpha \left[\frac{\varphi_n}{\delta_n}\right] =: \left[\frac{\mathcal{D}^\alpha \varphi_n}{\delta_n}\right] \quad (\alpha \in \mathbb{R}).$$

Given $\left[\frac{\varphi_n}{\delta_n}\right] \in \mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ and $\varpi \in \mathcal{I}_{c,d}$. Then, $\otimes$ can be extended to $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee)) \times \mathcal{I}_{c,d}$ by

$$\left[\frac{\varphi_n}{\delta_n}\right] \otimes \varpi =: \left[\frac{\varphi_n \otimes \varpi}{\delta_n}\right].$$

$\beta_n \xrightarrow{\delta} \beta$ in $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ if there can be (δ_n) in Δ satisfying $(\beta_n \otimes \delta_k), (\beta \otimes \delta_k) \in \mathcal{I}_{c,d}$ $(k, n \in \mathbb{N})$ and $(\beta_n \otimes \delta_k) \to (\beta \otimes \delta_k)$ in $\mathcal{I}_{c,d}$ as $n \to \infty$ $(k \in \mathbb{N})$. This can be expressed to mean :

$\beta_n \xrightarrow{\delta} \beta$ $(n \to \infty)$ in $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ if there are $\varphi_{n,k}$ and $\varphi_k \in \mathcal{I}_{c,d}$, and $(\delta_k) \in \Delta$ where $\beta_n = \left[\frac{\varphi_{n,k}}{\delta_k}\right]$, $\beta = \left[\frac{\varphi_k}{\delta_k}\right]$ and for each $k \in \mathbb{N}$ we have $f_{n,k} \to f_k$ as $n \to \infty$ in $\mathcal{I}_{c,d}$.

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$\beta_n \xrightarrow{\Delta} \beta$ in $\mathcal{B}((\mathcal{I}_{c,d},\otimes);(\mathcal{D},\curlyvee))$, in a sense of Δ, if there can be $(\delta_n)\in\Delta$ where $(\beta_n-\beta)\otimes\delta_n\in\mathcal{I}_{c,d}$ $(\forall n\in\mathbb{N})$ and that $(\beta_n-\beta)\otimes\delta_n\to 0$ as $n\to\infty$ in $\mathcal{I}_{c,d}$.

By techniques similar to above, the space $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$ can similarly be generated. In $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$, addition and multiplication by a scalar has the following meanings

$$\left[\frac{\varphi_n}{\delta_n}\right]+\left[\frac{\psi_n}{\varepsilon_n}\right]=:\left[\frac{\varphi_n\curlyvee\delta_n+\psi_n\curlyvee\delta_n}{\delta_n\curlyvee\varepsilon_n}\right]\text{ and }\rho\left[\frac{\varphi_n}{\delta_n}\right]=:\left[\frac{\alpha\varphi_n}{\delta_n}\right]\quad(\rho\in\mathbb{C}).$$

We extend $\curlyvee$ and the differentiation to $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$ as

$$\left[\frac{\varphi_n}{\delta_n}\right]\curlyvee\left[\frac{\psi_n}{\varepsilon_n}\right]=\left[\frac{\varphi_n\curlyvee\psi_n}{\delta_n\curlyvee\varepsilon_n}\right],\ \mathcal{D}^{\alpha}\left[\frac{\varphi_n}{\delta_n}\right]=\left[\frac{\mathcal{D}^{\alpha}\varphi_n}{\delta_n}\right],$$

α being real number.

Given $\left[\frac{\varphi_n}{\delta_n}\right]\in\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$ and $\varpi\in\mathcal{I}_{c,d}$. We define $\curlyvee$ for $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))\times\mathcal{I}_{c,d}$ as

$$\left[\frac{\varphi_n}{\delta_n}\right]\curlyvee\varpi=:\left[\frac{\varphi_n\curlyvee\varpi}{\delta_n}\right].$$

Convergence in $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$ is as follows :

$\beta_n\xrightarrow{\delta}\beta$ $(n\to\infty)$ in $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$ if and only if there can be (δ_n) in Δ such that $(\beta_n\curlyvee\delta_k),(\beta\curlyvee\delta_k)\in\mathcal{I}_{c,d}$ $(\forall k,n\in\mathbb{N})$ and $(\beta_n\curlyvee\delta_k)\to(\beta\curlyvee\delta_k)$ in $\mathcal{I}_{c,d}$ as $n\to\infty$ $(\forall k\in\mathbb{N})$. Or, if there can be found $\varphi_{n,k},\varphi_k\in\mathcal{I}_{c,d}$, $(\delta_k)\in\Delta$, $\beta_n=\left[\frac{\varphi_{n,k}}{\delta_k}\right]$, $\beta=\left[\frac{\varphi_k}{\delta_k}\right]$ and $f_{n,k}\to f_k$ as $n\to\infty$ in $\mathcal{I}_{c,d}$ $(k\in\mathbb{N})$.

$\beta_n\xrightarrow{\Delta}\beta$ $(n\to\infty)$, in $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$, if there can be $(\delta_n)\in\Delta$ satisfying $(\beta_n-\beta)\curlyvee\delta_n\in\mathcal{I}_{c,d}$ and $(\beta_n-\beta)\curlyvee\delta_n\to 0$ as $n\to\infty$ in $\mathcal{I}_{c,d}$.

3 The Generalized χ_g^s Transform of $\mathcal{B}((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee))$

We devote this section to the definition of the generalized Stieltjes transform and to derive some desired properties. The following theorem specifies the relation between $\curlyvee$ and $\otimes$.

THEOREM 10 Given $\varphi\in\mathcal{I}_{c,d}$, then $\chi_g^s(\varphi\curlyvee\psi)(\omega)=\left(\chi_g^s(\varphi)\psi\right)(\omega)$ for every $\psi\in\mathcal{D}$.

PROOF Let $\varphi\in\mathcal{I}_{c,d}$ and $\psi\in\mathcal{D}$ be given. Then, by (1), we have

$$\begin{aligned}\chi_g^s(\varphi\curlyvee\psi)(\omega) &= \int_0^\infty \omega^{\square 1}H_{2,2}^{1,2}\left[\left(\frac{\xi}{\omega}\right)^{\lambda}\left|\begin{matrix}(a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1)\\(e_1,\gamma_1),(e_2,\gamma_2)\end{matrix}\right.\right]\\ &\quad\times(\varphi\curlyvee\psi)(\omega)\,\mathtt{d}\xi,\end{aligned}$$

which can be expressed after setting the variables and using Fubini's theorem as

$$\begin{aligned}\chi_g^s(\varphi\curlyvee\psi)(\omega) &= \int_0^\infty\psi(y)\int_0^\infty\omega^{\square 1}\\ &\quad\times H_{2,2}^{1,2}\left[\left(\frac{z}{y^{\square 1}\omega}\right)^{\lambda}\left|\begin{matrix}(a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1)\\(e_1,\gamma_1),(e_2,\gamma_2)\end{matrix}\right.\right]\varphi(z)\,\mathtt{d}z\mathtt{d}y.\end{aligned}\tag{14}$$

Simple motivation on (14) yields

$$\begin{aligned}\chi_g^s(\varphi \curlyvee \psi)(\omega) &= \int_0^\infty \psi(y) \int_0^\infty \left(y\omega^{\square 1}\right) \\ &\quad \times H_{2,2}^{1,2}\left[\left(\frac{z}{y^{\square 1}\omega}\right)^{\lambda} \middle| \begin{array}{l}(a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2)\end{array}\right]\varphi(z)\,\mathtt{d}z\mathtt{d}y.\end{aligned}$$

Hence, the above equation is interpreted to mean

$$\chi_g^s(\varphi \curlyvee \psi)(\omega) = \int_0^\infty \psi(y)\, y^{\square 1}\left(\chi_g^s\right)(\varphi)\left(y\omega^{\square 1}\right)\mathtt{d}y.$$

Hence, the proof of this theorem is finished.

In view of the preceeding result we give the definition of χ_g^s transform of $\left[\frac{\varphi_n}{\delta_n}\right]$ in the space $\mathcal{B}\left(\left(\mathcal{I}_{c,d},\curlyvee\right),\left(\mathcal{D},\curlyvee\right)\right)$ as

$$\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right]\right) =: \left[\frac{\chi_g^s\varphi_n}{\delta_n}\right] \tag{15}$$

which belongs to $\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right),\left(\mathcal{D},\curlyvee\right)\right)$ by means of Theorem 10.

Theorem 11 The operator $\widehat{\chi_g^s}$ is well - defined and linear, mapping from $\mathcal{B}\left(\left(\mathcal{I}_{c,d},\curlyvee\right);\left(\mathcal{D},\curlyvee\right)\right)$ into $\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$.

Proof Let $\left[\frac{\varphi_n}{\delta_n}\right] = \left[\frac{\psi_n}{\varepsilon_n}\right]$ in the sense of $\mathcal{B}\left(\left(\mathcal{I}_{c,d},\curlyvee\right);\left(\mathcal{D},\curlyvee\right)\right)$. Then, by the concept of equivalent classes, $\frac{\varphi_n}{\delta_n}$ and $\frac{\psi_n}{\varepsilon_n}$ are equivalent in $\mathcal{B}\left(\left(\mathcal{I}_{c,d},\curlyvee\right);\left(\mathcal{D},\curlyvee\right)\right)$. Thus, it has been obtained $\varphi_n \curlyvee \varepsilon_m = \psi_n \curlyvee \delta_m$.

Applying χ_g^s to the sides of the above equation and employing Theorem 10 imply

$$\chi_g^s\varphi_n \otimes \varepsilon_m = \chi_g^s\psi_n \otimes \delta_m \ \ (\forall n,m \in \mathbb{N}).$$

That is,

$$\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right] = \left[\frac{\chi_g^s\psi_n}{\varepsilon_n}\right].$$

To show that the $\widehat{\chi_g^s} : \mathcal{B}\left(\left(\mathcal{I}_{c,d},\curlyvee\right);\left(\mathcal{D},\curlyvee\right)\right) \to \mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ is linear, let $\rho_1 = \left[\frac{\varphi_n}{\delta_n}\right], \rho_2 = \left[\frac{\psi_n}{\varepsilon_n}\right] \in \mathcal{B}\left(\left(\mathcal{I}_{c,d},\curlyvee\right);\left(\mathcal{D},\curlyvee\right)\right)$. Then, addition of Boehmians of $\mathcal{B}\left(\left(\mathcal{I}_{c,d},\curlyvee\right);\left(\mathcal{D},\curlyvee\right)\right)$ and Equation 15, suggest to write

$$\widehat{\chi_g^s}(\rho_1+\rho_2) = \left[\frac{\chi_g^s(\varphi_n \curlyvee \varepsilon_n) + \chi_g^s(\psi_n \curlyvee \delta_n)}{\delta_n \curlyvee \varepsilon_n}\right].$$

By aid of Theorem 10, we obtain

$$\widehat{\chi_g^s}(\rho_1+\rho_2) = \left[\frac{\chi_g^s\varphi_n \otimes \varepsilon_n + \chi_g^s\psi_n \otimes \delta_n}{r_n \curlyvee \varepsilon_n}\right].$$

Employing the product $\otimes$ that assigned to the $\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ gives

$$\widehat{\chi_g^s}(\rho_1+\rho_2) = \left[\frac{\chi_g^s\varphi_n}{\delta_n}\right] + \left[\frac{\chi_g^s\psi_n}{\varepsilon_n}\right].$$

Hence, we have obtained that

$$\widehat{\chi_g^s}\left(\rho_1+\rho_2\right)=\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right]\right)+\widehat{\chi_g^s}\left(\left[\frac{\psi_n}{\varepsilon_n}\right]\right).$$

Also, it is easy for readers to check that

$$\lambda\widehat{\chi_g^s}\left(\rho_1\right)=\widehat{\chi_g^s}\left(\lambda\rho_1\right)\ \ (\lambda\in\mathbb{C}).$$

Hence, the proof of this theorem is finished.

Theorem 12 The mapping $\widehat{\chi_g^s}:\mathcal{B}\left(\left(\mathcal{I}_{c,d},\curlyvee\right);\left(\mathcal{D},\curlyvee\right)\right)\to\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ is an isomorphism.

Proof Given $\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]=\left[\frac{\chi_g^s\psi_n}{\varepsilon_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$. Then, by virtue of Theorem 10, we get

$$\chi_g^s\varphi_n\otimes\varepsilon_m=\chi_g^s\psi_m\otimes\delta_n\ \ (m,n\in\mathbb{N}).$$

Once again, Theorem 10 implies

$$\chi_g^s\left(\varphi_n\otimes\varepsilon_m\right)=\chi_g^s\left(\psi_m\otimes\delta_n\right).$$

Hence $\varphi_n\otimes\varepsilon_m=\psi_m\otimes\delta_n$. Therefore,

$$\left[\frac{\varphi_n}{\delta_n}\right]=\left[\frac{\psi_n}{\varepsilon_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\curlyvee\right);\left(\mathcal{D},\curlyvee\right)\right).$$

This proves that the above mapping is an injection. surjectivity of $\widehat{\chi_g^s}$ is obvious.
The proof is finished.

Definition 13 Let $\rho^*\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right),\ \rho^*=\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]$. Then, we the inverse mapping $\widehat{\chi_g^s}$ is defined as

$$\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\right)=\left[\frac{\left(\chi_g^s\right)^{\square 1}\left(\chi_g^s\varphi_n\right)}{\delta_n}\right]=\left[\frac{\varphi_n}{\delta_n}\right],$$

for each $(\delta_n)\in\Delta$.

Theorem 14 Let $\rho^*=\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ for some $\left[\frac{\varphi_n}{\delta_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ and $\phi,\psi\in\mathcal{D}$. Then we have

(i) $\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\otimes\phi\right)=\left[\frac{\varphi_n}{\delta_n}\right]\curlyvee\phi$,

(ii) $\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right]\curlyvee\psi\right)=\rho^*\otimes\psi$.

Proof Assume $\rho^*=\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ be given. Then, by Theorem 10, we write

$$\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\otimes\phi\right)=\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\left[\frac{\chi_g^s\varphi_n\otimes\phi}{\delta_n}\right]\right)=\left[\frac{\left(\chi_g^s\right)^{\square 1}\left(\chi_g^s\varphi_n\otimes\phi\right)}{\delta_n}\right].$$

Hence,

$$\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\otimes\phi\right)=\left[\frac{\varphi_n\curlyvee\phi}{\delta_n}\right].$$

Therefore,

$$\left(\widehat{\chi_g^s}\right)^{\square 1}(\rho^* \otimes \phi) = \left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \phi.$$

To prove the second identity, we apply Theorem 10 to get

$$\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \psi\right) = \widehat{\chi_g^s}\left(\left[\frac{\varphi_n \curlyvee \psi}{\delta_n}\right]\right) = \rho^* \otimes \psi.$$

This finishes the proof of the theorem.

Conclusion : This paper provides some integral products which were implemented to extend a new type of Stieltjes transforms enfolding Fox's H-functions as kernels to generalized functions. The generalized Sitieltjes transform was formed to satisfy the desired properties of the classical transform. It may be concluded here that the employed Stieltjes transform method is a very efficient technique in extending integral transforms to generalized functions and could lead to a promising approach for many integrals of special functions kernels .

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Decision making based on interval-valued intuitionistic fuzzy soft sets and its algorithm *

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Abstract: This paper investigates an approach to interval-valued intuitionistic fuzzy soft sets in decision making by means of grey relational analysis and D-S theory of evidence. An algorithm based on this approach in decision making is presented.

Keywords: Interval-valued intuitionistic fuzzy soft set; Decision making; Grey relational analysis; D-S theory of evidence.

1 Introduction

In 1999, Molodtsov [18] initiated soft sets as a mathematical tool for dealing with vagueness and uncertainties. Compared with some traditional tools for dealing with uncertainties, such as probability theory, fuzzy set theory [32], rough set theory [23], soft set theory has the advantage of freeing from the inadequacy of the parametrization tools of those theories.

Recently, many efforts have been devoted to further generalizations and extensions of Molodtsov's soft sets. Maji et al. [19, 20] defined fuzzy soft sets and intuitionistic fuzzy soft sets by combining soft sets with fuzzy sets and and intuitionistic fuzzy sets, Yang et al. [31] defined the interval-valued fuzzy soft sets. Jiang et al. [7] proposed a more general soft set model called interval-valued intuitionistic fuzzy soft set, which is a substantial and important combination of the soft set and the interval-valued intuitionistic fuzzy set. The intuitionistic fuzzy soft set theory makes descriptions of the objective world more realistic, practical and accurate in some cases, making it very promising.

With the rapid development of soft set theory, there has been some progress on the practical applications, especially the use of soft sets in decision making. Roy et al. [25] discussed score value as the evaluation basis to find an optimal choice object in fuzzy soft sets. But Kong et al. [10] argued that the Roy's method was incorrect by using a counter example to discuss two evaluation bases

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of choice value and score value, and they proposed a revised algorithm. Later Feng et al. [5] applied level soft sets to discuss fuzzy soft sets based decision making and subsequently extended the approach to interval-valued fuzzy soft set based decision making [6]. Jiang et al. [8] generalize the approach to solve intuitionistic fuzzy soft sets. Based on Feng' works, Basu et al. [2] further investigated the previous methods to fuzzy soft sets in decision making and introduced the mean potentiality approach, which was showed more efficient and more accurate than the previous methods. Zhang [36] proposed a rough set approach to intuitionistic fuzzy soft set based decision making. Li et al. [15] investigated decision making based on intuitionistic fuzzy soft sets. Li et al. [16] considered fuzzy soft set based decision making for applications in medical diagnosis. Ma et al. [22] presented the algorithm to solve decision making problems based on interval-valued intuitionistic fuzzy soft sets. Qin et al. [24] present an adjustable approach to interval-valued intuitionistic fuzzy soft set based decision making by using reduct intuitionistic fuzzy soft sets and level soft sets of intuitionistic fuzzy soft sets.

All of the above methods for soft sets in decision making are mainly based on the level soft set to obtain useful information such as choice values and score values. However, the existing methods have their limitations. For example, it is very difficult for decision maker to select a suitable level soft set to reduce subjectivity and uncertainty (see [36]). Moreover, there has been rather little work completed for interval-valued intuitionistic fuzzy soft set based decision making. Then it is necessary to pay attention to this issue.

Grey relational analysis, initiated by Deng [4], is an important method to reflect uncertainty in grey system theory, which is utilized for generalizing estimates under small samples and uncertain conditions. It has been successfully applied in solving decision-making problems [9, 27, 28, 35]. D-S theory of evidence, proposed by Dempster [3] and Shafer [26], is a powerful method for combining accumulative evidence of changing prior opinions in the light of new evidences [26]. Compared to probability theory, this theory captures more information to support decision making by identifying the uncertain and unknown evidence. It provides a mechanism to derive solutions from various vague evidences without knowing much prior information. Therefore, combining both theories enables the decision makers to take advantage of both methods' merits and make evaluation experts to deal with uncertainty and risk confidently. The hybrid model is effective and practical under uncertainty [27, 29]. It is very meaningful to extend the hybrid model to interval-valued fuzzy soft set based decision making Thus, this not only allows us to avoid selecting a suitable level soft set, but also helps reducing humanistic and subjective in nature to raise the choices decision level.

The purpose of this paper is to investigate decision making based on the interval-valued intuitionistic fuzzy soft sets.

2 Preliminaries

Throughout this paper, U denotes an initial universe, E denotes the set of all possible parameters, 2^U denotes the family of all subsets of U. We only consider the case where U and E are both nonempty finite sets. Int$[0,1]$ denotes a set of all closed subintervals of $[0,1]$.

2.1 Interval-valued intuitionistic fuzzy soft sets

Definition 2.1 ([1]). *An interval-valued intuitionistic fuzzy set $\widetilde{X}$ over U is an object having the form $\widetilde{X} = \{(x, \mu_{\widetilde{X}}(x), \nu_{\widetilde{X}}(x)) \mid x \in U\}$ $(e \in A)$, where $\mu_{\widetilde{X}} : U \to Int[0,1]$ and $\nu_{\widetilde{X}} : U \to Int\ [0,1]$ satisfy $0 \leqslant sup\ \mu_{\widetilde{X}}(x) + sup\ \nu_{\widetilde{X}}(x) \leqslant 1$ for all $x \in U$.*

$\mu_{\widetilde{X}}(x)$ and $\nu_{\widetilde{X}}(x)$ are called the membership degree and non-membership degree of the element $x \in U$ to $\widetilde{X}$.

The set of all interval-valued intuitionistic fuzzy subsets of U is denoted by $IVIF(U)$.

Definition 2.2 ([18]). *Let $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to 2^U$.*

Definition 2.3 ([7]). *Let $A \subseteq E$. A pair (F, A) is called an interval-valued intuitionistic fuzzy soft set over U, where F is a mapping given by $F : A \to IVIF(U)$.*

In other words, an interval-valued intuitionistic fuzzy soft set over U is a parameterized family of interval-valued intuitionistic fuzzy subsets of U. For any $e \in A$, $F(e)$ is referred as the set of e-approximate elements of (F, A) and can be written as:

$$F(e) = \{(x, \mu_{F(e)}(x), \nu_{F(e)}(x)) \mid x \in U\}\ (e \in A),$$

where $\mu_{F(e)} : U \to Int[0,1]$ and $\nu_{F(e)} : U \to Int[0,1]$ satisfy $0 \leqslant sup\ \mu_{F(e)}(x) + sup\ \nu_{F(e)}(x) \leqslant 1$. $\mu_{F(e)}(x)$ and $\nu_{F(e)}(x)$ are called the membership degree and non-membership degree that x holds e, respectively. $\pi_{F(e)}(x) = 1 - \mu_{F(e)}(x) - \nu_{F(e)}(x)$ is called the hesitating degree of x holds e.

The set of all interval-valued intuitionistic fuzzy soft subsets of U is denoted by $IVIFS(U)$.

Example 2.4. *Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a set of houses and let $A = \{e_1, e_2, e_3, e_4,\} \subseteq E$ be a set of status of houses where e_j $(j = 1, 2, 3, 4)$ stand for the parameters "beautiful", "modern", "cheap" and "in the green surroundings", respectively.*

Now, we consider an interval-valued intuitionistic fuzzy soft set (F, A) over U, which describes "the attractiveness of the houses" to this decision maker and its tabular representation is shown in Table 1.

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Obviously, we can see that the precise evaluation for each object on each parameter is unknown while the lower and upper limits of such an evaluation are given. For example, we cannot present the precise membership degree and non-membership degree of how beautiful house h_1 is, however, house h_1 is at least beautiful on the membership degree of 0.6 *and it is at most beautiful on the membership degree of* 0.8*; house h_1 is not at least beautiful on the non-membership degree of 0.1 and it is not at most beautiful on the non-membership degree of 0.2.*

Table 1: Tabular representation of the interval-valued intuitionistic soft set (F, A)

	e_1	e_2	e_3	e_4
h_1	[0.6, 0.8],[0.1,0.2]	[0.7, 0.8],[0.15,0.2]	[0.75, 0.85],[0.1,0.15]	[0.8, 0.9],[0.01,0.1]
h_2	[0.8, 0.9],[0.05,0.1]	[0.6, 0.7],[0.15,0.21]	[0.5, 0.6],[0.2,0.35]	[0.65, 0.75],[0.2,0.25]
h_3	[0.6, 0.7],[0.2,0.25]	[0.5, 0.7],[0.2,0.3]	[0.6, 0.8],[0.1,0.18]	[0.66, 0.77],[0.2,0.22]
h_4	[0.65, 0.78],[0.15,0.21]	[0.7, 0.75],[0.15,0.25]	[0.68, 0.75],[0.1,0.2]	[0.69, 0.78],[0.1,0.2]

2.2 Basic concepts of D-S theory of evidence

D-S theory of evidence is a new important reasoning method under uncertainty. It has an advantage to deal with subjective judgments and to synthesize the uncertainty knowledge [34].

A frame of discernment, denoted Θ, is a finite nonempty set of mutually exclusive and exhaustive hypotheses, denoted $\{A_1, A_2, \cdots, A_n\}$ and $A_i \cap A_j = \emptyset$. 2^Θ denotes the set of all subsets of Θ.

Definition 2.5 ([26])**.** *Let Θ be a frame of discernment. A basic probability assignment function (or Mass function) on Θ is defined a mapping $m : 2^\Theta \to [0, 1]$, m satisfies*

$$m(\emptyset) = 0 \ , \ \sum_{A \subseteq \Theta} m(A) = 1 \ for \ A \in 2^\Theta.$$

For any $A \subseteq \Theta$, A is called as focal elements if $m(A) > 0$, $m(A)$ represents the belief measuser that one is willing to commit exactly to A, given a certain piece of evidence.

Definition 2.6 ([26])**.** *Let Θ be the frame of discernment and $m : 2^\Theta \to [0, 1]$ be a Mass function. Then a belief function on Θ is defined a mapping Bel : $2^\Theta \to [0, 1]$, Bel satisfies*

$$Bel(\emptyset) = 0, \ Bel(\Theta) = 1, \ Bel(A) = \sum_{B \subseteq A} m(B) \ for \ A \subseteq \Theta.$$

$Bel(A)$ can be interpreted as a global belief measure that the hypothesis A is true, and represents the imprecision and uncertainty in the decision-making process. In the case of single hypothesis, $Bel(A) = m(A)$.

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Definition 2.7 ([26]). *Let Θ be the frame of discernment. Suppose there are two Mass functions are m_1 and m_2 over Θ, induced by two independent items of evidences $A_1, A_2, \cdots, A_s$ and $B_1, B_2, \cdots, B_t$, respectively. D-S rule of evidence combination is defined and denoted as follows:*

$$m(A) = m_1 \oplus m_2(A) = \begin{cases} \frac{1}{1-K} \sum\limits_{A_i \cap B_j = A} m_1(A_i) m_2(B_j), \ \forall\ A \subseteq \Theta, A \neq \emptyset, \\ 0, \ A = \emptyset, \end{cases}$$

where $K = \sum\limits_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j) < 1$.

K is called the conflict probability and reflects the extent of the conflict between the evidences. Coefficient $\frac{1}{1-K}$ is called normalized factor, its role is to avoid the probability of assigning non-0 to empty set $\emptyset$ in the combination.

D-S rule of evidence combination can be generalized to multiple Mass functions, the belief measure resulting from the combination of multiply evidences A_i is as follows:

$$m_1 \oplus m_2 \cdots \oplus m_n(A) = \frac{1}{1-K} \sum_{\bigcap_{i=1}^n A_i = A, A_i \subset \Theta} m_1(A_1) m_2(A_2) \cdots m_n(A_n),$$

where $K = \sum\limits_{\bigcap_{i=1}^n A_i = \emptyset, A_i \subset \Theta} m_1(A_1) m_2(A_2) \cdots m_n(A_n) < 1$.

D-S rule of evidence combination can increase belief measure of hypotheses and reduce the uncertain degree to improve reliability.

Example 2.8. *Let $\Theta = \{A_1, A_2\}$ be the frame of discernment. Suppose there are two Mass functions m_1 and m_2 over Θ, induced by the independent items of evidences A_1, A_2, given by*

$m_1(A_1) = 0.3, \quad m_1(A_2) = 0.4, \quad m_1(\Theta) = 0.3,$

$m_2(A_1) = 0.4, \quad m_2(A_2) = 0.3, \quad m_2(\Theta) = 0.3.$

Combining the two evidences by D-S rule of evidence combination leads to:

$m(A_1) = m_1 \oplus m_2(A_1) = \frac{m_1(A_1)m_2(A_1) + m_1(A_1)m_2(\Theta) + m_1(\Theta)m_2(A_1)}{1-K} = 0.44,$

$m(A_2) = m_1 \oplus m_2(A_2) = \frac{m_1(A_2)m_2(A_2) + m_1(A_2)m_2(\Theta) + m_1(\Theta)m_2(A_2)}{1-K} = 0.44,$

$m(\Theta) = m_1 \oplus m_2(\Theta) = \frac{m_1(\Theta)m_2(\Theta)}{1-K} = 0.12,$

where $K = m_1(A_1)m_2(A_2) + m_1(A_2)m_2(A_1) = 0.25$.

3 An approach to interval-valued intuitionistic fuzzy soft sets in decision making

Recently, research on soft sets based decision making has attracted more and more attention. The works of Roy et al. [10, 25, 5, 2, 11] are fundamental and significant. Later other authors like Qin et al. further studied and proposed an adjustable approach to interval-valued intuitionistic fuzzy soft set based decision making using the level soft sets and reductions . Generally, there does not exist

any unique or uniform criterion for the evaluation of decision alternatives under uncertain condition. However, it is very difficult for decision makers to select suitable level soft sets and discuss reduct intuitionistic fuzzy soft sets.

Now we investigate interval-valued intuitionistic fuzzy soft sets based decision making by means of grey relational analysis and D-S theory of evidence. It is divided three phases: First, grey relational analysis is applied to calculate the grey mean relational degree and the uncertain degree of each parameter is obtained. Second, the corresponding Mass function with respect to each parameter is constructed by the uncertain degree of each parameter. Third, we apply D-S rule of evidence combination to aggregate individual alternatives into a collective alternative, by which the candidate alternatives are ranked and the best alternative is obtained.

In the following, we consider the decision making problem with m mutually exclusive alternatives x_i and n evaluation parameters (or indexes) e_j. d_{ij} denotes the degree that the alternative x_i satisfies the parameter e_j. Put

$$\Theta = \{x_1, x_2, \cdots, x_m\} \; and \; A = \{e_1, e_2, \cdots, e_n\}.$$

Define $F : A \to IVIF(\Theta)$ by $F(e_j) = \{(x_i, \mu_{F(e_j)}(x_i), \nu_{F(e_j)}(x_i)) \mid x_i \in \Theta\}$ $(e_j \in A)$ where $\mu_{F(e_j)} : U \to Int[0,1]$ and $\nu_{F(e_j)} : U \to Int[0,1]$ satisfy $0 \leqslant sup\ \mu_{F(e_j)}(x_i) + sup\ \nu_{F(e_j)}(x_i) \leqslant 1$. Then (F, A) is an interval-valued intuitionistic fuzzy soft set over Θ. Denote $\mu_{F(e_j)}(x_i) = [\mu_{ij}^-, \mu_{ij}^+]$, $\nu_{F(e_j)}(x_i) = [\nu_{ij}^-, \nu_{ij}^+]$, $a_{ij} = (\mu_{F(e_j)}(x_i), \nu_{F(e_j)}(x_i))$. $D = (a_{ij})_{m\times n}$ is called an interval-valued intuitionistic fuzzy soft decision matrix induced by (F, A). Here, we see the set of parameters as a item of evidences information.

The key to solve decision problems by using D-S theory of evidence is how to obtain the uncertain degree of evidences (or parameters).

First, inspired by Xu [12], we define the score function of as follows.

Definition 3.1. *Suppose that (F, A) is an interval-valued intuitionistic fuzzy soft over Θ. Suppose that $D = (a_{ij})_{m\times n}$ is an interval-valued intuitionistic fuzzy soft decision matrix induced by (F, A). Denote $\mu_{F(e_j)}(x_i) = [\mu_{ij}^-, \mu_{ij}^+]$, $\nu_{F(e_j)}(x_i) = [\nu_{ij}^-, \nu_{ij}^+]$, $a_{ij} = (\mu_{F(e_j)}(x_i), \nu_{F(e_j)}(x_i))$. Then score function of d_{ij} is defined and denoted as*

$$s(a_{ij}) = (\mu_{ij}^- + \mu_{ij}^+ - \nu_{ij}^- - \nu_{ij}^+)/2 + \alpha(\mu_{ij}^+ + \nu_{ij}^+ - \mu_{ij}^- - \nu_{ij}^-)/2.$$

By Definition 4.1, we can convert d_{ij} into real numbers. $s(a_{ij})$ presents the global degree that the alternative x_i holds the parameter e_j. Obviously, $0 \leqslant s(a_{ij}) \leqslant 1$. α is called a risk factor. For $\alpha = 0, > 0, < 0$, they imply the attitude of decision makers for risk is neutral, positive, oppose, respectively. Decision makers can select a α value according to their risk preference. In this paper, we pick $\alpha = 0$.

To obtain Mass functions of each alternative with respect to each parameter, we consider score function values may be negative, so we should normalize the

score function values by the following formula:

$$d_{ij} = \frac{s(a_{ij}) - min_{1\leqslant i\leqslant m}s(a_{ij})}{max_{1\leqslant i\leqslant m}s(a_{ij}) - min_{1\leqslant i\leqslant m}S(a_{ij})}, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

Hence, we can get normalized matrix of score function values $D = (d_{ij})_{m\times n}$.

Next, inspired by the paper [12], we define the grey mean relational degree and the uncertain degree of the parameter as follows.

Definition 3.2. *Let $\Theta = \{x_1, x_2, \cdots, x_m\}$, $A = \{e_1, e_2, \cdots, e_n\}$ and let (F, A) be an intuitionistic fuzzy soft set on Θ. Suppose that $D = (d_{ij})_{m\times n}$ is normalized matrix of score function values. For any i, j, denote*

$$\widetilde{d_i} = \frac{1}{n}\sum_{j=1}^{n} d_{ij}, \quad \triangle d_{ij} = |d_{ij} - \widetilde{d_i}|,$$

$$r_{ij} = \frac{min_{1\leqslant j\leqslant n}min_{1\leqslant i\leqslant m}\ \triangle d_{ij} + \rho\ max_{1\leqslant j\leqslant n}max_{1\leqslant i\leqslant m}\ \ \triangle d_{ij}}{\triangle d_{ij} + \rho\ max_{1\leqslant j\leqslant n}max_{1\leqslant i\leqslant m}\ \triangle d_{ij}}, \ \rho \in (0, 1),$$

$$DOI(e_j) = \frac{1}{m}(\sum_{i=1}^{m}(r_{ij})^q)^{\frac{1}{q}} \ (j = 1, 2, \cdots, n).$$

r_{ij} is called the grey mean relational degree between d_{ij} and $\widetilde{d_i}$. $DOI(e_j)$ is called q order uncertain degree of the parameter e_j.

ρ aims to expand or compress the range of the grey relational coefficient. Decision makers can select q, ρ values according to different circumstance. To obtain strong distinguishing effectiveness, we pick $q = 2$, $\rho = 0.5$ in this paper. We call $DOI(e_j)$ the uncertain degree of e_j for short.

It is worthy to notice that the method to obtain the uncertain degree varies from different situation in Definition 4.2. General speaking, since a index (or parameter) is specially more matching the mean of the index set than other indexes, it contains more satisfying information for decision making and the uncertain degree of the index information is lower. Then, in this paper we just consider grey mean relational degree between d_{ij} and $\widetilde{d_i}$.

Definition 3.3 ([36])**.** *Let $X = (x_1, x_2, \cdots, x_m)$ be a finite difference information sequence, where there exists $x_{i_k} \neq 0$ for $k = 1, 2, \cdots, m$ and $1 \leqslant i_k \leqslant m$. Then the information structure image sequence $Y = (y_1, y_2, \cdots, y_m)$ is given by $y_i = \frac{x_i}{\sum_{i=1}^{m} x_i}$.*

In the normalized matrix of score function values $D = (d_{ij})_{m\times n}$, the information structure image sequence with respect to a parameter e_j is denoted by $d_j = \{\widetilde{d_{1j}}, \widetilde{d_{2j}}, \widetilde{d_{3j}}, \cdots, \widetilde{d_{mj}}\}$, where $\widetilde{d_{ij}} = \frac{d_{ij}}{\sum_{i=1}^{m} d_{ij}}$. Then we obtain an information structure image matric $\widetilde{D} = (\widetilde{d_{ij}})_{m\times n}$ induced by d_j $(j = 1, 2, \cdots, n)$.

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D-S theory of evidence is a powerful method for combining accumulative evidence of changing prior opinions in the light of new evidences [26]. The primary procedure of combining the known evidences or information with other evidences is to construct suitable Mass functions of evidences.

Now, by the uncertain degree of each parameter, we can obtain Mass function of each alternative with respect to each parameter.

Theorem 3.4. *Let $\Theta = \{x_1, x_2, \cdots, x_m\}$, $A = \{e_1, e_2, \cdots, e_n\}$ and let (F, A) be an intuitionistic fuzzy soft set on Θ. Suppose that $D = (d_{ij})_{m\times n}$ is the normalized matrix of score function values and $DOI(e_j)$ is the uncertain degree of e_j. Denote $\widetilde{d_{ij}} = \frac{d_{ij}}{\sum\limits_{i=1}^{m} d_{ij}}$. For any i, j, we define functions $m_{e_j}(j = 1, 2, \cdots, n)$ with respect to the parameter e_j, it satisfies:*

$$m_{e_j}(x_i) = \widetilde{d_{ij}}\ (1 - DOI(e_j)), \quad m_{e_j}(\Theta) = 1 - \sum_{i=1}^{m} m_j(i).$$

Then $m_{e_j}(j = 1, 2, \cdots, n)$ are Mass functions.

In a normalized matrix of score function values $D = (d_{ij})_{m\times n}$, denote $m_{e_j}(x_i)$, $m_{e_j}(\Theta)$ by $m_j(i)$ and $m_j(m+1)$, respectively. $m_j(i)$ implies the belief measure that holds the alternative x_i with the parameter e_j and $m_j(m+1)$ implies the belief measure of the whole uncertainty with parameter e_j.

Next, using D-S rule of evidence combination to compose m_j $(j = 1, 2, \cdots, n)$, we get the belief measure of each alternative with all the parameters, by which the candidate alternatives are ranked and thus the best alternative is obtained.

4 Algorithm

4.1 Algorithm

Based on the above analysis, the detailed step-wise procedure as an algorithm is given as follows:

Input: An interval-value intuitionistic fuzzy soft set (F, A).

Output: The optimal decision-making results.

Step 1. Input an interval-value intuitionistic fuzzy soft set (F, A) and construct an interval-value intuitionistic fuzzy soft decision matrix induced by (F, A).

Step 2. Compute the normalized matrix of score function values ($D = (d_{ij})_{m\times n}$).

Step 3. Compute the mean of all the score function values ($\widetilde{d_i}$) with respect to each alternative.

Step 4. Compute the difference information between d_{ij} and $\widetilde{d_i}$.

Step 5. Compute the gray mean relational degree between d_{ij} and $\widetilde{d_i}$.

Step 6. Compute the uncertain degree $DOI(e_j)$ of each parameter e_j.

Step 7. Compute the information structure image sequence $\widetilde{d_{ij}}$ with respect to each parameter e_j by Definition 3.3.

Step 8. Compute Mass function values of the alternative x_i and Θ with respect to the parameter e_j by Theorem 3.4.

Step 9. Compute belief measure of each alternative x_i by combining these Mass functions $m_{e_j}(j = 1, 2, \cdots, n)$ respectively by Definition 2.8.

Step 10. The optimal decision is to select x_k if $c_k = max_i \{Bel(x_i)\}$. k has more than one value then any one of x_k may be optimal choices .

4.2 An illustrative example

Suppose that a fund manager in a wealth management wants to invest a company. Suppose that the set of four potential investment companies $U = \{x_1, x_2, x_3, x_4\}$ which are characterized by a set of parameters $A = \{e_1, e_2, e_3, e_4\}$. For $i = 1, 2, 3, 4$, the parameters e_i stand for "risk", "growth ,"socio-political issues" ,and "environmental impacts", respectively. The fund manager provide his/her assessment of each investment company on each parameter as an interval-valued intuitionistic fuzzy soft set (F, A). Its tabular representation is shown in Table 2.

Table 2: Tabular representation of the interval-valued intuitionistic soft set (F, A)

	e_1	e_2	e_3	e_4
x_1	[0.4, 0.5],[0.3,0.4]	[0.4, 0.6],[0.2,0.4]	[0.1, 0.3],[0.5,0.6]	[0.5, 0.7],[0.2,0.3]
x_2	[0.4, 0.5],[0.4,0.5]	[0.5, 0.8],[0.1,0.2]	[0.3, 0.6],[0.3,0.4]	[0.6, 0.7],[0.1,0.3]
x_3	[0.3, 0.5],[0.4,0.5]	[0.1, 0.3],[0.2,0.4]	[0.7, 0.8],[0.1,0.2]	[0.5, 0.7],[0.1,0.2]
x_4	[0.2, 0.4],[0.4,0.5]	[0.6, 0.7],[0.2,0.3]	[0.5, 0.6],[0.2,0.3]	[0.7, 0.8],[0.1,0.2]

Now, we suppose that the four mutually exclusive and exhaustive investment companies consist a frame of discernment, denoted $\Theta = \{x_1, x_2, x_3, x_4\}$. And we consider the set of parameters $A = \{e_1, e_2, e_3, e_4\}$ as a set of evidences.

Step 1. Construct an interval-valued intuitionistic fuzzy soft decision matrix induced by (F, A) as follows:

$$\begin{pmatrix} ([0.4,0.5],[0.3,0.4]) & ([0.4,0.6],[0.2,0.4]) & ([0.1,0.3],[0.5,0.6]) & ([0.5,0.7],[0.2,0.3]) \\ ([0.4,0.5],[0.4,0.5]) & ([0.5,0.8],[0.1,0.2]) & ([0.3,0.6],[0.3,0.4]) & ([0.6,0.7],[0.1,0.3]) \\ ([0.3,0.5],[0.4,0.5]) & ([0.1,0.3],[0.2,0.4]) & ([0.7,0.8],[0.1,0.2]) & ([0.5,0.7],[0.1,0.2]) \\ ([0.2,0.4],[0.4,0.5]) & ([0.6,0.7],[0.2,0.3]) & ([0.5,0.6],[0.2,0.3]) & ([0.7,0.8],[0.1,0.2]) \end{pmatrix}$$

Step 2. Compute the normalized matrix of score function values as follows:

$$D = (d_{ij})_{4\times 4} = \begin{pmatrix} 1.0000 & 0.5000 & 0 & 0 \\ 0.6000 & 1.0000 & 0.4737 & 0.4000 \\ 0.4000 & 0 & 1.0000 & 0.4000 \\ 0 & 0.8333 & 0.6842 & 1.0000 \end{pmatrix}$$

Step 3. Compute the mean of all parameters with respect to each investment company x_i as follows:

$$\widetilde{d_1} = 0.3750,\ \widetilde{d_2} = 0.6184,\ \widetilde{d_3} = 0.4500, \widetilde{d_4} = 0.6294$$

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Step 4. Compute the difference information between d_{ij} and $\widetilde{d_i}$, and construct the difference matrix as follows:

$$\triangle D = \begin{pmatrix} 0.6250 & 0.1250 & 0.3750 & 0.3750 \\ 0.0184 & 0.3816 & 0.1447 & 0.2184 \\ 0.0500 & 0.4500 & 0.5500 & 0.0500 \\ 0.6294 & 0.2039 & 0.0548 & 0.3706 \end{pmatrix}$$

Step 5. Compute the gray mean relational degree between d_{ij} and $\widetilde{d_i}$ based on $\triangle D$ as follows:

$$(r_{ij})_{4\times 4} = \begin{pmatrix} 0.3545 & 0.7576 & 0.4830 & 0.4830 \\ 1.0000 & 0.4784 & 0.7251 & 0.6248 \\ 0.9134 & 0.4356 & 0.3852 & 0.9134 \\ 0.3528 & 0.6423 & 0.9015 & 0.4861 \end{pmatrix}$$

Step 6. Compute the uncertain degree of each parameter e_j by Definition 3.2 as follows:

$$DOI(e_1) = 0.3609,\ \ DOI(e_2) = 0.2963,\ \ DOI(e_3) = 0.3279,\ \ DOI(e_4) = 0.3254.$$

Step 7. Compute the information structure image sequence with respect to each parameter and construct the matrix as follows:

$$\widetilde{D} = (\widetilde{d_{ij}})_{4\times 4} = \begin{pmatrix} 0.5000 & 0.2143 & 0 & 0 \\ 0.3000 & 0.4286 & 0.2195 & 0.2222 \\ 0.2000 & 0 & 0.4634 & 0.2222 \\ 0 & 0.3571 & 0.3171 & 0.5556 \end{pmatrix}$$

Step 8. Let $2^\Theta = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \Theta\}$. Compute Mass function values of x_i and Θ with respect to the parameter e_j by Theorem 3.4:

$$(m_j(i))_{4\times 4} = \begin{pmatrix} 0.3195 & 0.1508 & 0 & 0 \\ 0.1917 & 0.3016 & 0.1475 & 0.1499 \\ 0.1278 & 0 & 0.3115 & 0.1499 \\ 0 & 0.2513 & 0.2131 & 0.3748 \end{pmatrix}$$

and

$$m_1(5) = 0.3609, \quad m_2(5) = 0.2963, \quad m_3(5) = 0.3279, \quad m_4(5) = 0.3254,$$

$$\frac{1}{4}\sum_{j=1}^{4} m_j(5) = 0.3276.$$

Step 9. We combine these Mass functions and compute each belief measure of each candidate x_i respectively as follows:

$$Bel(\{x_1\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_1\}) = 0.1098,$$
$$Bel(\{x_2\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_2\}) = 0.3298,$$
$$Bel(\{x_3\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_3\}) = 0.1700,$$
$$Bel(\{x_4\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_4\}) = 0.3309,$$
$$Bel(\{x_5\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\Theta) = 0.0595.$$

Then the final rang order is $x_4 \succ x_2 \succ x_3 \succ x_1$.

Step 10. x_4 is the optimal investment company for $max_i \{Bel(x_i)\} = 0.3309$.

From the above results, the belief measure of the uncertainty with respect to the whole candidates Θ is declined from 0.3276 to 0.0595, after applying grey relational analysis to construct the corresponding Mass functions for different evidences and then using the rule of evidence combination to compose these information. This implies the above algorithm is effective and practical under uncertainties. It not only allows us to avoid selecting the suitable level soft set, but also helps reducing humanistic and subjective in nature to raise the choices decision level. Moreover, it broadens the application field of the grey system theory and D-S theory of evidence.

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PRODUCT-TYPE OPERATORS FROM WEIGHTED ZYGMUND SPACES TO BLOCH-ORLICZ SPACES

YONG YANG AND ZHI-JIE JIANG

ABSTRACT. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The boundedness and compactness of the product-type operators $D^n M_u C_\varphi$, $D^n C_\varphi M_u$, $M_u D^n C_\varphi$, $C_\varphi D^n M_u$, $M_u C_\varphi D^n$ and $C_\varphi M_u D^n$ from weighted Zygmund spaces to Bloch-Orlicz spaces are characterized by constructing some test functions in weighted Zygmund spaces.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. For $\alpha > 0$, the weighted Zygmund space $\mathcal{Z}^\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$b_{\mathcal{Z}^\alpha}(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha |f''(z)| < \infty.$$

It is a Banach space with the norm

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + b_{\mathcal{Z}^\alpha}(f).$$

If $\alpha = 1$, then it becomes the famous Zygmund space, usually denoted by $\mathcal{Z}$. For some results of weighted Zygmund spaces and some concrete operators on them, see, for example, [9, 22, 24, 43, 56] and the references therein.

Next we introduce the Bloch-Orlicz space which was defined by Ramos Fernández in [32]. Let Ψ be a Young's function, i.e., Ψ is a strictly increasing convex function on $[0, +\infty)$ such that $\Psi(0) = 0$ and $\lim_{t\to+\infty} \Psi(t) = +\infty$. The Bloch-Orlicz space $\mathcal{B}^\Psi$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)\Psi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. The Minkowski's functional

$$\|f\|_\Psi = \inf\left\{k > 0 : S_\Psi\left(\frac{f'}{k}\right) \le 1\right\}$$

defines a seminorm for $\mathcal{B}^\Psi$, where

$$S_\Psi(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)\Psi(|f(z)|).$$

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$\mathcal{B}^\Psi$ becomes a Banach space with the norm $\|f\|_{\mathcal{B}^\Psi} = |f(0)| + \|f\|_\Psi$. Ramos Fernández in [32] proved that it is isometrically equal to a special μ_Ψ-Bloch space, where

$$\mu_\Psi(z) = \frac{1}{\Psi^{-1}(\frac{1}{1-|z|^2})}, \;\; z \in \mathbb{D}.$$

Consequently, a equivalent norm on $\mathcal{B}^\Psi$ is given by $\|f\|_{\mathcal{B}^\Psi} = |f(0)| + b_{\mathcal{B}^\Psi}(f)$, where

$$b_{\mathcal{B}^\Psi}(f) = \sup_{z\in\mathbb{D}} \mu_\Psi(z)|f'(z)|.$$

Clearly, the quantity $b_{\mathcal{B}^\Psi}(f)$ is a seminorm on the space $\mathcal{B}^\Psi$ and a norm on the quotient space $\mathcal{B}^\Psi/\mathbb{P}_0$, where $\mathbb{P}_0$ is the set of all constant functions. The Bloch-Orlicz space generalizes some spaces. For example, if $\Psi(t) = t^p$ with $p > 0$, then $\mathcal{B}^\Psi$ coincides with the weighted Bloch space $\mathcal{B}^\alpha$, where $\alpha = 1/p$; if $\Psi(t) = t\log(1+t)$, then $\mathcal{B}^\Psi$ coincides with the Log-Bloch space (see [2]).

Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator $W_{\varphi,u}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,u}f(z) = u(z)f(\varphi(z)), \;\; z \in \mathbb{D}.$$

If $u \equiv 1$, it becomes the composition operator, usually denoted by C_φ. If $\varphi(z) = z$, it becomes the multiplication operator, usually denoted by M_u. Since $W_{\varphi,u} = M_uC_\varphi$, it is a product-type operator. For some studies on weighted composition operators, see, for example, [1, 4, 7, 10, 19, 22, 29, 42, 49, 50] and the references therein.

Let $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The nth differentiation operator D^n on $H(\mathbb{D})$ is defined by

$$D^nf(z) = f^{(n)}(z), \;\; z \in \mathbb{D},$$

where $f^{(0)} = f$. If $n = 1$, it is the well-known differentiation operator D. Zhu in [57] introduced the following, so-called, generalized weighted composition operator:

$$D^n_{\varphi,u}f(z) = u(z)f^{(n)}(\varphi(z)), \;\; z \in \mathbb{D}.$$

If $n = 0$, it becomes the weighted composition operator. Since $D^n_{\varphi,u} = M_uC_\varphi D^n$, it is also a product-type operator. For generalized weighted composition operators, see, for example, [3, 28, 47, 53, 54, 59, 60] and the references therein. Before the operator $D^n_{\varphi,u}$ some other product-type operators were introduced and studied. For example, the next product-type operators

$$M_uC_\varphi D,\; C_\varphi M_uD,\; M_uDC_\varphi,\; C_\varphi DM_u,\; DC_\varphi M_u,\; DM_uC_\varphi$$

were studied by Sharma in [34]. They were also studied on weighted Bergman spaces by Stević et al. in [51] and [52]. However, a normally systematic study of product-type operators started by Stević et al. since the publication of papers [21] and [25]. Before that there were a few papers in the topic, e.g., [8]. The publication of paper [21] first attracted some attention in product-type operators DC_φ and $C_\varphi D$ (see, e.g., [23, 30, 39, 41] and the references therein). The publication of paper [25] attracted some attention in product-type operators involving integral-type ones (see, e.g., [16, 26, 37, 43, 48] and the references therein). Recently there is a great interest in various product-type operators between two given spaces of holomorphic functions (see, e.g., [11, 12, 17, 31, 33, 36, 38, 40, 45, 57] and the references therein).

Before this paper some product-type operators from Zygmund spaces or weighted Zygmund spaces to some other spaces were studied, for example, in [3, 13, 14, 18, 27]. In this paper we consider the following product-type operators:

$$D^nM_uC_\varphi,\; D^nC_\varphi M_u,\; M_uD^nC_\varphi,\; C_\varphi D^nM_u,\; M_uC_\varphi D^n,\; C_\varphi M_uD^n. \tag{1}$$

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The boundedness and compactness of operators in (1) from Zygmund spaces to Bloch-Orlicz spaces were characterized in [14]. As a continuation and completeness of our work, we consider the same problems for operators in (1) from weighted Zygmund spaces with $\alpha \neq 1$ to Bloch-Orlicz spaces. Because these operators are more complicated than those above mentioned, we need seek some other test functions in weighted Zygmund spaces to achieve our objective.

Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is bounded if there exists a positive constant K such that $\|Lf\|_Y \leq K\|f\|_X$ for all $f \in X$. The operator $L : X \to Y$ is compact if it maps bounded sets into relatively compact sets. The norm of the operator $L : X \to Y$ is defined by

$$\|L\|_{X\to Y} = \sup_{\|f\|_X \leq 1} \|Lf\|_Y.$$

In this paper, the letter C denotes a positive constant which may differ from one occurrence to the other. The notation $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$. When $a \lesssim b$ and $b \lesssim a$, we write $a \asymp b$.

2. Preliminaries and test functions

We first state the following result which was essentially proved in [35] and [46].

Lemma 2.1. *For $\alpha > 0$ and $f \in \mathcal{Z}^\alpha$. Then*

(a) For $0 < \alpha < 1$, $|f(z)| \leq \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}}$ and $|f'(z)| \leq \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}}$.
(b) For $\alpha = 1$, $|f(z)| \leq \|f\|_{\mathcal{Z}}$ and $|f'(z)| \leq \|f\|_{\mathcal{Z}} \log \frac{e}{1-|z|^2}$.
(c) For $1 < \alpha < 2$, $|f(z)| \leq \frac{1}{(\alpha-1)(2-\alpha)}\|f\|_{\mathcal{Z}^\alpha}$ and $|f'(z)| \leq \frac{2}{\alpha-1}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-1}}$.
(d) For $\alpha = 2$, $|f(z)| \leq 2\|f\|_{\mathcal{Z}^2} \log \frac{e}{1-|z|^2}$ and $|f'(z)| \leq \frac{e}{1-|z|^2}\|f\|_{\mathcal{Z}^2}$.
(e) For $\alpha > 2$, $|f(z)| \leq \frac{1}{(\alpha-1)(\alpha-2)}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-2}}$ and $|f'(z)| \leq \frac{2}{\alpha-1}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-1}}$.

The following result directly follows from the corresponding result for the Bloch type spaces when a function f is replaced by f' (see, e.g., [55]).

Lemma 2.2. *For each $k \in \mathbb{N}$ and $k \geq 2$, there exists a positive constant C_k independent of $f \in \mathcal{Z}^\alpha$ and $z \in \mathbb{D}$ such that*

$$|f^{(k)}(z)| \leq \frac{C_k\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha+k-2}}.$$

Let $w \in \mathbb{D}$ and $i \in \mathbb{N}_0$. It is easily shown that the next function is in the space $\mathcal{Z}^\alpha$

$$r_{w,i}(z) = \frac{(1-|w|^2)^{2+i}}{(1-\overline{w}z)^{\alpha+i}},\ z \in \mathbb{D}.$$

The following result provides the needed test functions for the cases $0 < \alpha < 1$, $1 < \alpha < 2$, $\alpha = 2$ and $\alpha > 2$.

Lemma 2.3. *(a) If $0 < \alpha < 1$, then for each fixed $k \in \{2, 3, \ldots, n+1\}$, there exist constants $a_{0,k}$, $a_{1,k}$, $\ldots$, $a_{n+1,k}$ such that the function*

$$f_{w,k}(z) = \sum_{i=0}^{n+1} a_{i,k} r_{w,i}(z)$$

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satisfies

$$f_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\alpha+k-2}} \quad \textit{and} \quad f_{w,k}^{(j)}(w) = 0 \tag{2}$$

for each $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$.

(b) If $1 < \alpha \leq 2$*, then for each fixed* $k \in \{1, 2, \ldots, n+1\}$*, there exist constants* $b_{0,k}$*,* $b_{1,k}$*, ...,* $b_{n+1,k}$ *such that the function*

$$g_{w,k}(z) = \sum_{i=0}^{n+1} b_{i,k} r_{w,i}(z)$$

satisfies

$$g_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\alpha+k-2}} \quad \textit{and} \quad g_{w,k}^{(j)}(w) = 0 \tag{3}$$

for each $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$.

(c) If $\alpha > 2$*, then for each fixed* $k \in \{0, 1, \ldots, n+1\}$*, there exist constants* $c_{0,k}$*,* $c_{1,k}$*, ...,* $c_{n+1,k}$ *such that the function*

$$h_{w,k}(z) = \sum_{i=0}^{n+1} c_{i,k} r_{w,i}(z)$$

satisfies

$$h_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\alpha+k-2}} \quad \textit{and} \quad h_{w,k}^{(j)}(w) = 0 \tag{4}$$

for each $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$.

Proof. (a). From a calculation, it follows that (2) is equivalent to the following system

$$\begin{cases} \sum\limits_{i=0}^{n+1} (\alpha+i) a_{i,k} = 0 \\ \sum\limits_{i=0}^{n+1} (\alpha+i)(\alpha+i+1) a_{i,k} = 0 \\ \qquad \cdots\cdots\cdots \\ \sum\limits_{i=0}^{n+1} \prod\limits_{j=0}^{k-1} (\alpha+i+j) a_{i,k} = 1 \\ \qquad \cdots\cdots\cdots \\ \sum\limits_{i=0}^{n+1} \prod\limits_{j=0}^{n} (\alpha+i+j) a_{i,k} = 0. \end{cases} \tag{5}$$

Hence, we only need to prove that there exist constants $a_{0,k}$, $a_{1,k}$, ..., $a_{n+1,k}$ such that the system (5) holds. By Lemma 3 in [47], the determinant of the system (5) equals to $\prod_{j=1}^{n+1} j!$, which is different from zero. So there exist constants $a_{0,k}$, $a_{1,k}$, ..., $a_{n+1,k}$ such that the system (5) holds. Results (b) and (c) can be proved similarly, so we omit. □

Let $w \in \mathbb{D}$ and

$$q_w(z) = \Big(1 + \log^2 \frac{e}{1-\overline{w}z}\Big) \log^{-1} \frac{e}{1-|w|^2}.$$

Lemma 2.4. *For the function q_w, it follows that*

$$q_w^{(k)}(w) = c_k \frac{\overline{w}^k}{(1-|w|^2)^k} + d_k \frac{\overline{w}^k}{(1-|w|^2)^k} \log^{-1} \frac{e}{1-|w|^2}, \tag{6}$$

where $c_k > 0$ for each $k \geq 1$, $d_1 = 0$ and $d_k > 0$ for each $k \geq 2$.

Proof. By a direct computation, we have

$$q'_w(z) = 2\frac{\overline{w}}{1-\overline{w}z} \log \frac{e}{1-\overline{w}z} \log^{-1} \frac{e}{1-|w|^2}, \tag{7}$$

and

$$q''_w(z) = 2\frac{\overline{w}^2}{(1-\overline{w}z)^2} \log \frac{e}{1-\overline{w}z} \log^{-1} \frac{e}{1-|w|^2} + 2\frac{\overline{w}^2}{(1-\overline{w}z)^2} \log^{-1} \frac{e}{1-|w|^2}. \tag{8}$$

Also, from a direct computation, we see that for $k \geq 2$

$$\begin{aligned} q_w^{(k)}(z) = {} & 2(k-1)!\frac{\overline{w}^k}{(1-\overline{w}z)^k} \log \frac{e}{1-\overline{w}z} \log^{-1} \frac{e}{1-|w|^2} \\ & + \big[k-1+2(k-1)!\big] \frac{\overline{w}^k}{(1-\overline{w}z)^k} \log^{-1} \frac{e}{1-|w|^2}. \end{aligned} \tag{9}$$

Set $c_k = 2(k-1)!$, $d_1 = 0$ and $d_k = k-1+2(k-1)!$ for $k \geq 2$. Then (6) follows from (7)-(9). □

Remark 2.1. *Let X_w be the functions in Lemmas 2.3 and 2.4. Then*

$$\sup_{w \in \mathbb{D}} \|X_w\|_{\mathcal{Z}^\alpha} \lesssim 1, \tag{10}$$

and $X_w \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$. In fact, if X_w are the functions in Lemma 2.3, then this remark follows from the facts that $\sup_{w\in\mathbb{D}} \|r_{w,i}\|_{\mathcal{Z}^\alpha} \lesssim 1$ and $r_{w,i} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$; if X_w is the function in Lemma 2.4, then it follows from [44].

Stević in [47] used Faà di Bruno's formula of the following version

$$(f \circ \varphi)^{(n)}(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z)) B_{n,k}(\varphi'(z), \ldots, \varphi^{(n-k+1)}(z)), \tag{11}$$

where $B_{n,k}(x_1, ..., x_{n-k+1})$ is the Bell polynomial. See [15] for the Faà di Bruno's formula. For $n \in \mathbb{N}$ the sum can go from $k = 1$ since $B_{n,0}(\varphi'(z), ..., \varphi^{(n-k+1)}(z)) = 0$, however we will keep the summation since for $n = 0$ the only existing term $B_{0,0}$ is equal to 1. From (11) and the Leibniz formula the next lemma follows.

Lemma 2.5. *Let f, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then*

$$\big(u(z)f(\varphi(z))\big)^{(n+1)} = \sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big).$$

3. Boundedness the product-type operators

We first characterize the boundedness of the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$.

Theorem 3.1. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, C_{n+1}^j the binomial coefficient and $0 < \alpha < 1$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.
(b) The functions u and φ satisfy the following conditions:

$$I_0 := \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z), \dots, \varphi^{(j+1)}(z)\big)\Big| < \infty,$$

$$I_1 := \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=1}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,1}\big(\varphi'(z), \dots, \varphi^{(j)}(z)\big)\Big| < \infty,$$

and

$$I_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \infty$$

for each $k \in \{2, 3, \dots, n+1\}$.

Moreover, if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^n M_u C_\varphi\big\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0} \asymp \sum_{k=0}^{n+1} I_k.$$

Proof. $(a) \Rightarrow (b)$. Let $h_k(z) = z^k \in \mathcal{Z}$, $k = 0, 1, \dots, n+1$. Then applying the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ to the function h_0, we have

$$I_0 = \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}(\varphi'(z), \dots, \varphi^{(j+1)}(z))\Big| \le C\big\|D^n M_u C_\varphi\big\|. \quad (12)$$

By the fact $\|\varphi\|_\infty \le 1$, the boundedness of $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$, the triangle inequality and (12), we have

$$I_1 \le I_0 + C\|D^n M_u C_\varphi\|. \quad (13)$$

Assume now that we have proved the following inequalities

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=l}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,l}(\varphi'(z), \dots, \varphi^{(j-l+1)}(z))\Big| \le C\|D^n M_u C_\varphi\| \quad (14)$$

for each $l \in \{0, 1, ..., k-1\}$ and a $k \le n+1$. Applying Lemma 2.5 to the function h_k, and noticing that $h_k^{(s)}(z) \equiv 0$ for $s > k$, we get

$$\begin{aligned}\big(D^n M_u C_\varphi h_k\big)'(z) &= \sum_{j=0}^{k} h_k^{(j)}(\varphi(z)) \sum_{i=j}^{n+1} C_{n+1}^i u^{(n+1-i)}(z) B_{i,j}(\varphi'(z), \dots, \varphi^{(i-j+1)}(z)) \\ &= \sum_{j=0}^{k} k\cdots(k-j+1)(\varphi(z))^{k-j} \sum_{i=j}^{n+1} C_{n+1}^i u^{(n+1-i)}(z) B_{i,j}(\varphi'(z), \dots, \varphi^{(i-j+1)}(z)). \quad (15)\end{aligned}$$

From (15), the boundedness of function φ and the triangle inequality, by noticing that the coefficient at

$$\sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))$$

is independent of z and finally using hypothesis (14) we easily obtain

$$L_k := \sup_{z\in\mathbb{D}} \mu_\Psi(z) \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)) \Big| \le C\|D^n M_u C_\varphi\|. \tag{16}$$

By induction we see that (16) holds for each $k \in \{0, 1, \dots, n+1\}$.

For a fixed $w \in \mathbb{D}$ and a fixed $k \in \{2, 3, \dots, n+1\}$, by Lemma 2.3 (a) there exists a function

$$f_{w,k}(z) = \sum_{i=0}^{n+1} a_{i,k} r_{\varphi(w),i}(z)$$

such that

$$f_{w,k}^{(k)}(\varphi(w)) = \frac{\overline{\varphi(w)}^{k}}{(1-|\varphi(w)|^2)^{\alpha+k-2}} \quad \text{and} \quad f_{w,k}^{(j)}(\varphi(w)) = 0 \tag{17}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$, and

$$\sup_{w\in\mathbb{D}} \|f_{w,k}\|_{\mathcal{Z}^\alpha} \le C. \tag{18}$$

Then by (17), (18) and the boundedness of $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} I_k(w) &:= \frac{\mu_\Psi(w)|\varphi(w)|^k \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(w) B_{j,k}\big(\varphi'(w), \dots, \varphi^{(j-k+1)}(w)\big) \Big|}{(1-|\varphi(w)|^2)^{\alpha+k-2}} \\ &\le \|D^n M_u C_\varphi f_{w,k}\|_{\mathcal{B}^\Psi} \le C\|D^n M_u C_\varphi\|. \end{aligned} \tag{19}$$

From (19) we see that

$$\sup_{z\in\mathbb{D}} I_k(z) \le C\big\|D^n M_u C_\varphi\big\|,$$

which leads to

$$\sup_{|\varphi(z)|>1/2} \frac{\mu_\Psi(z) \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)) \Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} \le C\|D^n M_u C_\varphi\|. \tag{20}$$

On the other hand, by (16) we have

$$\sup_{|\varphi(z)|\le 1/2} \frac{\mu_\Psi(z) \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)) \Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} \le C\|D^n M_u C_\varphi\|. \tag{21}$$

Hence from (20) and (21) we obtain

$$I_k \le C\|D^n M_u C_\varphi\| < \infty. \tag{22}$$

$(b) \Rightarrow (a)$. By Lemmas 2.1, 2.2 and 2.5, for all $f \in \mathcal{Z}^\alpha$ we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \Big(\frac{1}{1-\alpha}(I_0+I_1) + \sum_{k=2}^{n+1} C_k I_k\Big)\|f\|_{\mathcal{Z}^\alpha}. \qquad (23)
\end{aligned}$$

It is clear that

$$|(D^n M_u C_\varphi f)(0)| \le C\|f\|_{\mathcal{Z}^\alpha}. \qquad (24)$$

Hence, from (23) and (24) it follows that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

Clearly, if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0$ is also bounded. By the definition of the norm in the quotient spaces, and using the same functions in the proofs of (12), (13) and (22), we obtain

$$I_k \le C\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0},$$

for each $k \in \{0,1,2,\dots,n+1\}$, and then

$$\sum_{k=0}^{n+1} I_k \le C\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0}. \qquad (25)$$

By (23) we have

$$\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0} \le C\sum_{k=0}^{n+1} I_k. \qquad (26)$$

The asymptotic expression of $\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0}$ follows from (25) and (26). □

Remark 3.1. *In fact, from the fact $z^k \in \mathcal{Z}^\alpha$, in the proof of Theorem 3.1 we have seen that if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then $L_k < \infty$ for all $\alpha > 0$.*

Theorem 3.2. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, C_{n+1}^j the binomial coefficient and $1 < \alpha < 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.
(b) The functions u and φ are such that $I_0 < \infty$ and for each $k \in \{1,2,\dots,n+1\}$

$$M_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \infty.$$

Moreover, if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^n M_u C_\varphi\big\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0} \asymp I_0 + \sum_{k=1}^{n+1} M_k.$$

Proof. $(a) \Rightarrow (b)$. Let $h_0(z) \equiv 1 \in \mathcal{Z}^\alpha$. Then $I_0 < \infty$. For a fixed $w \in \mathbb{D}$ and each fixed $k \in \{1, 2, \dots, n+1\}$, by Lemma 2.3 (b) there exists a function

$$g_{w,k}(z) = \sum_{i=0}^{n+1} b_{i,k} r_{\varphi(w),i}(z)$$

such that

$$g_{w,k}^{(k)}(\varphi(w)) = \frac{\overline{\varphi(w)}^k}{(1-|\varphi(w)|^2)^{\alpha+k-2}} \quad \text{and} \quad g_{w,k}^{(j)}(\varphi(w)) = 0 \tag{27}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$. Moreover,

$$\sup_{w\in\mathbb{D}} \|g_{w,k}\|_{\mathcal{Z}^\alpha} \le C. \tag{28}$$

Then from (27), (28) and the boundedness of $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} M_k(w) &:= \frac{\mu_\Psi(w)|\varphi(w)|^k \Big| \sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(w) B_{j,k}\big(\varphi'(w), \dots, \varphi^{(j-k+1)}(w)\big)\Big|}{(1-|\varphi(w)|^2)^{\alpha+k-2}} \\ &\le \|D^n M_u C_\varphi g_{\varphi(w),k}\|_{\mathcal{B}^\Psi} \le C\|D^n M_u C_\varphi\|. \end{aligned} \tag{29}$$

From (29) we see

$$\sup_{z\in\mathbb{D}} M_k(z) \le C\big\|D^n M_u C_\varphi\big\|, \tag{30}$$

and then

$$\sup_{|\varphi(z)|>1/2} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} \le C\|D^n M_u C_\varphi\|. \tag{31}$$

On the other hand, by using the fact $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$, we get

$$\sup_{|\varphi(z)|\le 1/2} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} \le C\|D^n M_u C_\varphi\|. \tag{32}$$

Hence from (31) and (32) we see that $M_k < \infty$ for each $k \in \{1, 2, ..., n+1\}$.

$(b) \Rightarrow (a)$. By Lemmas 2.1, 2.2 and 2.5, for all $f \in \mathcal{Z}^\alpha$ we have

$$\begin{aligned} &\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f)'(z)\big| \\ &= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big| \\ &\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big| \\ &\le \Big(\frac{I_0}{(\alpha-1)(2-\alpha)} + \frac{2M_1}{\alpha-1} + \sum_{k=2}^{n+1} C_k M_k\Big)\|f\|_{\mathcal{Z}^\alpha}. \end{aligned} \tag{33}$$

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It is clear that

$$|(D^n M_u C_\varphi f)(0)| \leq C\|f\|_{\mathcal{Z}^\alpha}. \tag{34}$$

Hence from (33) and (34) it follows that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. Similarly is obtained the asymptotic formula of $\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0}$, hence we omit. □

Theorem 3.3. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, C_{n+1}^j the binomial coefficient and $\alpha = 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded.

(b) The functions u and φ satisfy the following conditions:

$$R_0 := \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z), \dots, \varphi^{(j+1)}(z)\big)\Big| \log\frac{e}{1-|\varphi(z)|^2} < \infty.$$

and for each $k \in \{1, 2, \dots, n+1\}$

$$R_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^k} < \infty.$$

Moreover, if the operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^n M_u C_\varphi\big\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0} \asymp \sum_{k=0}^{n+1} R_k.$$

Proof. $(a) \Rightarrow (b)$. By using Lemma 2.3 (b), we can prove that $R_k < \infty$ for each $k \in \{1, 2, \dots, n+1\}$, so we do not give the proof again. For a fixed $w \in \mathbb{D}$, by Lemma 2.4 there exists a function

$$s_{\varphi(w)}(z) = p_{\varphi(w)}(z) + \sum_{i=0}^{n+1} d_i r_{\varphi(w),i}(z)$$

such that

$$s_{\varphi(w)}(\varphi(w)) = \log\frac{e}{1-|\varphi(w)|^2} \quad \text{and} \quad s_{\varphi(w)}^{(j)}(\varphi(w)) = 0 \tag{35}$$

for each $j \in \{1, 2, \dots, n+2\}$, moreover, $\sup_{w\in\mathbb{D}} \|s_{\varphi(w)}\|_{\mathcal{Z}^2} \leq C$. Then from these and the boundedness of $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} R_0(w) &:= \mu_\Psi(w)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(w) B_{j,0}\big(\varphi'(w), \dots, \varphi^{(j+1)}(w)\big)\Big| \log\frac{e}{1-|\varphi(w)|^2} \\ &\leq \|D^n M_u C_\varphi s_{\varphi(w)}\|_{\mathcal{B}^\Psi} \leq C\big\|D^n M_u C_\varphi\big\|. \end{aligned} \tag{36}$$

Then from (36) it follows that $R_0 < \infty$.

$(b) \Rightarrow (a)$. From Lemmas 2.1, 2.2 and 2.5, for all $f \in \mathcal{Z}^2$ we have

$$\begin{aligned} &\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f)'(z)\big| \\ &= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big| \end{aligned}$$

$$\leq \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f^{(k)}(\varphi(z))\big| \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big) \Big|$$

$$\leq \Big(2R_0 + eR_1 + \sum_{k=2}^{n+1} C_k R_k\Big) \|f\|_{\mathcal{Z}^2}. \tag{37}$$

It is clear that

$$|(D^n M_u C_\varphi f)(0)| \leq C\|f\|_{\mathcal{Z}^2}. \tag{38}$$

Hence from (37) and (38) it follows that the operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded. The asymptotic expression of $\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0}$ can be similarly obtained. □

Theorem 3.4. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $\alpha > 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

(b) The functions u and φ satisfy

$$S_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z) \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big) \Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \infty, \ k = 0, \dots, n+1.$$

Moreover, if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^n M_u C_\varphi\big\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0} \asymp \sum_{k=0}^{n+1} S_k.$$

Proof. Similarly to the proofs of Theorems 3.1-3.3, this result can be proved. □

Remark 3.2. *By using the similar methods and techniques, the boundedness of the operators $D^n C_\varphi M_u$, $C_\varphi D^n M_u$, $M_u D^n C_\varphi$, $M_u C_\varphi D^n$ and $C_\varphi M_u D^n$ from weighted Zygmund spaces to Bloch-Orlicz spaces can be characterized, so we omit.*

4. Compactness of the product-type operators

The first result is an alternative to Proposition 3.11 in [5], which characterizes the compactness in terms of sequential convergence. So the proof is omitted.

Lemma 4.1. *Let $T \in \{D^n M_u C_\varphi, D^n C_\varphi M_u, M_u D^n C_\varphi, C_\varphi D^n M_u, M_u C_\varphi D^n, C_\varphi M_u D^n\}$. Then the bounded operator $T : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact if and only if for every bounded sequence $\{f_j\}$ in $\mathcal{Z}^\alpha$ such that $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, it follows that* $\lim_{j\to\infty} \|Tf_j\|_{\mathcal{B}^\Psi} = 0$.

The following lemma was proved in [46].

Lemma 4.2. *(a) If $0 < \alpha < 2$ and $\{f_j\}$ is a bounded sequence in $\mathcal{Z}^\alpha$ which uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j \to \infty$, then*

$$\lim_{j\to\infty} \sup_{z\in\mathbb{D}} |f_j(z)| = 0.$$

(b) If $0 < \alpha < 1$ and $\{f_j\}$ is a bounded sequence in $\mathcal{Z}^\alpha$ which uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j \to \infty$, then

$$\lim_{j\to\infty} \sup_{z\in\mathbb{D}} |f_j'(z)| = 0.$$

Now we characterize the compactness of the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$.

Theorem 4.1. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $0 < \alpha < 1$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact.

(b) The functions u and φ satisfy $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$, and for each $k \in \{2, 3, \dots, n+1\}$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} = 0.$$

Proof. $(a) \Rightarrow (b)$. Suppose that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact. Clearly the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. By Remark 2.1, $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$. Consider a sequence $\{\varphi(z_i)\}$ in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1$ as $i \to \infty$. If such a sequence does not exist, then the last condition in (b) obviously holds. Without loss of generality, we may suppose that $|\varphi(z_i)| > 1/2$ for all $i \in \mathbb{N}$. For each fixed $k \in \{2, 3, \dots, n+1\}$, using this sequence we define the function sequence $f_{i,k}(z) = f_{\varphi(z_i),k}(z),\ i \in \mathbb{N}$. Then by Lemma 2.3 (a) we have that $\sup_{i\in\mathbb{N}} \|f_{i,k}\|_{\mathcal{Z}^\alpha} \le C$ and $f_{i,k} \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$, moreover

$$f_{i,k}^{(k)}(\varphi(z_i)) = \frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^{\alpha+k-2}} \quad \text{and} \quad f_{i,k}^{(j)}(\varphi(z_i)) = 0 \tag{39}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$. By Lemma 4.1 and (39), we have

$$\lim_{i\to\infty} \frac{\mu_\Psi(z_i)\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,k}(\varphi'(z_i), \dots, \varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^{\alpha+k-2}} = 0. \tag{40}$$

$(b) \Rightarrow (a)$. We first check that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. We observe that the last condition in (b) implies that for every $\varepsilon > 0$, there is an $\eta \in (0, 1)$ such that for all $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$ and for each $k \in \{2, 3, \dots, n+1\}$

$$\frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \varepsilon. \tag{41}$$

From the fact $L_k < \infty$ for each $k \in \{2, 3, \dots, n+1\}$, and (41), we have

$$I_k \le \varepsilon + \frac{L_k}{(1-\eta^2)^{\alpha+k-2}}. \tag{42}$$

From (42) and the fact $L_k < \infty$, it follows that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

To prove that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact, by Lemma 4.1 we just need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{Z}^\alpha$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{Z}^\alpha} \le M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

For such chosen ε and η, by using (39), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|\big(D^n M_u C_\varphi f_i\big)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z)\sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z),\ldots,\varphi^{(j+1)}(z)\big)\Big|\big|f_i(\varphi(z))\big| \\
&\quad + \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=1}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,1}\big(\varphi'(z),\ldots,\varphi^{(j)}(z)\big)\Big|\big|f_i'(\varphi(z))\big| \\
&\quad + \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z)\sum_{k=2}^{n+1}\big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le L_0 \sup_{z\in\mathbb{D}}\big|f_i(\varphi(z))\big| + L_1 \sup_{z\in\mathbb{D}}\big|f_i'(\varphi(z))\big| + \sum_{k=2}^{n+1} L_k \sup_{|z|\le\eta}\big|f_i^{(k)}(z)\big| + C\varepsilon. \qquad (43)
\end{aligned}$$

From (43), Lemma 4.2 and the fact $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ implies that for each $k \in \mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$, we finally get

$$\lim_{i\to\infty} \sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| = 0. \qquad (44)$$

It is clear that

$$\lim_{i\to\infty} \big|(D^n M_u C_\varphi f_i)(0)\big| = 0. \qquad (45)$$

From (44) and (45) we obtain

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

This shows that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact. □

Theorem 4.2. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $1 < \alpha < 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact.

(b) The functions u and φ are such that $L_k < \infty$ for each $k \in \{0, 1, \ldots, n+1\}$, and for each $k \in \{1, 2, \ldots, n+1\}$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)\big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} = 0.$$

Proof. $(a) \Rightarrow (b)$. Suppose that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact. Obviously the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. Then $L_k < \infty$ for each $k \in \{0, 1, \ldots, n+1\}$. Consider a sequence $\{\varphi(z_i)\}_{i\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1$ as $i \to \infty$. If such a sequence does not exist, then the last condition in (b) obviously holds. Without loss of generality,

we may suppose that $|\varphi(z_i)| > 1/2$ for all $i \in \mathbb{N}$. For each fixed $k \in \{1, 2, \ldots, n+1\}$, by using this sequence we define the function sequence $g_{i,k}(z) = g_{\varphi(z_i),k}(z)$, $i \in \mathbb{N}$. Then from Lemma 2.3 (b) we see that $\sup_{i\in\mathbb{N}} \|g_{i,k}\|_{\mathcal{Z}^\alpha} \le C$ and $g_{i,k} \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$, moreover

$$g_{i,k}^{(k)}(\varphi(z_i)) = \frac{\overline{\varphi(z_i)}^k}{(1 - |\varphi(z_i)|^2)^{\alpha+k-2}} \quad \text{and} \quad g_{i,k}^{(j)}(\varphi(z_i)) = 0 \tag{46}$$

for each $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$. From Lemma 4.1 and (46), for each fixed $k \in \{1, 2, \ldots, n+1\}$ we have

$$\lim_{i\to\infty} \frac{\mu_\Psi(z_i)\big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,k}(\varphi'(z_i), \ldots, \varphi^{(j-k+1)}(z_i))\big|}{(1 - |\varphi(z_i)|^2)^{\alpha+k-2}} = 0. \tag{47}$$

$(b) \Rightarrow (a)$. We first check that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. We observe that the last condition in (b) implies that for every $\varepsilon > 0$, there is an $\eta \in (0, 1)$ such that for all $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$ and for each $k \in \{1, 2, \ldots, n+1\}$

$$\frac{\mu_\Psi(z)\big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z))\big|}{(1 - |\varphi(z)|^2)^{\alpha+k-2}} < \varepsilon. \tag{48}$$

From the fact $L_k < \infty$ for each $k \in \{0, 1, \ldots, n+1\}$, and (48), we have

$$M_k \le \varepsilon + \frac{L_k}{(1 - \eta^2)^{\alpha+k-2}}. \tag{49}$$

From (49) and the fact $I_0 = L_0 < \infty$, it follows that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

In order to prove that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact, by Lemma 4.1 we just need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{Z}^\alpha$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{Z}^\alpha} \le M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then $\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0$. For such chosen ε and η, by using (46), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z), \ldots, \varphi^{(j+1)}(z)\big)\Big|\big|f_i(\varphi(z))\big| \\
&\quad + \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z) \sum_{k=1}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le L_0 \sup_{z\in\mathbb{D}} \big|f_i(\varphi(z))\big| + \sum_{k=1}^{n+1} L_k \sup_{|z|\le\eta} \big|f_i^{(k)}(z)\big| + C\varepsilon.
\end{aligned} \tag{50}$$

From (50), Lemma 4.2 and the fact $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ implies that for each $k \in \mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$, we

get

$$\lim_{i\to\infty}\sup_{z\in\mathbb{D}}\mu_\Psi(z)\big|(D^nM_uC_\varphi f_i)'(z)\big| = 0. \tag{51}$$

It is clear that

$$\lim_{i\to\infty}\big|(D^nM_uC_\varphi f_i)(0)\big| = 0. \tag{52}$$

From (51) and (52) we obtain

$$\lim_{i\to\infty}\|D^nM_uC_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

This shows that the operator $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact. $\square$

Theorem 4.3. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $\alpha = 2$. Then the following statements are equivalent.*

(a) The operator $D^nM_uC_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is compact.
(b) The functions u and φ are such that $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$,

$$\lim_{|\varphi(z)|\to 1}\mu_\Psi(z)\Big|\sum_{j=0}^{n+1}C_{n+1}^j u^{(n+1-j)}(z)B_{j,0}\big(\varphi'(z),\dots,\varphi^{(j+1)}(z)\big)\Big|\log\frac{e}{1-|\varphi(z)|^2} = 0,$$

and for each $k \in \{1, 2, \dots, n+1\}$

$$\lim_{|\varphi(z)|\to 1}\frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1}C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^k} = 0.$$

Proof. $(a) \Rightarrow (b)$. Suppose that the operator $D^nM_uC_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is compact. Clearly the operator $D^nM_uC_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded. Then $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$. Consider a sequence $\{\varphi(z_i)\}_{i\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1$ as $i \to \infty$. If such a sequence does not exist, then the last two conditions in (b) obviously hold. Without loss of generality, we may suppose that $|\varphi(z_i)| > 1/2$ for all $i \in \mathbb{N}$. For each fixed $k \in \{1, 2, \dots, n+1\}$, by using this sequence we define the function sequence $g_{i,k}(z) = g_{\varphi(z_i),k}(z)$, $i \in \mathbb{N}$. Then from Lemma 2.3 (b) we see that $\sup_{i\in\mathbb{N}}\|g_{i,k}\|_{\mathcal{Z}^2} \le C$ and $g_{i,k} \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$, moreover

$$g_{i,k}^{(k)}(\varphi(z_i)) = \frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^k} \quad\text{and}\quad g_{i,k}^{(j)}(\varphi(z_i)) = 0 \tag{53}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$. From Lemma 4.1 and (53), for each fixed $k \in \{1, 2, \dots, n+1\}$ we have

$$\lim_{i\to\infty}\frac{\mu_\Psi(z_i)\Big|\sum_{j=k}^{n+1}C_{n+1}^j u^{(n+1-j)}(z_i)B_{j,k}(\varphi'(z_i),\dots,\varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^k} = 0. \tag{54}$$

Now consider another function sequence $q_i(z) = q_{\varphi(z_i)}(z)$. Then by Lemma 2.4 we have

$$q_i^{(k)}(\varphi(z_i)) = c_k\frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^k} + d_k\frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^k}\log^{-1}\frac{e}{1-|\varphi(z_i)|^2}, \tag{55}$$

where $c_k > 0$ for each $k \geq 1$, $d_1 = 0$ and $d_k > 0$ for each $k \geq 2$. Moreover, $\sup_{i\in\mathbb{N}} \|q_i\|_{\mathcal{Z}^2} \leq C$, and $q_i \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$. From Lemma 4.1, we get

$$\lim_{i\to\infty} \|D^n M_u C_\varphi q_i\|_{\mathcal{B}^\Psi} = 0. \tag{56}$$

By (55) and the triangle inequality, we have

$$\begin{aligned}
&\mu_\Psi(z_i)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,0}\big(\varphi'(z_i), \dots, \varphi^{(j+1)}(z_i)\big)\Big|\Big(\log\frac{e}{1-|\varphi(z_i)|^2} + \log^{-1}\frac{e}{1-|\varphi(z_i)|^2}\Big)\\
&\leq \|D^n M_u C_\varphi q_i\|_{\mathcal{B}^\Psi} + \sum_{k=1}^{n+1}\frac{c_k\mu_\Psi(z_i)|\varphi(z_i)|^k\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,k}(\varphi'(z_i), \dots, \varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^k}\\
&+\sum_{k=1}^{n+1}\frac{d_k\mu_\Psi(z_i)|\varphi(z_i)|^k\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,k}(\varphi'(z_i), \dots, \varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^k}\log^{-1}\frac{e}{1-|\varphi(z_i)|^2}.
\end{aligned} \tag{57}$$

Therefore, taking the limit in (57) as $i \to \infty$, from (54), (56) and the fact

$$\log^{-1}\frac{e}{1-|\varphi(z_i)|^2} \to 0 \ \text{ as } \ i \to \infty,$$

we get

$$\lim_{i\to\infty} \mu_\Psi(z_i)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,0}\big(\varphi'(z), \dots, \varphi^{(j+1)}(z_i)\big)\Big|\log\frac{e}{1-|\varphi(z_i)|^2} = 0.$$

$(b) \Rightarrow (a)$. We first check that $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded. We observe that the conditions in (b) imply that for every $\varepsilon > 0$, there is an $\eta \in (0, 1)$, such that for any $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$

$$\mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z), \dots, \varphi^{(j+1)}(z)\big)\Big|\log\frac{e}{1-|\varphi(z)|^2} < \varepsilon \tag{58}$$

and

$$\frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^k} < \varepsilon \tag{59}$$

for each $k \in \{1, 2, \dots, n+1\}$. From the fact $L_0 < \infty$ and (58), we see

$$R_0 \leq \varepsilon + L_0 \log\frac{e}{1-\eta^2}. \tag{60}$$

From (59) and the fact $L_k < \infty$ for each $k \in \{1, 2, \dots, n+1\}$, we see

$$R_k \leq \varepsilon + \frac{L_k}{(1-\eta^2)^k}. \tag{61}$$

Then from (60), (61) and Theorem 3.3, it follows that $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded.

In order to prove that $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is compact, by Lemma 4.1 we just need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{Z}^2$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{Z}^2} \leq M$ and $f_i \to 0$ uniformly

on any compact subset of $\mathbb{D}$ as $i \to \infty$, then $\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0$. For such chosen ε and η, by using (58), (59), Lemma 2.1 and Lemma 2.2, we have

$$
\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big| \Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z),\ldots,\varphi^{(j+1)}(z)\big)\Big|\big|f_i(\varphi(z))\big| \\
&\quad + \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z) \sum_{k=1}^{n+1} \big|f_i^{(k)}(\varphi(z))\big| \Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sum_{k=0}^{n+1} L_k \sup_{|z|\le\eta} \big|f_i^{(k)}(z)\big| + C\varepsilon. \qquad (62)
\end{aligned}
$$

From (62) and the fact $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ implies that for each $k \in \mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$, we get

$$\lim_{i\to\infty} \sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| = 0. \qquad (63)$$

It is clear that

$$\lim_{i\to\infty} \big|(D^n M_u C_\varphi f_i)(0)\big| = 0. \qquad (64)$$

From (63) and (64) we obtain

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

Hence this shows that the operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is compact. □

Theorem 4.4. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $\alpha > 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact.

(b) The functions u and φ are such that $L_k < \infty$ and for each $k \in \{0, 1, \ldots, n+1\}$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} = 0.$$

Proof. Similarly to the proofs of Theorems 4.1-4.3, this result can be proved. □

Remark 4.1. *By using the similar methods and techniques, the compactness of the operators $D^n C_\varphi M_u$, $C_\varphi D^n M_u$, $M_u D^n C_\varphi$, $M_u C_\varphi D^n$ and $C_\varphi M_u D^n$ from weighted Zygmund spaces to Bloch-Orlicz spaces can be characterized, so we omit.*

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Union soft p-ideals and union soft sub-implicative ideals in BCI-algebras

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Abstract. The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notion of union soft p-ideals(sub-implicative ideals) are introduced, and related properties are investigated. Conditions for a union soft ideal to be a union soft p-ideal(sub-implicative ideal) are established. Characterizations of a union soft p-ideal(sub-implicative ideal) are considered, and a new union soft p-ideal(sub-implicative ideal) from an old one is constructed.

1. Introduction

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [16]. In response to this situation Zadeh [17] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [18]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [14]. Maji et al. [13] and Molodtsov [14] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [14] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [13] described the application of soft set theory to a decision making problem. Maji et

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al. [12] also studied several operations on the theory of soft sets. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun [9] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [1, 3, 6, 8, 10] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we discuss applications of the union soft sets in p-ideals of BCI-algebras. We introduce the notion of union soft p-ideals, and investigated related properties. We provide conditions for a union soft ideal to be a union soft p-ideal, and establish characterizations of a union soft p-ideal. We construct a new union soft p-ideal from an old one.

Secondly, we define the notion of union soft sub-implicative ideals, and investigated related properties. We provide conditions for a union soft ideal to be a union soft sub-implicative ideal, and study characterizations of a union soft sub-implicative ideal. We find a new union soft sub-implicative ideal from an old one.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)\ (x * x = 0)$,
(a4) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(a5) $(\forall x \in X)(0 * x = 0)$,

then X is called a BCK-algebra. A BCI-algebra X is said to be *p-semisimple* if $0 * (0 * x) = x$ for all $x \in X$. A BCI-algebra X is said to be *implicative* if $(x * (x * y)) * (y * x) = y * (y * x)$ for all $x, y \in X$.

In any BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X has the following properties:

(b1) $(\forall x \in X)\ (x * 0 = x)$,
(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(b3) $(\forall x, y \in X)\ (0 * (x * y) = (0 * x) * (0 * y))$,
(b4) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$.
(b5) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,
(b6) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$,
(b7) $(\forall x, y, z \in X)\ (0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))$,

(b8) $(\forall x, y \in X)$ $(0 * (0 * (x * y)) = (0 * y) * (0 * x))$.

A non-empty subset S of a BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset A of a BCI-algebra X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,
(c2) $(\forall x \in X)$ $(\forall y \in A)$ $(x * y \in A \Rightarrow x \in A)$.

Note that every ideal A of a BCI-algebra X satisfies:

$$(\forall x \in X)\,(\forall y \in A)\,(x \leq y \Rightarrow x \in A).$$

A non-empty subset A of a BCI-algebra X is called a *p-ideal* ([15]) of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)((x * z) * (y * z) \in A \text{ and } y \in A \Rightarrow x \in A)$.

Note that any p-ideal is an ideal, but the converse is not true in general.

Theorem 2.1. ([15]) *An ideal I of a BCI-algebra X is a p-ideal if and only if $0 * (0 * x) \in I$ implies $x \in I$ for any $x \in X$.*

For any elements x and y of a BCI-algebra X, $x^n * y$ denotes $x * (x * \cdots * (x * (x * y \cdots)$ in which x occurs n times. A non-empty subset A of a BCI-algebra X is called a *sub-implicative ideal* ([11]) of X if it satisfies (c1) and

(c4) $(\forall x, y, z \in X)((x^2 * y) * (y * x)) * z \in A \text{ and } z \in A \Rightarrow y^2 * x \in A)$.

Note that any sub-implicative ideal is an ideal, but the converse is not true in general.

Theorem 2.2. ([11]) *An ideal I of a BCI-algebra X is a sub-implicative ideal if and only if $(x^2 * y) * (y * x) \in I$ implies $y^2 * x \in I$ for any $x, y \in X$.*

We refer the reader to the book [7] for further information regarding BCI-algebras. A soft set theory is introduced by Molodtsov [14].

In what follows, let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe*. Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.3. ([14]) A soft set $\mathscr{F}_A$ over U is defined to be the set of ordered pairs

$$\mathscr{F}_A := \{(x, f_A(x)) : x \in E,\ f_A(x) \in \mathscr{P}(U)\},$$

where $f_A : E \to \mathscr{P}(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

The function f_A is called the approximate function of the soft set $\mathscr{F}_A$. The subscript A in the notation f_A indicates that f_A is the approximate function of $\mathscr{F}_A$.

In what follows, denote by $S(U)$ the set of all soft sets over U.

Definition 2.4. ([12]) For two soft sets $\mathscr{F}_A$ and $\mathscr{G}_B$ over a common universe U, we say that $\mathscr{F}_A$ is a *soft subset* of $\mathscr{G}_B$, denoted by $\mathscr{F}_A \tilde{\subset} \mathscr{G}_B$, if it satisfies:

(i) $A \subset B$,
(ii) For every $\epsilon \in A$, $\mathscr{F}(\epsilon)$ and $\mathscr{G}(\epsilon)$ are identical approximations.

Let $\mathscr{F}_A \in S(U)$ and let $\tau \subseteq U$. Then the *τ-exclusive set* of $\mathscr{F}_A$ is defined to be the set

$$e(\mathscr{F}_A;\tau) := \{x \in A \mid f_A(x) \subseteq \tau\}.$$

Obviously, we have the following properties:

(1) $e(\mathscr{F}_A;U) = A$,
(2) $f_A(x) = \cap\{\tau \subseteq U \mid x \in e(\mathscr{F}_A;\tau)\}$,
(3) $(\forall \tau_1, \tau_2 \subseteq U)\,(\tau_1 \subseteq \tau_2 \ \Rightarrow\ e(\mathscr{F}_A;\tau_1) \subseteq e(\mathscr{F}_A;\tau_2))$.

3. Union soft p-ideals

Definition 3.1. ([9]) Let $(U,E) = (U,X)$ where X is a BCI-algebra. Given a subalgebra A of E, we let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft deal* over U (briefly, *U-soft ideal*) if the approximate function f_A of $\mathscr{F}_A$ satisfies:

$$(\forall x \in A)\,(f_A(0) \subseteq f_A(x)), \tag{3.1}$$

$$(\forall x, y \in A)\,(f_A(x) \subseteq f_A(x * y) \cup f_A(y)). \tag{3.2}$$

Definition 3.2. Let $(U,E) = (U,X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft p-ideal* over U (briefly, *U-soft p-ideal*) if the approximate function f_A of $\mathscr{F}_A$ satisfies (3.1) and

$$(\forall x, y, z \in A)\,(f_A(x) \subseteq f_A((x * z) * (y * z)) \cup f_A(y)). \tag{3.3}$$

Example 3.3. Let $(U,E) = (U,X)$ where $X = \{0, 1, a, b, c\}$ is a BCI-algebra ([10]) with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let τ_1, τ_2 and τ_3 be subsets of U such that $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0,\tau_1), (1,\tau_2), (a,\tau_3), (b,\tau_3), (c,\tau_2)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft p-ideal over U.

Theorem 3.4. *Let $(U,E) = (U,X)$ where X is a BCI-algebra. Then every U-soft p-ideal is a U-soft ideal.*

Proof. Let $\mathscr{F}_A$ be a U-soft p-ideal over U where A is a subalgebra of E. Taking $z := 0$ in (3.3) and using (b1) we obtain

$$\begin{aligned} f_A(x) \subseteq & f_A((x * 0) * (y * 0)) \cup f_A(y) \\ = & f_A(x * y)) \cup f_A(y) \end{aligned}$$

for all $x, y \in A$. Therefore $\mathscr{F}_A$ is a U-soft ideal over U. □

The following example shows that the converse of Theorem 3.4 is not true.

Example 3.5. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3, 4\}$ is a BCI-algebra ([9]) with the following Cayley table:

$*$	0	1	2	a	b
0	0	0	0	a	b
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $\tau_1, \tau_2, \tau_3, \tau_4$ and τ_5 be subsets of U such that $\tau_1 \subsetneq \tau_3 \subsetneq \tau_4 \subsetneq \tau_5$ and $\tau_1 \subsetneq \tau_2 \subsetneq \tau_5$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (1, \tau_2), (2, \tau_3), (a, \tau_4), (b, \tau_5)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft ideal over U. But it is not a U-soft p-ideal over U, since

$$f_E(b) = \tau_5 \nsubseteq \tau_4 = \tau_1 \cup \tau_4 = f_E((b * b) * (a * b)) \cup f_E(a).$$

We provide some conditions for a U-soft ideal to be a U-soft p-ideal over U.

Theorem 3.6. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(1) *$\mathscr{F}_A$ is a U-soft p-ideal over U,*
(2) *$\mathscr{F}_A$ is a U-soft ideal over U and its approximate function f_A satisfies*

$$(\forall x, y, z \in A) \left(f_A(x * y) \subseteq f_A((x * z) * (y * z)) \right). \tag{3.4}$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. By Theorem 3.4, $\mathscr{F}_A$ is a U-soft deal over U. Using (a1) and (b2), we have $0 = ((x * z) * (x * y)) * (y * z) = ((x * z) * (y * z)) * (x * y)$ for any $x, y, z \in A$. Hence $((x * y) * (x * y)) * [((x * z) * (y * z)) * (x * y)] = 0 * 0 = 0$. It follows from (3.3) and (3.1) that

$$\begin{aligned} f_A(x * y) \subseteq & f_A((x * y) * (x * y)) * [((x * z) * (y * z)) * (x * y)]) \cup f_A((x * z) * (y * z)) \\ = & f_A(0) \cup f_A((x * z) * (y * z)) \\ = & f_A((x * z) * (y * z)). \end{aligned}$$

Hence (3.4) holds.

Conversely, suppose that $\mathscr{F}_A$ is a U-soft ideal over U satisfying (3.4). Using (3.2) and (3.4), we have $f_A(x) \subseteq f_A(x*y) \cup f_A(y) \subseteq f_A((x*z)*(y*z)) \cup f_A(y)$ for any $x, y, z \in A$. Hence $\mathscr{F}_A$ is a U-soft p-ideal over U. This completes the proof. □

Lemma 3.7. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. If $\mathscr{F}_A$ is a U-soft ideal over U, then the approximate function f_A of $\mathscr{F}_A$ satisfies the following condition:*

$$(\forall x \in A)(f_A(0*(0*x)) \subseteq f_A(x)).$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft ideal over U. Note that $0 = (0*x)*(0*x) = (0*(0*x))*x$. Using (3.2) and (3.1), we have

$$\begin{aligned} f_A(0*(0*x)) \subseteq & f_A((0*(0*x)*x)) \cup f_A(x) \\ = & f_A(0) \cup f_A(x) \\ = & f_A(x) \end{aligned}$$

for any $x \in A$. This completes the proof. □

Theorem 3.8. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is a U-soft p-ideal over U,*
(ii) *$\mathscr{F}_A$ is a U-soft ideal over U and its approximate function f_A satisfies*

$$(\forall x \in A)\,(f_A(x) \subseteq f_A(0*(0*x)))\,. \tag{3.5}$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. By Theorem 3.4, $\mathscr{F}_A$ is a U-soft deal over U. It follows from (3.3) and (3.1) that

$$\begin{aligned} f_A(x) \subseteq & f_A((x*x)*(0*x)) \cup f_A(0) \\ = & f_A(0*(0*x)) \end{aligned}$$

for any $x \in A$. Hence (3.5) holds.

Conversely, suppose that $\mathscr{F}_A$ is a U-soft ideal over U satisfying (3.5). By Lemma 3.7, we obtain $f_A(0*(0*((x*z)*(y*z)))) \subseteq f_A((x*z)*(y*z))$. It follows from (b7) and (b8) that $0*(0*(x*y)) = (0*y)*(0*x) = 0*(0*(x*z)*(y*z)))$. Using (3.5), we have $f_A(x*y) \subseteq f_A(0*(0*(x*y))) \subseteq f_A((x*z)*(y*z))$. By Theorem 3.6, $\mathscr{F}_A$ is a U-soft p-ideal over U. □

Lemma 3.9. ([9]) *Let $(U, E) = (U, X)$ where X is a BCI-algebra, Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is an U-soft ideal over U,*
(ii) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a ideal of A for any $\tau \subseteq U$.*

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Theorem 3.10. *Let $(U,E)=(U,X)$ where X is a BCI-algebra, Given a subalgebra A of E. let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is a U-soft p-ideal over U,*

(ii) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a p-ideal of A for any $\tau \subseteq U$.*

Proof. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. Then $\mathscr{F}_A$ is a U-soft ideal over U by Theorem 3.4. Hence $e(\mathscr{F}_A;\tau)$ is an ideal of A for all $\tau \subseteq U$ by Lemma 3.9. Let $\tau \subseteq U$ and let $x,y,z \in A$ be such that $(x*z)*(y*z) \in e(\mathscr{F}_A;\tau)$ and $y \in e(\mathscr{F}_A;\tau)$. Then $f_A((x*z)*(y*z)) \subseteq \tau$, $f_A(y) \subseteq \tau$, and so

$$f_A(x) \subseteq f_A((x*z)*(y*z)) \cup f_A(y) \subseteq \tau.$$

Hence $x \in e(\mathscr{F}_A;\tau)$. Thus $e(\mathscr{F}_A;\tau)$ is a p-ideal of A.

Conversely, suppose that the nonempty τ-exclusive set of $\mathscr{F}_A$ is a p-ideal of A for any $\tau \subseteq U$. Then $e(\mathscr{F}_A;\tau)$ is an ideal of A for all $\tau \subseteq U$. Hence $\mathscr{F}_A$ is a U-soft ideal over U by Lemma 3.9. Let $x \in A$ be such that $f_A(0*(0*x))=\tau$. Then $0*(0*x) \in e(\mathscr{F}_A;\tau)$, and so $x \in e(\mathscr{F}_A;\tau)$ by Theorem 2.1. Hence $f_A(x) \subseteq f_A(0*(0*x))$. It follows from Theorem 3.8 that $\mathscr{F}_A$ is a U-soft p-ideal over U. □

The p-ideals $e(\mathscr{F}_A;\tau)$ in Theorem 3.10 are called the *exclusive p-ideals* of $\mathscr{F}_A$.

Theorem 3.11. *Let $(U,E)=(U,X)$ and $\mathscr{F}_A \in S(U)$ where X is a BCI-algebra and A is a subalgebra of E. For a subset τ of U, define a soft set $\mathscr{F}_A^*$ over U by*

$$f_A^*: E \to \mathscr{P}(U),\ x \mapsto \begin{cases} f_A(x) & \text{if } x \in e(\mathscr{F}_A;\tau), \\ U & \text{otherwise.} \end{cases}$$

If $\mathscr{F}_A$ is a U-soft p-ideal over U, then so is $\mathscr{F}_A^*$.

Proof. If $\mathscr{F}_A$ is a U-soft p-ideal over U, then $e(\mathscr{F}_A;\tau)$ is a p-ideal of A for any $\tau \subseteq U$. Hence $0 \in e(\mathscr{F}_A;\tau)$, and so $f_A^*(0)=f_A(0) \subseteq f_A(x) \subseteq f_A^*(x)$ for all $x \in A$. Let $x,y,z \in A$. If $(x*z)*(y*z) \in e(\mathscr{F}_A;\tau)$ and $y \in e(\mathscr{F}_A;\tau)$, then $x \in e(\mathscr{F}_A;\tau)$ and so

$$\begin{aligned} f_A^*(x) =& f_A(x) \\ \subseteq& f_A((x*z)*(y*z)) \cup f_A(y) \\ =& f_A^*((x*z)*(y*z)) \cup f_A^*(y). \end{aligned}$$

If $(x*y)*(y*z) \notin e(\mathscr{F}_A;\tau)$ or $y \notin e(\mathscr{F}_A;\tau)$, then $f_A^*((x*z)*(y*z))=U$ or $f_A^*(y)=U$. Hence

$$f_A^*(x) \subseteq U = f_A^*((x*z)*(y*z)) \cup f_A^*(y).$$

This shows that $\mathscr{F}_A^*$ is a U-soft p-ideal over U. □

Theorem 3.12. *Let $(U,E)=(U,X)$ where X is a BCI-algebra. Then any p-ideal of E can be realized as an exclusive p-ideal of some U-soft p-ideal over U.*

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Proof. Let A be a p-ideal of E. For any subset $\tau \subsetneq U$, let $\mathscr{F}_A$ be a soft set over U defined by

$$f_A : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases}$$

Obviously, $f_A(0) \subseteq f_A(x)$ for all $x \in E$. For any $x, y, z \in E$, if $(x * z) * (y * z) \in A$ and $y \in A$ then $x \in A$. Hence

$$f_A((x * z) * (y * z)) \cup f_A(y) = \tau = f_A(x).$$

If $(x * z) * (y * z) \notin A$ or $y \notin A$ then $f_A((x * z) * (y * z)) = U$ or $f_A(y) = U$. It follows from (3.3) that

$$f_A(x) \subseteq U = f_A((x * z) * (y * z)) \cup f_A(y).$$

Therefore $\mathscr{F}_A$ is a U-soft p-ideal over U, and clearly $e(\mathscr{F}_A; \tau) = A$. This completes the proof. □

Example 3.13. Let $(U, E) = (U, X)$ where X is a BCI-algebra.
(1) $B(X) := \{x \in X | 0 * x = 0\}$. Then $B(X)$ is a p-ideal ([15]) of X. For any subset $\tau \subsetneq U$, let $\mathscr{F}_{B(X)}$ be a soft set over U defined by

$$f_{B(X)} : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in B(X), \\ U & \text{if } x \notin B(X). \end{cases}$$

Then $\mathscr{F}_{B(X)}$ is a U-soft p-ideal over U.
(2) $T_n(X) := \{x \in X | 0 * x^n = 0\}$, where $0 * x^n = (\cdots (0 * x) * \cdots) * x$ in which x appears n-times. Then $T_n(X)$ is a p-ideal ([15]) of X. For any subset $\tau \subsetneq U$, let $\mathscr{G}_{T_n(X)}$ be a soft set over U defined by

$$g_{T_n(X)} : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in T_n(X), \\ U & \text{if } x \notin T_n(X). \end{cases}$$

Then $\mathscr{G}_{T_n(X)}$ is a U-soft p-ideal over U.

Theorem 3.14. [Extension property] *Let $(U, E) = (U, X)$ where X is a p-semisimple BCI-algebra. Given subalgebras A and B of E, let $\mathscr{F}_A, \mathscr{F}_B \in S(U)$ such that*

(i) $\mathscr{F}_A \tilde{\subset} \mathscr{F}_B$,
(ii) $\mathscr{F}_B$ *a U-soft ideal over U.*

If $\mathscr{F}_A$ is a U-soft p-ideal over U, then so is $\mathscr{F}_B$.

Proof. Let $\tau \subseteq U$ be such that $e(\mathscr{F}_B; \tau) \neq \emptyset$. It follows from the condition (ii) and Lemma 3.9 that $e(\mathscr{F}_B; \tau)$ is an ideal. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. Then $e(\mathscr{F}_A; \tau)$ is a p-ideal for every $\tau \subseteq U$ by Theorem 3.10. Let $x \in E$ and $\tau \subseteq U$ be such that $0 * (0 * x) \in e(\mathscr{F}_B; \tau)$. Since X is a p-semisimple BCI-algebra, $0 * (0 * x) = x$. Hence $x \in e(\mathscr{F}_B; \tau)$. Thus $e(\mathscr{F}_B; \tau)$ is a p-ideal by Theorem 2.1. By Theorem 3.10, $\mathscr{F}_B$ is a U-soft p-ideal over U. □

4. Union soft sub-implicative ideals

Definition 4.1. Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft sub-implicative ideal* over U (briefly, *U-soft sub-implicative ideal*) if the approximate function f_A of $\mathscr{F}_A$ satisfies (3.1) and

$$(\forall x, y, z \in A)\left(f_A(y^2 * x) \subseteq f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z)\right). \tag{4.1}$$

Example 4.2. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2\}$ is a BCI-algebra ([11]) with the following Cayley table:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Let τ_1 and τ_2 be subsets of U such that $\tau_1 \subsetneq \tau_2$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (1, \tau_1), (2, \tau_2)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft sub-implicative ideal over U.

Theorem 4.3. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then every U-soft sub-implicative ideal is a U-soft ideal.*

Proof. Let $\mathscr{F}_A$ be a U-soft sub-implicative ideal over U where A is a subalgebra of E. Taking $y := x$ in (4.1) we obtain

$$\begin{aligned} f_A(x) &= f_A(x^2 * x) \\ &\subseteq f_A(((x^2 * x) * (x * x)) * z) \cup f_A(z) \\ &= f_A(x * z)) \cup f_A(z) \end{aligned}$$

for all $x, z \in A$. Therefore $\mathscr{F}_A$ is a U-soft ideal over U. □

The following example shows that the converse of Theorem 4.3 is not true.

Example 4.4. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3, 4\}$ is a BCI-algebra ([11]) with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	c
b	b	b	0	c
c	c	c	c	0

Let τ_1 and τ_2 be subsets of U such that $\tau_1 \subsetneq \tau_2$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (a, \tau_2), (b, \tau_2), (c, \tau_2)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft ideal over U. But it is not a U-soft sub-implicative ideal over U, since

$$f_E(a^2 * b) = f_E(a) = \tau_2 \nsubseteq \tau_1 = f_E(((b^2 * a) * (a * b)) * 0) \cup f_E(0).$$

Proposition 4.5. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. If $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, then the approximate function f_A of $\mathscr{F}_A$ satisfies the following condition:*

$$(\forall x, y \in A) \left(f_A(y^2 * x) \subseteq f_A((x^2 * y) * (y * x))\right). \tag{4.2}$$

.

Proof. Assume that $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U. For any $x, y \in A$, we have

$$\begin{aligned} f_A(y^2 * x) \subseteq & f_A(((x^2 * y) * (y * x)) * 0) \cup f_A(0) \\ = & f_A((x^2 * y) * (y * x)). \end{aligned}$$

This completes the proof. □

We provide conditions for a U-soft BCI-ideal to be a U-soft sub-implicative ideal over U.

Theorem 4.6. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. If $\mathscr{F}_A$ is a U-soft ideal over U satisfying the condition (4.2), then $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U.*

Proof. Assume that $\mathscr{F}_A$ is a U-soft ideal over U satisfying the condition (4.2). For any $x, y \in A$, we have

$$\begin{aligned} f_A(y^2 * x) \subseteq & f_A((x^2 * y) * (y * x)) \\ \subseteq & f_A((x^2 * y) * (y * x)) * z) \cup f_A(z) \end{aligned}$$

which proves the condition (4.1). This completes the proof. □

Corollary 4.7. *Let $(U, E) = (U, X)$ where X is an implicative BCI-algebra. Then every U-soft sub-implicative ideal is a U-soft ideal.*

Theorem 4.8. *Let $(U, E) = (U, X)$ where X is a p-semisimple BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. The notions of a U-soft ideal over U and a U-soft sub-implicative ideal over U coincide.*

Proof. Note that $x^2 * y = y$ for all $x, y \in X$, since X is a p-semsimple BCI-algebra. Assume that $\mathscr{F}_A$ is a U-soft ideal over U. For any $x, y, z \in A$, we have

$$\begin{aligned} f_A(y^2 * x) = & f_A(x) \\ \subseteq & f_A(x * z) \cup f_A(z) \\ = & f_A((y^2 * x) * z)) \cup f_A(z) \\ = & f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z). \end{aligned}$$

Therefore $\mathscr{F}_A$ ia a U-soft sub-implicative ideal over U. □

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Theorem 4.9. *Let $(U,E)=(U,X)$ where X is a BCI-algebra. Then every U-soft p-ideal is a U-soft sub-implicative ideal.*

Proof. Let $\mathscr{F}_A$ be a U-soft p-ideal over U, where A is a subalgebra of E. Then $\mathscr{F}_A$ is a U-soft ideal over U. Then $\mathscr{F}_A$ is a U-soft ideal over U by Theorem 3.4. Note that

$$\begin{aligned}(0^2*(y^2*x))*((x^2*y)*(y*x)) &= 0*((x^2*y)*(y*x))*(0*(y^2*x))\\ &=[(0*(x^2*y))*(0*(y*x))]*(0*(y^2*x))\\ &=[((0*x)*(0*(x*y)))*(0*(y*x))]*[(0*y)*(0*(y*x))]\\ &\le((0*x)*(0*(x*y)))*(0*y)\\ &=((0*x)*(0*y))*(0*(x*y))\\ &=0.\end{aligned}$$

For any $x,y\in A$, we have

$$\begin{aligned}f_A(y^2*x) &\subseteq f_A(0^2*(y^2*x))\\ &\subseteq f_A((0^2*(y^2*x))*((x^2*y)*(y*x)))\cup f_A((x^2*y)*(y*x))\\ &\subseteq f_A(0)\cup((x^2*y)*(y*x))\\ &=f_A((x^2*y)*(y*x)).\end{aligned}$$

It follows from Theorem 4.6 that $\mathscr{F}_A$ is a U-soft sub-implicaticve ideal over U. ☐

The converse of Theorem 4.9 may not be true in general as seen in the following example.

Example 4.10. Let $(U,E)=(U,X)$ where $X=\{0,a,1,2,3\}$ is a BCI-algebra with the following Cayley table:

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Let τ_1,τ_2 and τ_3 be subsets of U such that $\tau_1\subsetneq\tau_2\subsetneq\tau_3$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E=\{(0,\tau_1),(a,\tau_2),(1,\tau_3),(2,\tau_3),(3,\tau_3)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft sub-implicative ideal over U. But it is not a U-soft p-ideal over U, since

$$f_E(a)=\tau_2\nsubseteq\tau_1=f_E((a*1)*(0*1))\cup f_E(0).$$

Theorem 4.11. *Let $(U,E)=(U,X)$ where X is a BCI-algebra, Given a subalgebra A of E. let $\mathscr{F}_A\in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is a U-soft sub-implicative ideal over U,*
(ii) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a sub-implicative ideal of A for any $\tau\subseteq U$.*

Proof. Assume that $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U. Then $\mathscr{F}_A$ is a U-soft ideal over U by Theorem 4.3. Hence $e(\mathscr{F}_A;\tau)$ is an ideal of A for all $\tau \subseteq U$ by Lemma 3.9. Let $\tau \subseteq U$ and let $x, y, z \in A$ be such that $((x^2 * y) * (y * x)) * z \in e(\mathscr{F}_A;\tau)$ and $z \in e(\mathscr{F}_A;\tau)$. Then $f_A(((x^2 * y) * (y * x)) * z) \subseteq \tau$, $f_A(z) \subseteq \tau$, and so

$$f_A(y^2 * x) \subseteq f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z) \subseteq \tau.$$

Hence $y^2 * x \in e(\mathscr{F}_A;\tau)$. Thus $e(\mathscr{F}_A;\tau)$ is a sub-implicative ideal of A.

Conversely, suppose that the nonempty τ-exclusive set of $\mathscr{F}_A$ is a sub-implicative ideal of A for any $\tau \subseteq U$. Then $e(\mathscr{F}_A;\tau)$ is an ideal of A for all $\tau \subseteq U$. Hence $\mathscr{F}_A$ is a U-soft ideal over U by Lemma 3.9. Let $x, y \in A$ be such that $f_A((x^2 * y) * (y * x)) = \tau$. Then $(x^2 * y) * (y * x) \in e(\mathscr{F}_A;\tau)$, and so $y^2 * x \in e(\mathscr{F}_A;\tau)$ by Theorem 2.2. Hence $f_A(y^2 * x) \subseteq f_A((x^2 * y) * (y * x))$. It follows from Theorem 4.6 that $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U. □

The sub-implicative ideals $e(\mathscr{F}_A;\tau)$ in Theorem 4.11 are called the *exclusive sub-implicative ideals* of $\mathscr{F}_A$.

Theorem 4.12. *Let $(U, E) = (U, X)$ and $\mathscr{F}_A \in S(U)$ where X is a BCI-algebra and A is a subalgebra of E. For a subset τ of U, define a soft set $\mathscr{F}_A^*$ over U by*

$$f_A^* : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} f_A(x) & \text{if } x \in e(\mathscr{F}_A;\tau), \\ U & \text{otherwise.} \end{cases}$$

If $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, then so is $\mathscr{F}_A^*$.

Proof. If $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, then $e(\mathscr{F}_A;\tau)$ is a sub-implicative ideal of A for any $\tau \subseteq U$. Hence $0 \in e(\mathscr{F}_A;\tau)$, and so $f_A^*(0) = f_A(0) \subseteq f_A(x) \subseteq f_A^*(x)$ for all $x \in A$. Let $x, y, z \in A$. If $((x^2 * y) * (y * x) * z) \in e(\mathscr{F}_A;\tau)$ and $z \in e(\mathscr{F}_A;\tau)$, then $y^2 * x \in e(\mathscr{F}_A;\tau)$ and so

$$\begin{aligned} f_A^*(y^2 * x) =& f_A(y^2 * x) \\ \subseteq& f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z) \\ =& f_A^*(((x^2 * y) * (y * x)) * z) \cup f_A^*(z). \end{aligned}$$

If $((x^2 * y) * (y * x)) * z \notin e(\mathscr{F}_A;\tau)$ or $z \notin e(\mathscr{F}_A;\tau)$, then $f_A^*((x^2 * y) * (y * x)) * z)) = U$ or $f_A^*(z) = U$. Hence

$$f_A^*(x) \subseteq U = f_A^*(((x^2 * y) * (y * x)) * z) \cup f_A^*(z).$$

This shows that $\mathscr{F}_A^*$ is a U-soft sub-implicative ideal over U. □

Theorem 4.13. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then any sub-implicative ideal of E can be realized as an exclusive sub-implicative ideal of some U-soft sub-implicative ideal over U.*

Proof. Let A be a sub-implicative ideal of E. For any subset $\tau \subsetneq U$, let $\mathscr{F}_A$ be a soft set over U defined by

$$f_A : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases}$$

Obviously, $f_A(0) \subseteq f_A(x)$ for all $x \in E$. For any $x, y, z \in E$, if $((x^2 * y) * (y * x)) * z \in A$ and $z \in A$, then $y^2 * x \in A$. Hence $f_A(y^2 * x) = \tau = f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z)$. If $((x^2 * y) * (y * x)) * z \notin A$ or $z \notin A$, then $f_A(((x^2 * y) * (y * x)) * z) = U$ or $f_A(z) = U$. It follows from (4.1) that

$$f_A(y^2 * x) \subseteq U = f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z).$$

Therefore $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, and clearly $e(\mathscr{F}_A; \tau) = A$. This completes the proof. □

Example 4.14. Let $(U, E) = (U, X)$ where X is a BCI-algebra and let $B(X) := \{x \in X | 0 * x = 0\}$. Then $B(X)$ is a sub-implicative ideal ([11]) of X. For any subset $\tau \subsetneq U$, let $\mathscr{F}_{B(X)}$ be a soft set over U defined by

$$f_{B(X)} : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in B(X), \\ U & \text{if } x \notin B(X). \end{cases}$$

Then it is easy to see that $\mathscr{F}_{B(X)}$ is a U-soft sub-implicative ideal over U.

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On interval-valued fuzzy rough approximation operators *

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Abstract: Rough approximation operators based on approximation spaces are a key concept of rough set theory. This paper investigates rough approximation operators in interval-valued fuzzy (for short, IVF) environment by using constructive and axiomatic approaches. Moreover, IVF pseudo-closure operators are considered.

Keywords: IVF set; IVF relation; IVF approximate space; IVF rough set; IVF rough approximation operators.

1 Introduction

Rough set theory was proposed by Pawlak [16] as a mathematical tool for data reasoning. It may be seen as an extension of classical set theory, has been proved to be an effective approach to deal with intelligent systems characterized by insufficient and incomplete information, and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [17, 18, 19, 20]. The foundation of its object classification is an equivalence relation. The upper and lower approximation operations are two core notions of this theory. They can also be seen as the closure operator and the interior operator of the topology induced by an equivalence relation on the universe, respectively. In the real world, the equivalence

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relation is, however, too restrictive for many practical applications. To address this issue, many interesting and meaningful extensions of Pawlak's rough sets have been presented in the literature. Equivalence relations can be replaced by tolerance relations [23], similarity relations [24], binary relations [7, 27].

Various fuzzy generalizations of rough approximations have been proposed in the literature [1, 2, 6, 10, 11, 15, 21, 26, 29]. The most common fuzzy rough set is obtained by replacing the crisp binary relations with fuzzy relations on the universe and the crisp subsets with the fuzzy sets.

There are mainly two approaches to the development of rough set theory. One is the constructive approach in which rough approximation operators are constructed by means of relations, partitions, coverings, neighborhood systems and so on. The constructive approach is suitable for practical applications of rough sets. The other one is the axiomatic approach. In this approach, a set of axioms is used to characterize rough approximation operators that guarantee the existence of certain types of relations which produce the same operators. This approach is appropriate for studying algebra structures of rough sets. Under this point of view, rough set theory may be interpreted as an extension of set theory with two additional unary operators.

As a generalization of Zadeh's fuzzy set, interval-valued fuzzy (IVF, for short) sets were introduced by Gorzalczany [4] and Turksen [25], and they were applied to the fields of approximate inference, signal transmission and controller, etc. Mondal et al. [14] defined topology of IVF sets and studied their properties.

By integrating Pawlak rough set theory with IVF set theory, Sun et al. [22] introduced IVF rough sets based on an IVF approximation space, defined IVF information systems and discussed their attribute reduction. Gong et al. [5] studied the knowledge discovery in IVF information systems. Zhang et al. [30] discussed $(\mathcal{I},\mathcal{T})$-IVF rough sets based on an IVF approximation space on two universes of discourse.

The purpose of this paper is to investigate IVF rough approximation operators by using constructive and axiomatic approaches.

2 Preliminaries

Throughout this paper, " interval-valued fuzzy " denote briefly by " IVF ". U denotes a nonempty finite set called the universe of discourse. I denotes $[0,1]$ and $[I]$ denotes $\{[a,b] : a,b \in I \ and \ a \leq b\}$. $\mathscr{P}(U)$ denotes the family of all subsets of U. $F^{(i)}(U)$ denotes the family of all IVF sets in U. $\bar{a}$ denotes $[a,a]$ for each $a \in [0,1]$.

2.1 IVF sets

For any $[a_j,b_j] \in [I](\ j=1,2)$, we define

$$[a_1,b_1]=[a_2,b_2] \iff a_1=a_2, b_1=b_2;$$

$$[a_1,b_1]\leq[a_2,b_2] \iff a_1\leq a_2, b_1\leq b_2;$$

$$[a_1, b_1] < [a_2, b_2] \iff [a_1, b_1] \leq [a_2, b_2] \text{ and } [a_1, b_1] \neq [a_2, b_2];$$

$$\bar{1} - [a_1, b_1] \text{ or } [a_1, b_1]^c = [1 - b_1, 1 - a_1].$$

Obviously, $([a, b]^c)^c = [a, b]$ for each $[a, b] \in [I]$.

Definition 2.1 ([4, 25])**.** *For each* $\{[a_j, b_j] : j \in J\} \subseteq [I]$, *we define*

$$\bigvee_{j\in J}[a_j, b_j] = [\bigvee_{j\in J} a_j, \bigvee_{j\in J} b_j] \text{ and } \bigwedge_{j\in J}[a_j, b_j] = [\bigwedge_{j\in J} a_j, \bigwedge_{j\in J} b_j],$$

where $\bigvee\limits_{j\in J} a_j = sup\ \{a_j : j \in J\}$ *and* $\bigwedge\limits_{j\in J} a_j = inf\ \{a_j : j \in J\}$.

Definition 2.2 ([4, 25])**.** *An IVF set A in U is defined by a mapping $A : U \to [I]$. Denote*

$$A(x) = [A^-(x), A^+(x)] \quad (x \in U).$$

Then $A^-(x)$ (resp. $A^+(x)$) is called the lower (resp. upper) degree to which x belongs to A. A^- (resp. A^+) is called the lower (resp. upper) IVF set of A.

The set of all IVF sets in U is denoted by $F^{(i)}(U)$.

Let $a, b \in I$. $\widetilde{[a, b]}$ represents the IVF set which satisfies $\widetilde{[a, b]}(x) = [a, b]$ for each $x \in U$. We denoted $\widetilde{[a, a]}$ by $\tilde{a}$.

We recall some basic operations on $F^{(i)}(U)$ as follows ([4, 25]): for any $A, B \in F^{(i)}(U)$ and $[a, b] \in [I]$,

(1) $A = B \iff A(x) = B(x)$ for each $x \in U$.
(2) $A \subseteq B \iff A(x) \leq B(x)$ for each $x \in U$.
(3) $A = B^c \iff A(x) = B(x)^c$ for each $x \in U$.
(4) $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in U$.
(5) $(A \cup B)(x) = A(x) \vee B(x)$ for each $x \in U$.

Moreover,

$$(\bigcup_{j\in J} A)(x) = \bigvee_{j\in J} A(x) \text{ and } (\bigcap_{j\in J} A)(x) = \bigwedge_{j\in J} A(x),$$

where $\{A_j : j \in J\} \subseteq F^{(i)}(U)$.

(6) $([a, b]A)(x) = [a, b] \wedge [A^-(x), A^+(x)]$ for each $x \in U$.

Obviously,

$$A = B \iff A^- = B^- \text{ and } A^+ = B^+ \ ; \ (\widetilde{[a, b]})^c = \widetilde{[a, b]^c} \quad ([a, b] \in [I]).$$

Definition 2.3 ([14])**.** *$A \in F^{(i)}(U)$ is called an IVF point in U, if there exist $[a, b] \in [I] - \{\bar{0}\}$ and $x \in U$ such that*

$$A(y) = \begin{cases} [a, b], & y = x, \\ \bar{0}, & y \neq x. \end{cases}$$

We denote A by $x_{[a,b]}$.

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If $[a,b]=\bar{1}$, then

$$x_{\bar{1}}(y)=\begin{cases}\bar{1}, & y=x,\\ \bar{0}, & y\neq x.\end{cases}$$

Remark 2.4. $A=\bigcup_{x\in U}(A(x)x_{\bar{1}})$.

2.2 Definition of IVF rough approximation operators

Recall that R is called an IVF relation on U if $R\in F^{(i)}(U\times U)$.

Definition 2.5 ([7, 22])**.** *Let R be an IVF relation on U. Then R is called*
(1) *reflexive, if $R(x,x)=\bar{1}$ for each $x\in U$.*
(2) *transitive, if $R(x,z)\geq R(x,y)\wedge R(y,z)$ for any $x,y,z\in U$.*
(3) *preorder, if R is reflexive and transitive.*

Definition 2.6 ([22])**.** *Let R be an IVF relation on U. The pair (U,R) is called an IVF approximation space. For each $A\in F^{(i)}(U)$, the IVF lower and the IVF upper approximation of A with respect to (U,R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two IVF sets and are respectively defined as follows:*

$$\underline{R}(A)(x)=\bigwedge_{y\in U}(A(y)\vee(\bar{1}-R(x,y)))\ (x\in U)$$

and

$$\overline{R}(A)(x)=\bigvee_{y\in U}(A(y)\wedge R(x,y))\ (x\in U).$$

The the pair $(\underline{R}(A),\overline{R}(A))$ is called the IVF rough set of A with respect to (U,R).

$\underline{R}:F^{(i)}(U)\to F^{(i)}(U)$ and $\overline{R}:F^{(i)}(U)\to F^{(i)}(U)$ are called the IVF lower approximation operator and the IVF upper approximation operator, respectively. In general, we refer to $\underline{R}$ and $\overline{R}$ as the IVF rough approximation operators.

Remark 2.7. *Let (U,R) be an IVF approximation space. Then*
(1) *for each $x,y\in U$,*

$$\overline{R}(x_{\bar{1}})(y)=R(y,x)\quad and\quad \underline{R}((x_{\bar{1}})^c)(y)=\bar{1}-R(y,x).$$

(2) *for each $[a,b]\in[I]$, $\underline{R}(\widetilde{[a,b]})\supseteq\widetilde{[a,b]}\supseteq\overline{R}(\widetilde{[a,b]})$.*

Proposition 2.8 ([22])**.** *Let (U,R) be an IVF approximation space. Then for each $A\in F^{(i)}(U)$,*

$$(\underline{R}(A))^-=\underline{R^+}(A^-),\ (\underline{R}(A))^+=\underline{R^-}(A^+),$$

$$(\overline{R}(A))^-=\overline{R^-}(A^-)\quad and\quad (\overline{R}(A))^+=\overline{R^+}(A^+).$$

3 IVF rough approximation operators

In this section, we deeply investigate IVF rough approximation operators.

3.1 Construction of IVF rough approximation operators

Theorem 3.1 ([28]). *Let (U, R) be an IVF approximation space. Then for any $A, B \in F^{(i)}(U)$, $\{A_j : j \in J\} \subseteq F^{(i)}(U)$ and $[a, b] \in [I]$,*

(1) $\underline{R}(\tilde{1}) = \tilde{1}$, $\overline{R}(\tilde{0}) = \tilde{0}$.

(2) $A \subseteq B \Longrightarrow \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$.

(3) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$.

(4) $\underline{R}(\bigcap\limits_{j\in J} A_j) = \bigcap\limits_{j\in J} \underline{R}(A_j)$, $\overline{R}(\bigcup\limits_{j\in J} A_j) = \bigcup\limits_{j\in J} \overline{R}(A_j)$.

(5) $\underline{R}(\widetilde{[a,b]} \cup A) = \widetilde{[a,b]} \cup \underline{R}(A)$, $\overline{R}([a,b]A) = [a,b]\overline{R}(A)$.

Theorem 3.2 ([28]). *Let (U, R) be an IVF approximation space. Then*

$$\begin{array}{lcl}
(1)\ R\ is\ reflexive & \Longleftrightarrow & (ALR)\ \forall A \in F^{(i)}(U), \underline{R}(A) \subseteq A. \\
 & \Longleftrightarrow & (AUR)\ \forall A \in F^{(i)}(U), A \subseteq \overline{R}(A). \\
(2)\ R\ is\ transitive & \Longleftrightarrow & (ALT)\ \forall A \in F^{(i)}(U), \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)). \\
 & \Longleftrightarrow & (AUT)\ \forall A \in F^{(i)}(U), \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A).
\end{array}$$

Corollary 3.3 ([28]). *Let (U, R) be an IVF approximation space. If R is pre-order, then*

$$\underline{R}(\underline{R}(A)) = \underline{R}(A) \quad and \quad \overline{R}(\overline{R}(A)) = \overline{R}(A) \quad (A \in F^{(i)}(U)).$$

Let $A \in F^{(i)}(U)$. Denote

$$\begin{array}{rcl}
A_\lambda & = & \{(x) \in U : A^-(x) \geq \lambda\} \quad (\lambda \in I), \\
A^\lambda & = & \{(x) \in U : A^+(x) \geq \lambda]\} \quad (\lambda \in I), \\
A_{\lambda^+} & = & \{(x) \in U : A^-(x) > \lambda\} \quad (\lambda \in [0, 1)), \\
A^{\lambda^+} & = & \{(x) \in U : A^+(x) > \lambda]\} \quad (\lambda \in [0, 1)).
\end{array}$$

Definition 3.4 ([4, 25]). *Let $A \in F^{(i)}(U)$ and $[\alpha, \beta] \in [I]$. Denote*

$$A_{[\alpha,\beta]} = \{x \in U : A^-(x) \geq \alpha, A^+(x) \geq \beta]\},$$

$$A_{[\alpha,\beta]^+} = \{x \in U : A(x) > [\alpha, \beta]\},$$

$$A_{(\alpha,\beta)} = \{x \in U : A^-(x) > \alpha, A^+(x) > \beta]\}.$$

Then $A_{[\alpha,\beta]}$ (resp.$A_{[\alpha,\beta]^+}, A_{(\alpha,\beta)}$) is called the $[\alpha, \beta]$-level (resp. strong $[\alpha, \beta]$-level, (α, β)-level) set of A.

Obviously, $A_{(\alpha,\beta)} \subseteq A_{[\alpha,\beta]^+} \subseteq A_{[\alpha,\beta]}$.

Proposition 3.5 ([4, 25]). *Let $A, B \in F^{(i)}(U)$ and $[\alpha, \beta] \in [I]$. Then*

(1) $A \subseteq B \Longrightarrow A_{[\alpha,\beta]^+} \subseteq B_{[\alpha,\beta]^+}$;

(2) $(A \cup B)_{[\alpha,\beta]^+} \supseteq A_{[\alpha,\beta]^+} \cup B_{[\alpha,\beta]^+}$;

(2) $(A \cap B)_{[\alpha,\beta]^+} = A_{[\alpha,\beta]^+} \cap B_{[\alpha,\beta]^+}$.

Let $R \in F^{(i)}(U \times U)$. Denote

$$\begin{aligned} R_\lambda &= \{(x,y) \in U \times U : R^-(x,y) \geq \lambda\} \quad (\lambda \in I), \\ R^\lambda &= \{(x,y) \in U \times U : R^+(x,y) \geq \lambda]\} \quad (\lambda \in I), \\ R_{\lambda^+} &= \{(x,y) \in U \times U : R^-(x,y) > \lambda\} \quad (\lambda \in [0,1)), \\ R^{\lambda^+} &= \{(x,y) \in U \times U : R^+(x,y) > \lambda]\} \quad (\lambda \in [0,1)), \\ R_{[\alpha,\beta]} &= \{(x,y) \in U \times U : R(x,y) \geq [\alpha,\beta]\} \quad ([\alpha,\beta] \in [I]), \\ R_{[\alpha,\beta]^+} &= \{(x,y) \in U \times U : R(x,y) > [\alpha,\beta]\} \quad (\alpha < 1, [\alpha,\beta] \in [I]). \end{aligned}$$

Proposition 3.6. *Let R be an IVF relation on U.*

(1) *If R is reflexive, then R_λ, R^λ, R_{λ^+}, R^{λ^+} and $R_{[\alpha,\beta]^+}$ are reflexive.*

(2) *If R is transitive, then R_λ, R^λ, R_{λ^+}, R^{λ^+} and $R_{[\alpha,\beta]^+}$ are transitive.*

Proof. (1) are obvious.

(2) For any $x, y, z \in U$, if $(x,y), (y,z) \in R_\lambda$, we have $R^-(x,y) \geq \lambda$ and $R^-(y,z) \geq \lambda$. Note that R is transitive. Then $R(x,z) \geq R(x,y) \wedge R(y,z)$ and so

$$R^-(x,z) \geq R^-(x,y) \wedge R^-(y,z) \geq \lambda.$$

Thus $(x,z) \in R_\lambda$. Hence R_λ is transitive.

Similarly, We can prove that R^λ, R_{λ^+} and R^{λ^+} are transitive.

For any $x, y, z \in U$, if $(x,y), (y,z) \in R_{[\alpha,\beta]^+}$, we have $R(x,y) > [\alpha,\beta]$ and $R(y,z) > [\alpha,\beta]$. Note that R is transitive. Then

$$R(x,z) \geq R(x,y) \wedge R(y,z) > [\alpha,\beta].$$

and so $(x,z) \in R_{[\alpha,\beta]^+}$. Hence $R_{[\alpha,\beta]^+}$ is transitive. □

Theorem 3.7. *Let (U,R) be an IVF approximation space. Then IVF rough approximation operator can be represented as follows: for each $A \in F^{(i)}(U)$,*

(1) $(\underline{R}(A))^- = \bigcup\limits_{\lambda\in I} \lambda \underline{R^{1-\lambda}}(A_\lambda) = \bigcup\limits_{\lambda\in I} \lambda \underline{R^{1-\lambda}}(A_{\lambda^+}),$
$= \bigcup\limits_{\lambda\in I} \lambda \underline{R^{(1-\lambda)^+}}(A_\lambda) = \bigcup\limits_{\lambda\in I} \lambda \underline{R^{(1-\lambda)^+}}(A_{\lambda^+});$

(2) $(\underline{R}(A))^+ = \bigcup\limits_{\lambda\in I} \lambda \underline{R_{1-\lambda}}(A^\lambda) = \bigcup\limits_{\lambda\in I)} \lambda \underline{R_{1-\lambda}}(A^{\lambda^+}),$
$= \bigcup\limits_{\lambda\in I} \lambda \underline{R_{(1-\lambda)^+}}(A^\lambda) = \bigcup\limits_{\lambda\in I} \lambda \underline{R_{(1-\lambda)^+}}(A^{\lambda^+});$

(3) $(\overline{R}(A))^- = \bigcup\limits_{\lambda\in I} \lambda \overline{R_\lambda}(A_\lambda) = \bigcup\limits_{\lambda\in I} \lambda \overline{R_{\lambda^+}}(A_\lambda),$
$= \bigcup\limits_{\lambda\in I} \lambda \overline{R_\lambda}(A_{\lambda^+}) = \bigcup\limits_{\lambda\in I} \lambda \overline{R_{\lambda^+}}(A_{\lambda^+});$

(4) $(\overline{R}(A))^+ = \bigcup\limits_{\lambda\in I} \lambda \overline{R^\lambda}(A^\lambda) = \bigcup\limits_{\lambda\in I} \lambda \overline{R^{\lambda^+}}(A^\lambda),$
$= \bigcup\limits_{\lambda\in I} \lambda \overline{R^\lambda}(A^{\lambda^+}) = \bigcup\limits_{\lambda\in I} \lambda \overline{R^{\lambda^+}}(A^{\lambda^+});$

Proof. (1) For each $x \in U$, by Proposition 2.10,

$$\begin{aligned}(\bigcup_{\lambda\in I}\lambda\underline{R^{1-\lambda}}(A_\lambda))(x) &= \bigvee\{\lambda\in I : x\in\underline{R^{1-\lambda}}(A_\lambda)\}\\ &= \bigvee\{\lambda\in I : (R^{1-\lambda})_s(x)\subseteq A_\lambda\}\\ &= \bigvee\{\lambda\in I : R^+(x,y)\geq 1-\lambda \textit{ implies } A^-(y)\geq\lambda\}\\ &= \bigvee\{\lambda\in I : 1-R^+(x,y)\leq\lambda \textit{ implies } A^-(y)\geq\lambda\}\\ &= \bigvee\{\lambda\in I : \bigwedge_{y\in U}(A^-(x)\vee(1-R^+(x,y)))\geq\lambda\}\\ &= \bigvee\{\lambda\in I : (\underline{R}(A))^-(x)\geq\lambda\}\\ &= (\underline{R}(A))^-(x).\end{aligned}$$

Then $(\underline{R}(A))^- = \bigcup_{\lambda\in I}\lambda\underline{R^{1-\lambda}}(A_\lambda)$.

Similarly, we can prove that

$$(\underline{R}(A))^- = \bigcup_{\lambda\in[0,1)}\lambda\underline{R^{1-\lambda}}(A_{\lambda^+}) = \bigcup_{\lambda\in(0,1]}\lambda\underline{R^{(1-\lambda)^+}}(A_\lambda) = \bigcup_{\lambda\in(0,1)}\lambda\underline{R^{(1-\lambda)^+}}(A_{\lambda^+}).$$

(2) The proof is similar to (1).

(3) For each $x \in U$, by Proposition 2.10,

$$\begin{aligned}(\bigcup_{\lambda\in I}\lambda(\overline{R_\lambda}(A_\lambda)))(x) &= \bigvee\{\lambda\in I : x\in\overline{R_\lambda}(A_\lambda)\}\\ &= \bigvee\{\lambda\in I : (R_\lambda)_s(x)\cap A_\lambda\neq\emptyset\}\\ &= \bigvee\{\lambda\in I : \exists y\in U, y\in A_\lambda\cap(R_\lambda)_s(x)\}\\ &= \bigvee\{\lambda\in I : \exists y\in U, A^-(y)\wedge R^-(x,y)\geq\lambda\}\\ &= \bigvee\{\lambda\in I : \bigvee_{y\in U}(A^-(y)\wedge R^-(x,y))\geq\lambda\}\\ &= \bigvee\{\lambda\in I : (\overline{R}(A))^-(x)\geq\lambda\}\\ &= (\overline{R}(A))^-(x).\end{aligned}$$

Then $\bigcup_{\lambda\in I}\lambda(\overline{R_\lambda}(A_\lambda)) = (\overline{R}(A))^-$.

Similarly, we can prove that

$$(\overline{R}(A))^- = \bigcup_{\lambda\in[0,1)}\lambda(\overline{R_{\lambda^+}}(A_\lambda)) = \bigcup_{\lambda\in[0,1)}\lambda(\overline{R_\lambda}(A_{\lambda^+})) = \bigcup_{\lambda\in[0,1)}\lambda(\overline{R_{\lambda^+}}(A_{\lambda^+})).$$

(4) The proof is similar to (3). $\square$

Theorem 3.8. *Let (U, R) be an IVF approximation space. Then IVF rough approximation operator can be represented as follows: for each $A \in F^{(i)}(U)$,*

$$\begin{aligned}\overline{R}(A) &= \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+})) = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]})) \\ &= \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]}}(A_{[\alpha,\beta]^+})).\end{aligned}$$

Proof. Denote $B = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+}))$. By Proposition 2.10,

$$\begin{aligned}B^-(x) &= \bigvee_{\alpha\in I} (\alpha \wedge \overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+})(x)) \\ &= \bigvee\{\alpha \in I : x \in (\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+}))\} \\ &= \bigvee\{\alpha \in I : (R_{[\alpha,\beta]^+})_s(x) \cap A_{[\alpha,\beta]^+} \neq \emptyset\} \\ &= \bigvee\{\alpha \in I : \exists y \in U, R(x,y) > [\alpha,\beta] \; and \; A(y) > [\alpha,\beta]\} \\ &= \bigvee\{\alpha \in I : \exists y \in U, A^-(y) \wedge R^-(x,y) > \alpha \; and \; A^+(y) \wedge R^+(x,y) \\ &\quad \geq \beta \; or \; A^-(y) \wedge R^-(x,y) \geq \alpha \; and \; A^+(y) \wedge R^+(x,y) > \beta\} \\ &= \bigvee_{y\in U} (A^-(y) \wedge R^-(x,y)) = (\overline{R}(A))^-(x).\end{aligned}$$

Then $(\overline{R}(A))^- = B^-$. Similarly, we can prove that $(\overline{R}(A))^+ = B^+$. Hence

$$\overline{R}(A) = B = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+})).$$

Similarly, we can prove that

$$\overline{R}(A) = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]})) = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]}}(A_{[\alpha,\beta]^+})).$$

□

3.2 Axiomatic characterizations of IVF rough approximation operators

In an axiomatic approach, rough sets are axiomatized by abstract operators. For the case of IVF rough sets, the primitive notion is the system $(F^{(i)}(U), \bigcap, \bigcup, c, L, H)$, where $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two IVF set operators. In this subsection, rough approximation operators in the IVF environment are characterized by some axioms.

Definition 3.9. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two IVF set operators. If*

$$(L(A))^c = H(A^c) \; (A \in F^{(i)}(U)),$$

then L, H are called two dual operators.

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Remark 3.10. *$L, H : F^{(i)}(U) \to F^{(i)}(U)$ are two dual operators iff $(H(A))^c = L(A^c)$ for each $A \in F^{(i)}(U)$.*

Theorem 3.11. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists an IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axioms $(AL1)$ and $(AL2)$, or equivalently, H satisfies axioms $(AU1)$ and $(AU2)$:*

$$(AL1) \quad L(\widetilde{[a,b]} \cup A) = \widetilde{[a,b]} \cup L(A) \ (A \in F^{(i)}(U), [a,b] \in [I]),$$

$$(AL2) \qquad L(A \cap B) = \underline{R}(A) \cap L(B) \ (A, B \in F^{(i)}(U);$$

$$(AU1) \qquad H([a,b]A) = [a,b]H(A) \ (A \in F^{(i)}(U), [a,b] \in [I]),$$

$$(AU2) \qquad H(A \cup B) = H(A) \cup H(B) \ (A, B \in F^{(i)}(U)).$$

Proof. Note that $L, H : F^{(i)}(U) \to F^{(i)}(U)$ are two dual operators. Then $(AL1)$ and $(AL2)$ are equivalent to $(AU1)$ and $(AU2)$. We only need to prove that $L = \underline{R}$ and $H = \overline{R}$ iff H satisfies axioms $(AU1)$ and $(AU2)$.

Necessity. This is obvious.

Sufficiency. Assume that the operator H satisfies axioms $(AU1)$ and $(AU2)$. Define an IVF relation R on U by

$$R(x, y) = H(y_{\bar{1}})(x) \quad (x, y \in U).$$

Let $A \in F^{(i)}(U)$. Note that

$$\begin{aligned} H(A)(x) &= H(\bigcup_{y \in U} (A(y)y_{\bar{1}}))(x) = (\bigcup_{y \in U} H(A(y)y_{\bar{1}}))(x) = (\bigcup_{y \in U} (A(y)H(y_{\bar{1}})))(x) \\ &= \bigvee_{y \in U} (A(y) \wedge H(y_{\bar{1}})(x)) = \bigvee_{y \in U} (A(y) \wedge R(x, y)) = \overline{R}(A)(x) \end{aligned}$$

for each $x \in U$. Then $H(A) = \overline{R}(A)$. By Theorem 3.1(3),

$$L(A) = (H(A^c))^c = (\overline{R}(A^c))^c = \underline{R}(A).$$

Thus $L = \underline{R}$, $H = \overline{R}$. □

Theorem 3.12. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists a reflexive IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axiom $(AL1)$, $(AL2)$ and (ALR), or equivalently, H satisfies axiom $(AU1)$, $(AU2)$ and (AUR):*

$$(ALR) \quad L(A) \subseteq A \quad (A \in F^{(i)}(U));$$

$$(AUR) \quad A \subseteq H(A) \quad (A \in F^{(i)}(U)).$$

Proof. This holds by Theorems 3.2(1) and 3.11. □

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Theorem 3.13. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists a symmetric IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axiom $(AL1)$, $(AL2)$ and (ALS), or equivalently, H satisfies axiom $(AU1)$, $(AU2)$ and (AUS):*

$$(ALS) \quad L((x_{\bar{1}})^c)(y) = L((y_{\bar{1}})^c)(x) \ (x, y \in U);$$
$$(ALS) \quad H(x_{\bar{1}})(y) = H(y_{\bar{1}})(x) \ (x, y \in U).$$

Proof. This hold by Remark 2.9(1) and Theorem 3.11. □

Theorem 3.14. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists a transitive IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axiom $(AL1)$, $(AL2)$ and (ALT), or equivalently, H satisfies axiom $(AU1)$, $(AU2)$ and (AUT):*

$$(ALT) \quad L(A) \subseteq L(L(A)) \quad (A \in F^{(i)}(U));$$
$$(AUT) \quad H(H(A)) \subseteq H(A) \quad (A \in F^{(i)}(U)).$$

Proof. This holds by Theorems 3.2(2) and 3.11. □

4 IVF pseudo-closure operators in IVF approximation spaces

In this section, we investigate IVF pseudo-closure operators in IVF approximation spaces.

For each $[a, b] \in [I], X \in \mathscr{P}(U)$, we define

$$([a, b]X)(x) = \begin{cases} [a, b], & x \in X, \\ \bar{0}, & x \in U - X. \end{cases}$$

Denote

$$\mathscr{E}(U) = \{[a, b]X : [a, b] \in [I], X \in \mathscr{P}(U)\}.$$

Then $\mathscr{E}(U) \subseteq F^{(i)}(U)$.

Definition 4.1. *Let τ be an IVF topology on U. Define*

$$S_\tau(A) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha, \beta]A_{[\alpha,\beta]}\} \quad (A \in F^{(i)}(U)).$$

Then $S_\tau : F^{(i)}(U) \to F^{(i)}(U)$ is called the IVF pseudo-closure operator induced by τ on U.

Theorem 4.2 ([25]). *Let $A \in F^{(i)}(U)$. Then*

$$A = \bigcup_{[\alpha,\beta]\in[I]} [\alpha, \beta]A_{[\alpha,\beta]} = \bigcup_{[\alpha,\beta]\in[I]} [\alpha, \beta]A_{(\alpha,\beta)}.$$

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Theorems 4.3(5) and 4.4 below illustrate the meaning on IVF pseudo-closure operators.

Theorem 4.3. *Let τ be an IVF topology on U and let S_τ be the IVF pseudo-closure operator induced by τ on U. Then for any $A, B \in F^{(i)}(U)$,*

(1) $S_\tau(\tilde{0}) = \tilde{0}$.

(2) $A \subseteq S_\tau(A) \subseteq cl_\tau(A)$.

(3) $S_\tau(A \cup B) \supseteq S_\tau(A) \cup S_\tau(B)$. $S_\tau(A \cap B) \subseteq S_\tau(A) \cap S_\tau(B)$.

(4) $A \in \tau^c \Longrightarrow S_\tau(A) = A$.

(5) *S_τ coincides with cl_τ as operators from $\mathscr{E}(U)$ to $F^{(i)}(U)$.*

Proof. (1) For any $[\alpha, \beta] \in [I]$ and $x \in U$, since

$$([\alpha,\beta]\tilde{0}_{[\alpha,\beta]})(x) = [\alpha,\beta] \wedge \tilde{0}_{[\alpha,\beta]}(x) = \begin{cases} [0,0] \wedge \bar{1} = \bar{0}, & [\alpha,\beta] = \bar{0}, \\ [\alpha,\beta] \wedge \bar{0} = \bar{0}, & [\alpha,\beta] \in [I] - \{\bar{0}\}. \end{cases}$$

we have $[\alpha,\beta]\tilde{0}_{[\alpha,\beta]} = \tilde{0}$. Thus

$$S_\tau(\tilde{0}) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]\tilde{0}_{[\alpha,\beta]}) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\tilde{0}) = \tilde{0}.$$

(2) By Theorem 4.2,

$$A = \bigcup_{[\alpha,\beta]\in[I]} [\alpha,\beta]A_{[\alpha,\beta]} \subseteq \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) = S_\tau(A) \quad \textit{and}$$

$$S_\tau(A) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) \subseteq cl_\tau(\bigcup_{[\alpha,\beta]\in[I]} [\alpha,\beta]A_{[\alpha,\beta]}) = cl_\tau(A).$$

(3) For any $A, B \in F^{(i)}(U)$, $[\alpha,\beta] \in [I]$ and $x \in U$ put

$$C(x) = \begin{cases} \bar{1}, & x \in A_{[\alpha,\beta]}, \\ \bar{0}, & x \in U - A_{[\alpha,\beta]} \end{cases}, \quad D(x) = \begin{cases} \bar{1}, & x \in B_{[\alpha,\beta]}, \\ \bar{0}, & x \in U - B_{[\alpha,\beta]}. \end{cases}$$

Obviously,

$$[\alpha,\beta]A_{[\alpha,\beta]} = \widetilde{[\alpha,\beta]} \cap C, \quad [\alpha,\beta]B_{[\alpha,\beta]} = \widetilde{[\alpha,\beta]} \cap D,$$

$$[\alpha,\beta](A_{[\alpha,\beta]} \cup B_{[\alpha,\beta]}) = \widetilde{[\alpha,\beta]} \cap (C \cup D)$$

and

$$[\alpha,\beta](A_{[\alpha,\beta]} \cap B_{[\alpha,\beta]}) = \widetilde{[\alpha,\beta]} \cap (C \cap D).$$

We can easily prove that

$$(A \cup B)_{[\alpha,\beta]} \supseteq A_{[\alpha,\beta]} \cup B_{[\alpha,\beta]} \quad \textit{and} \quad (A \cap B)_{[\alpha,\beta]} = A_{[\alpha,\beta]} \cap B_{[\alpha,\beta]}.$$

By Proposition 2.6(5),

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$$
\begin{aligned}
& S_\tau(A\cup B)\\
= & \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A\cup B)_{[\alpha,\beta]}) \supseteq \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A_{[\alpha,\beta]}\cup B_{[\alpha,\beta]}))\\
= & \bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]}\cap(C\cup D)) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau((\widetilde{[\alpha,\beta]}\cap C)\cup(\widetilde{[\alpha,\beta]}\cap D))\\
= & \bigcup_{[\alpha,\beta]\in[I]} (cl_\tau(\widetilde{[\alpha,\beta]}\cap C)\cup cl_\tau(\widetilde{[\alpha,\beta]}\cap D))\\
= & (\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]}\cap C)\ )\cup(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]}\cap D)\ )\\
= & (\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]})\ )\cup(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]B_{[\alpha,\beta]})\ )\\
= & S_\tau(A)\cup S_\tau(B).
\end{aligned}
$$

By Proposition 2.6(3),

$$
\begin{aligned}
& S_\tau(A\cap B)\\
= & \bigcap_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A\cap B)_{[\alpha,\beta]}) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A_{[\alpha,\beta]}\cap B_{[\alpha,\beta]}))\\
= & \bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]}\cap(C\cap D)) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau((\widetilde{[\alpha,\beta]}\cap C)\cap(\widetilde{[\alpha,\beta]}\cap D))\\
\subseteq & \bigcup_{[\alpha,\beta]\in[I]} (cl_\tau(\widetilde{[\alpha,\beta]}\cap C)\cap cl_\tau(\widetilde{[\alpha,\beta]}\cap D))\\
\subseteq & (\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]}\cap C)\ )\cap(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]}\cap D)\ )\\
= & (\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]})\ )\cap(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]B_{[\alpha,\beta]})\ )\\
= & S_\tau(A)\cap S_\tau(B).
\end{aligned}
$$

(4) By (2) and Proposition 2.6(6),

$$cl_\tau(A)\subseteq S(cl_\tau(A))\subseteq cl_\tau(cl_\tau(A)) = cl_\tau(A),$$

Note that $A\in\tau^c$. Then

$$S_\tau(A) = S_\tau(cl_\tau(A)) = cl_\tau(A) = A.$$

(5) Let $A\in\mathscr{E}(U)$. Then there exist $[a,b]\in[I]$ and $X\in\mathscr{P}(U)$ such that $A=[a,b]X$.

(i) If $[a,b]\neq\bar{0}$, then for each $x\in U$,

$$A_{[a,b]}(x) = ([a,b]X)_{[a,b]}(x) = \begin{cases}\bar{1}, & ([a,b]X)(x)\geq[a,b]\\ \bar{0}, & ([a,b]X)(x)\ngeq[a,b]\end{cases} = \begin{cases}\bar{1}, & x\in X,\\ \bar{0}, & x\in U-X.\end{cases}$$

Thus $A_{[a,b]} = X$. So

$$\begin{aligned} S_\tau(A) &= \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) \\ &\supseteq cl_\tau([a,b]A_{[a,b]}) = cl_\tau([a,b]X) = cl_\tau(A). \end{aligned}$$

By (2), $S_\tau(A) \subseteq cl(A)$. Thus $S_\tau(A) = cl_\tau(A)$.
(ii) If $[a,b] = \bar{0}$, then $A = \tilde{0}$. By (1), $S_\tau(\tilde{0}) = \tilde{0}$. Thus $S_\tau(A) = cl_\tau(A)$.
By (i) and (ii),

$$S_\tau \text{ coincides with } cl_\tau \text{ as operators from } \mathscr{E}(U) \text{ to } F^{(i)}(U).$$

□

Theorem 4.4. *Let (U,R) be an IVF approximation space. If R is preorder, then*

$$\overline{R}(A) = S_{\tau_R}(A) \quad (A \in \mathscr{E}(U)).$$

Proof. For each $A \in \mathscr{E}(U)$, by Theorems 3.11(3) and 4.3(5),

$$\overline{R}(A) = cl_{\tau_R}(A) = S_{\tau_R}(A).$$

□

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SOME WEIGHTED HERMITE-HADAMARD TYPE INEQUALITIES FOR GEOMETRICALLY–ARITHMETICALLY CONVEX FUNCTIONS ON THE CO-ORDINATES

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Abstract. In this paper, the concept of GA-convex functions on the co-ordinates is introduced. By using a concept of GA-convex functions on the co-ordinates, Hermite-Hadamard type inequalities for this class of functions are settled.

1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ forenamed as convex in the classical touch [24], if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x,\ y \in I$ and $\lambda \in [0,1]$.

Indeed, a vast literature has been written on inequalities using classical convexity but one of the most celebrated is the Hermite-Hadamard inequelity. This double inequality is stated as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $a,\ b \in I$ with $a < b$. Then f is convex on $[a,b]$ iff

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This also reveals that (1.1) can be compulsary as a adequate and sufficient condition to function f to be convex on $[a,b]$.

Hermite-Hadmard inequality (1.1) has recieved considerable attention of many reserchers because of its various applications and usefulness in the field of mathematical inequalities itself as well as in other areas of mathematics. The inequality (1.1) has been extended to various forms by using various generalizations of the definition of classical convex functions and it has also been refined under different hypotheses, see for instance [6, 9, 10, 11, 15, 24, 32] and the references therein.

As stated above the classical convexity has been generalized to different forms and we mention below one of the generalizations of the classical convexity which is known as GA-convexity.

Definition 1. [18, 19] *A function $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be GA-convex function on I if*

$$f\left(x^{\lambda}y^{1-\lambda}\right) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x,\ y \in I$ and $\lambda \in [0,1]$, where $x^{\lambda}y^{1-\lambda}$ and $\lambda f(x) + (1-\lambda)f(y)$ are respectively the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

For results on Hermite-Hadamard type inequalities on GA-convex functions and their applications we refere to a recent articles of Latif [15] and Zhang et al. [32].

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The definition of classical convexity for functions of of one variables was extended to functions two variables as follows.

Definition 2. [5, 6] *Let* $\Delta =: [a,b] \times [c,d] \subseteq \mathbb{R}^2$ *with* $a < b$ *and* $c < d$ *be a bidimensional interval. A mapping* $f : \Delta \to \mathbb{R}$ *is said to be convex on* Δ *if the inequality*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)$$

holds for all $(x,y),(z,w) \in \Delta$ *and* $\lambda \in [0,1]$.

The Definition 2 of convex functions on Δ was modified as co-ordinated convex functions by Dragomir in [5].

Definition 3. [5] *A function* $f : \Delta \to \mathbb{R}$ *is said to be convex on the co-ordinates on* Δ *if the partial mappings* $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ *and* $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ *are convex where defined for all* $x \in [a,b]$, $y \in [c,d]$.

Remark 1. [12] *It is clear that if a function* $f : \Delta \to \mathbb{R}$ *is convex on the co-ordinates on* Δ*. Then*

$$\begin{aligned}
&f(tx + (1-t)z, sy + (1-s)w) \\
&\leq tsf(x,y) + t(1-s)f(x,w) + s(1-t)f(z,y) + (1-t)(1-s)f(z,w),
\end{aligned}$$

holds for all $(t,s) \in [0,1] \times [0,1]$ *and* $x,z \in [a,b], y,w \in [c,d]$.

It is well-known that every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates but converse may not be true (see [5]).

The following inequalities of Hermite-Hadamard type for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ were established in [5, Theorem 1, page 778]:

Most recently, the notion of co-ordinated convexity has also been generalized in a diverse manner and as a result, the author [14] extended the defintion of GA-convex functions of one variable to GA-convex functions of two variables.

Definition 4. [14]*A function* $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ *is GA-convex on* Δ *if*

$$f\left(x^{\lambda} z^{1-\lambda}, y^{\lambda} w^{1-\lambda}\right) \leq \lambda f(x,y) + (1-\lambda) f(z,w)$$

holds for all $(x,y), (z,w) \in \Delta$ *and* $\lambda \in [0,1]$.

A modification in Definition 4 resulted in the notion of GA-convex functions on the co-ordinates on Δ.

Definition 5. [14] *A function* $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ *is said to be GA-convex on the co-ordinates on* Δ *if the partial mappings* $f_y : [a,b] \subseteq (0,\infty) \to \mathbb{R}$, $f_y(u) = f(u,y)$ *and* $f_x : [c,d] \subseteq (0,\infty) \to \mathbb{R}$, $f_x(v) = f(x,v)$ *are GA-convex where defined for all* $x \in [a,b]$, $y \in [c,d]$.

The following result holds as a consequence of the defintion of GA-convex fuctions on the co-ordinates on Δ.

Remark 2. *If a function* $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ *is GA-convex on the co-ordinates on* Δ*. Then*

$$\begin{aligned}
&f\left(x^t z^{1-t}, y^s w^{1-s}\right) \\
&\leq tf\left(x, y^s w^{1-s}\right) + (1-t) f\left(z, y^s w^{1-s}\right) \\
&\leq t\left[sf(x,y) + (1-s) f(x,w)\right] + (1-t)\left[sf(z,y) + (1-s) f(z,w)\right] \\
&\leq tsf(x,y) + t(1-s)f(x,w) + s(1-t)f(z,y) + (1-t)(1-s)f(z,w)
\end{aligned}$$

holds for all $(t,s) \in [0,1] \times [0,1]$ *and* $x,z \in [a,b], y,w \in [c,d]$.

In [13], some H-H type inequalities for GA-convex functions on the co-ordinates on Δ were also proved for GA-convex functions on the co-ordinates on Δ. For more results on H-H type inequalities for different generilazations of the defintion of of co-ordinated convex functions we refer the reader to [1], [2], [7]-[12], [16], [20]-[23], [27], [28] and closely related articles mentioned therein.

The main objective of the present paper is to establish some new weighted H-H type inequalities for the class of GA-convex functions on the co-ordinates on a rectangle from the plane in Section 2.

2. Weighted Inequalities for co-ordinated GA-convex functions

For the sake of convenience to the reader, we will use the following notations

$$L_1(t)=a^{\frac{1+t}{2}}b^{\frac{1-t}{2}},L_2(s)=c^{\frac{1+s}{2}}d^{\frac{1-s}{2}},U_1(t)=a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},U_2(s)=c^{\frac{1-s}{2}}d^{\frac{1+s}{2}}.$$

To obtain our main results, we first establish the following weighted identity.

Lemma 1. *Suppose that $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ has second order partial derivatives on Δ° and $[a,b]\times[c,d]\subseteq\Delta^\circ$ with $a<b$ and $c<d$. If $h:[a,b]\times[c,d]\to[0,\infty)$ is twice partially differentiable mapping and $f_{ts}\in L([a,b]\times[c,d])$, then we have*

$$\begin{aligned}
&\Phi(a,b,c,d;f,h)\\
&\quad=h(a,c)f(a,c)-h(a,d)f(a,d)-h(b,c)f(b,c)+h(b,d)f(b,d)\\
&\quad+\int_c^d h_y(a,y)f(a,y)\,dy-\int_c^d h_y(b,y)f(b,y)\,dy-\int_a^b h_x(x,d)f(x,d)\,dx\\
&\quad+\int_a^b h_x(x,c)f(x,c)\,dx+\int_a^b\int_c^d h_{xy}(x,y)f(x,y)\,dydx\\
&=\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\left[\int_0^1\int_0^1 L_1(t)L_2(s)h(L_1(t),L_2(s))f_{ts}(L_1(t),L_2(s))\,dsdt\right.\\
&\quad+\int_0^1\int_0^1 U_1(t)L_2(s)h(U_1(t),L_2(s))f_{ts}(U_1(t),L_2(s))\,dsdt\\
&\quad+\int_0^1\int_0^1 L_1(t)U_2(s)h(L_1(t),U_2(s))f_{ts}(L_1(t),U_2(s))\,dsdt\\
&\quad\left.+\int_0^1\int_0^1 U_1(t)U_2(s)h(U_1(t),U_2(s))f_{ts}(U_1(t),U_2(s))\,dsdt\right].\qquad(2.1)
\end{aligned}$$

Proof. By letting $x=a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}$, $y=c^{\frac{1+s}{2}}d^{\frac{1-s}{2}}$ and by integration by parts with respect to y and then with respect to x, we have

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 L_1(t)L_2(s)h(L_1(t),L_2(s))f_{ts}(L_1(t),L_2(s))\,dsdt\\
&\quad=\int_a^{\sqrt{ab}}\int_c^{\sqrt{cd}}h(x,y)f_{xy}(x,y)\,dydx=h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right)\\
&-h\left(a,\sqrt{cd}\right)f\left(a,\sqrt{cd}\right)-h\left(\sqrt{ab},c\right)f\left(\sqrt{ab},c\right)+h(a,c)f(a,c)+\int_c^{\sqrt{cd}}h_y(a,y)f(a,y)\,dy\\
&\quad-\int_c^{\sqrt{cd}}h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy-\int_a^{\sqrt{ab}}h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx\\
&\quad+\int_a^{\sqrt{ab}}h_x(x,c)f(x,c)\,dx+\int_a^{\sqrt{ab}}\int_c^{\sqrt{cd}}h_{xy}(x,y)f(x,y)\,dydx.\qquad(2.2)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 U_1(t)\,L_2(s)\,h(U_1(t),L_2(s))\,f_{ts}(U_1(t),L_2(s))\,dsdt \\
&= h\left(b,\sqrt{cd}\right)f\left(b,\sqrt{cd}\right)-h(b,c)\,f(b,c)-h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right) \\
&\quad +h\left(\sqrt{ab},c\right)f\left(\sqrt{ab},c\right)-\int_c^{\sqrt{cd}} h_y(b,y)\,f(b,y)\,dy \\
&\quad +\int_c^{\sqrt{cd}} h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy-\int_{\sqrt{ab}}^b h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx \\
&\quad +\int_{\sqrt{ab}}^b h_x(x,c)\,f(x,c)\,dx+\int_{\sqrt{ab}}^b\int_c^{\sqrt{cd}} h_{xy}(x,y)\,f(x,y)\,dydx, \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 L_1(t)\,U_2(s)\,h(L_1(t),U_2(s))\,f_{ts}(L_1(t),U_2(s))\,dsdt \\
&= h\left(\sqrt{ab},d\right)f\left(\sqrt{ab},d\right)-h(a,d)\,f(a,d)-h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right) \\
&\quad +h\left(a,\sqrt{cd}\right)f\left(a,\sqrt{cd}\right)-\int_c^{\sqrt{cd}} h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy \\
&\quad +\int_c^{\sqrt{cd}} h_y(a,y)\,f(a,y)\,dy-\int_a^{\sqrt{ab}} h_x(x,d)\,f(x,d)\,dx \\
&\quad +\int_a^{\sqrt{ab}} h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx+\int_a^{\sqrt{ab}}\int_{\sqrt{cd}}^d h_{xy}(x,y)\,f(x,y)\,dydx \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 U_1(t)\,U_2(s)\,h(U_1(t),U_2(s))\,f_{ts}(U_1(t),U_2(s))\,dsdt \\
&= h(b,d)\,f(b,d)-h\left(b,\sqrt{cd}\right)f\left(b,\sqrt{cd}\right)-h\left(\sqrt{ab},d\right)f\left(\sqrt{ab},d\right) \\
&\quad +h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right)-\int_{\sqrt{cd}}^d h_y(b,y)\,f(b,y)\,dy \\
&\quad +\int_{\sqrt{cd}}^d h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy-\int_{\sqrt{ab}}^b h_x(x,d)\,f(x,d)\,dx \\
&\quad +\int_{\sqrt{ab}}^b h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx+\int_{\sqrt{ab}}^b\int_{\sqrt{cd}}^d h_{xy}(x,y)\,f(x,y)\,dydx. \qquad (2.5)
\end{aligned}$$

Adding (2.2)-(2.5), we get the desired identity. This completes the proof of the lemma. □

Lemma 2. *Let $u,\ v>0$, $\eta,k\in\mathbb{R}$ and $\eta\neq 0$. Then*

$$\begin{aligned}
\zeta(u,v;k,\eta) &= \int_0^1 (1-kt)\,u^{\frac{1}{2}+\eta t}v^{\frac{1}{2}-\eta t}dt \\
&= \begin{cases} \frac{kv^{\frac{1}{2}-\eta}u^{\frac{1}{2}}[L(u^\eta,v^\eta)-u^\eta]}{\eta(\ln u-\ln v)}+v^{\frac{1}{2}-\eta}u^{\frac{1}{2}}L(u^\eta,v^\eta), & u\neq v, \\ \frac{u\left[1-(1-k)^2\right]}{2k}, & u=v, \end{cases}
\end{aligned}$$

where $L(u,v)$ is the logarithmic mean

$$L(u,v)=\begin{cases}\frac{v-u}{\ln v-\ln u}, & u\neq v,\\ u, & u=v.\end{cases}$$

Proof. The proof follows by integration by parts. □

Now we present some new weighted H-H type inequality for GA-convex functions on a rectangle from $\mathbb{R}^2$.

In what follows, we will use the following notation to make our presentation compact.

$$\begin{aligned}\sigma_1(u,v,z,w;q)=&\Big[\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(a,c)|^q\\&+\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\\&\times\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(b,d)|^q\Big]^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}\sigma_2(u,v,z,w;q)=&\Big[\zeta\left(u,v;1,-\frac{1}{2}\right)\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(a,c)|^q\\&+\zeta\left(u,v;1,-\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(u,v;-1,-\frac{1}{2}\right)\\&\times\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(u,v;-1,-\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(b,d)|^q\Big]^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}\sigma_3(u,v,z,w;q)=&\Big[\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;1,-\frac{1}{2}\right)|f_{ts}(a,c)|^q\\&+\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;-1,-\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\\&\times\zeta\left(z,w;1,-\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\zeta\left(z,w;-1,-\frac{1}{2}\right)|f_{ts}(b,d)|^q\Big]^{\frac{1}{q}}\end{aligned}$$

and

$$\begin{aligned}\sigma_4(u,v,z,w;q)=&\Big[\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(a,c)|^q\\&+\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta(u,v;-1)\\&\times\zeta(z,w;1)|f_{ts}(b,c)|^q+\zeta\left(u,v;-1,-\frac{1}{2}\right)\zeta\left(z,w;-1,-\frac{1}{2}\right)|f_{ts}(b,d)|^q\Big]^{\frac{1}{q}}.\end{aligned}$$

It is easy to observe that when $u=v=z=w=1$, then

$$\sigma_1(1,1,1,1;q)=\left[\frac{9}{4}|f_{ts}(a,c)|^q+\frac{3}{4}|f_{ts}(a,d)|^q+\frac{3}{4}|f_{ts}(b,c)|^q+\frac{1}{4}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},$$

$$\sigma_2(1,1,1,1;q)=\left[\frac{3}{4}|f_{ts}(a,c)|^q+\frac{1}{4}|f_{ts}(a,d)|^q\left|+\frac{9}{4}f_{ts}(b,c)\right|^q+\frac{3}{4}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},$$

$$\sigma_3(1,1,1,1;q)=\left[\frac{3}{4}|f_{ts}(a,c)|^q+\frac{9}{4}|f_{ts}(a,d)|^q+\frac{1}{4}|f_{ts}(b,c)|^q+\frac{3}{4}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}$$

and

$$\sigma_4(1,1,1,1;q)=\left[\frac{1}{4}\left|f_{ts}(a,c)\right|^q+\frac{3}{4}\left|f_{ts}(a,d)\right|^q+\frac{3}{4}\left|f_{ts}(b,c)\right|^q+\frac{9}{4}\left|f_{ts}(b,d)\right|^q\right]^{\frac{1}{q}}.$$

Theorem 1. *Let $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a twice partially differentiable mapping on Δ° and $[a,b]\times[c,d]\subseteq\Delta^\circ$ with $a<b$ and $c<d$. If $h:[a,b]\times[c,d]\to[0,\infty)$ is a twice partially differentiable mapping such that $f_{ts}\in L([a,b]\times[c,d])$ and $|f_{ts}|^q$ is GA-convex on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq1$, then we get hands on:*

$$\begin{aligned}
\left|\Phi(a,b,c,d;f,h)\right| &\leq\left(\frac{1}{4}\right)^{\frac{1}{q}+1}(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty\\
&\times\left\{\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(a,b,c,d;q)\right.\\
&+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(a,b,c,d;q)\\
&+\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(a,b,c,d;q)\\
&\left.+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(a,b,c,d;q)\right\},
\end{aligned}\tag{2.6}$$

where $\|h\|_\infty=\sup_{(x,y)\in[a,b]\times[c,d]}h(x,y)$ and $\zeta(u,v;k,\eta)$ is defined in Lemma 2.

Proof. By virtue of Lemma 1, we have

$$\begin{aligned}
&\left|\Phi(a,b,c,d;f,h)\right|\\
&\leq\frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty}{4}\left[\int_0^1\int_0^1L_1(t)L_2(s)\left|f_{ts}(L_1(t),L_2(s))\right|dsdt\right.\\
&\quad+\int_0^1\int_0^1U_1(t)L_2(s)\left|f_{ts}(U_1(t),L_2(s))\right|dsdt\\
&\quad+\int_0^1\int_0^1L_1(t)U_2(s)\left|f_{ts}(L_1(t),U_2(s))\right|dsdt\\
&\quad\left.+\int_0^1\int_0^1U_1(t)U_2(s)\left|f_{ts}(U_1(t),U_2(s))\right|dsdt\right].
\end{aligned}\tag{2.7}$$

Now by using Hölder's inequality for double integrals and by the GA-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq1$, we acquire

$$\begin{aligned}
&\int_0^1\int_0^1L_1(t)L_2(s)\left|f_{ts}(L_1(t),L_2(s))\right|dsdt\\
&\leq\left(\int_0^1\int_0^1L_1(t)L_2(s)dsdt\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1L_1(t)L_2(s)\left|f_{ts}(L_1(t),L_2(s))\right|^qdsdt\right)^{\frac{1}{q}}\\
&\leq\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;-1,\frac{1}{2}\right)\right.\\
&\quad\times\left|f_{ts}(a,c)\right|^q+\zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)\left|f_{ts}(a,d)\right|^q+\zeta\left(a,b;1,\frac{1}{2}\right)\\
&\quad\left.\times\zeta\left(c,d;-1,\frac{1}{2}\right)\left|f_{ts}(b,c)\right|^q+\zeta\left(a,b;1,\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)\left|f_{ts}(b,d)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Correspondingly

$$\begin{aligned}
&\int_0^1\int_0^1 U_1(t)\,L_2(s)\,|f_{ts}(U_1(t),L_2(s))|\,dsdt \\
&\leq \left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;-1,\frac{1}{2}\right)\right. \\
&\quad\times|f_{ts}(a,c)|^q+\zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(a,b;-1,-\frac{1}{2}\right) \\
&\quad\left.\times\zeta\left(c,d;-1,\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(a,b;-1,-\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1 L_1(t)\,U_2(s)\,|f_{ts}(L_1(t),U_2(s))|\,dsdt \\
&\leq \left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;1,-\frac{1}{2}\right)\right. \\
&\quad\times|f_{ts}(a,c)|^q+\zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(a,b;1,\frac{1}{2}\right) \\
&\quad\left.\times\zeta\left(c,d;1,-\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(a,b;1,\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

by similar argument

$$\begin{aligned}
&\int_0^1\int_0^1 U_1(t)\,U_2(s)\,|f_{ts}(U_1(t),U_2(s))|\,dsdt \\
&\leq \left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;1,-\frac{1}{2}\right)\right. \\
&\quad\times|f_{ts}(a,c)|^q+\zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(a,b;-1,-\frac{1}{2}\right) \\
&\quad\left.\times\zeta\left(c,d;1,-\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(a,b;-1,-\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Using the above four inequalities in (2.7) and by resolution, it reveals (2.6) and proof is completed. □

Corollary 1. *Suppose the assumptions of Theorem 1 are met and if $q=1$,then*

$$\begin{aligned}
|\Phi(a,b,c,d;f,h)| &\leq \frac{(\ln b-\ln a)(\ln d-\ln c)}{16}\|h\|_\infty \\
&\quad\times\{\sigma_1(a,b,c,d;1)+\sigma_2(a,b,c,d;1)+\sigma_3(a,b,c,d;1)+\sigma_4(a,b,c,d;1)\}. \quad (2.8)
\end{aligned}$$

Corollary 2. *If we consider* $h(x,y)=\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}$, $(x,y)\in[a,b]\times[c,d]$ *in Theorem 1, then*

$$\begin{aligned}
&\left|\Phi\left(a,b,c,d;f,\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\right)\right|\\
&\leq\left(\frac{1}{4}\right)^{\frac{1}{q}+1}\left\{\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(a,b,c,d;q)\right.\\
&\quad+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(a,b,c,d;q)\\
&\quad+\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(a,b,c,d;q)\\
&\quad\left.+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(a,b,c,d;q)\right\}. \qquad (2.9)
\end{aligned}$$

Theorem 2. *Suppose* $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a twice partially differentiable mapping on* Δ° *and* $[a,b]\times[c,d]\subseteq\Delta^\circ$ *with* $a<b$ *and* $c<d$. *Further let* $h:[a,b]\times[c,d]\to[0,\infty)$ *be a twice partially differentiable mapping. If* $f_{ts}\in L([a,b]\times[c,d])$ *and* $|f_{ts}|^q$ *is GA-convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q>1$, *then we have inequality of the form:*

$$\begin{aligned}
&|\Phi(a,b,c,d;f,h)|\leq\left(\frac{1}{4}\right)^{1+\frac{1}{q}}(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty\\
&\times\left\{\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(1,1,1,1;q)\right.\\
&+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(1,1,1,1;q)\\
&+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(1,1,1,1;q)\\
&\left.+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(1,1,1,1;q)\right\}, \qquad (2.10)
\end{aligned}$$

where $\|h\|_\infty=\sup_{(x,y)\in[a,b]\times[c,d]}h(x,y)$ *and* $\zeta(u,v;k,\eta)$ *is defined in Lemma 2.*

Proof. From Lemma 1, we may write

$$\begin{aligned}
&|\Phi(a,b,c,d;f,h)|\\
&\leq\frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty}{4}\left[\int_0^1\int_0^1 L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|\,dsdt\right.\\
&\quad+\int_0^1\int_0^1 U_1(t)L_2(s)|f_{ts}(U_1(t),L_2(s))|\,dsdt\\
&\quad+\int_0^1\int_0^1 L_1(t)U_2(s)|f_{ts}(L_1(t),U_2(s))|\,dsdt\\
&\quad\left.+\int_0^1\int_0^1 U_1(t)U_2(s)|f_{ts}(U_1(t),U_2(s))|\,dsdt\right]. \qquad (2.11)
\end{aligned}$$

Now by using Hölder's inequality for double integrals, Lemma 2 and by the GA-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$, consequently we

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have

$$\begin{aligned}&\int_0^1\int_0^1 L_1(t)\,L_2(s)\,|f_{ts}(L_1(t),L_2(s))|\,dsdt\\&\leq\left[\int_0^1\int_0^1 (L_1(t)\,L_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(L_1(t),L_2(s))|^q\,dsdt\right]^{\frac{1}{q}}\\&\qquad\leq\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\\&\quad\times\left[\frac{9}{16}|f_{ts}(a,c)|^q+\frac{3}{16}|f_{ts}(a,d)|^q+\frac{3}{16}|f_{ts}(b,c)|^q+\frac{1}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}.\end{aligned}$$

In addition

$$\begin{aligned}&\int_0^1\int_0^1 U_1(t)\,L_2(s)\,|f_{ts}(U_1(t),L_2(s))|\,dsdt\\&\leq\left[\int_0^1\int_0^1 (U_1(t)\,L_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(U_1(t),L_2(s))|^q\,dsdt\right]^{\frac{1}{q}}\\&\qquad\leq\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\\&\quad\times\left[\frac{3}{16}|f_{ts}(a,c)|^q+\frac{1}{16}|f_{ts}(a,d)|^q+\frac{9}{16}|f_{ts}(b,c)|^q+\frac{3}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}&\int_0^1\int_0^1 L_1(t)\,U_2(s)\,|f_{ts}(L_1(t),U_2(s))|\,dsdt\\&\leq\left[\int_0^1\int_0^1 (L_1(t)\,U_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(L_1(t),U_2(s))|^q\,dsdt\right]^{\frac{1}{q}}\\&\qquad\leq\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\\&\quad\times\left[\frac{3}{16}|f_{ts}(a,c)|^q+\frac{9}{16}|f_{ts}(a,d)|^q+\frac{1}{16}|f_{ts}(b,c)|^q+\frac{3}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},\end{aligned}$$

equivalently

$$\begin{aligned}&\int_0^1\int_0^1 U_1(t)\,U_2(s)\,|f_{ts}(U_1(t),U_2(s))|\,dsdt\\&\leq\left[\int_0^1\int_0^1 (U_1(t)\,U_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(U_1(t),U_2(s))|^q\,dsdt\right]^{\frac{1}{q}}\\&\qquad\leq\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\\&\quad\times\left[\frac{1}{16}|f_{ts}(a,c)|^q+\frac{3}{16}|f_{ts}(a,d)|^q+\frac{3}{16}|f_{ts}(b,c)|^q+\frac{9}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}.\end{aligned}$$

Using the above four inequalities in (2.11) and simplifying, we get the required inequality (2.10). □

Corollary 3. *If we take* $h(x,y) = \frac{1}{(\ln b - \ln a)(\ln d - \ln c)}$, $(x,y) \in [a,b] \times [c,d]$ *in Theorem 2, then*

$$\begin{aligned}
&\left|\Phi\left(a,b,c,d;f,\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\right)\right| \\
&\leq \left(\frac{1}{4}\right)^{1+\frac{1}{q}}\left\{\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(1,1,1,1;q)\right. \\
&\quad+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(1,1,1,1;q) \\
&\quad+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(1,1,1,1;q) \\
&\left.+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(1,1,1,1;q)\right\}. \quad (2.12)
\end{aligned}$$

We shall use the following notation for the next theorem and its related corollary.

$$\begin{aligned}
\Delta_1(a,b,c,d;q) &= (\theta(q))^{\frac{2}{q}}|f_{ts}(a,c)|^q+(\theta(q))^{\frac{1}{q}}|f_{ts}(a,d)|^q \\
&\quad+(\theta(q))^{\frac{1}{q}}|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q,
\end{aligned}$$

$$\begin{aligned}
\Delta_2(a,b,c,d;q) &= (\theta(q))^{\frac{1}{q}}|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q \\
&\quad+(\theta(q))^{\frac{2}{q}}|f_{ts}(b,c)|^q+(\theta(q))^{\frac{1}{q}}|f_{ts}(b,d)|^q,
\end{aligned}$$

$$\begin{aligned}
\Delta_3(a,b,c,d;q) &= (\theta(q))^{\frac{1}{q}}|f_{ts}(a,c)|^q+(\theta(q))^{\frac{2}{q}}|f_{ts}(a,d)|^q \\
&\quad+|f_{ts}(b,c)|^q+(\theta(q))^{\frac{1}{q}}|f_{ts}(b,d)|^q
\end{aligned}$$

and

$$\begin{aligned}
\Delta_4(a,b,c,d;q) &= |f_{ts}(a,c)|^q+(\theta(q))^{\frac{1}{q}}|f_{ts}(a,d)|^q \\
&\quad+(\theta(q))^{\frac{1}{q}}|f_{ts}(b,c)|^q+(\theta(q))^{\frac{2}{q}}|f_{ts}(b,d)|^q,
\end{aligned}$$

where $\theta(q) = 2^{q+1}-1$.

Theorem 3. *Let* $f : \Delta \subseteq (0,\infty)\times(0,\infty) \to \mathbb{R}$ *be a twice partially differentiable mapping on* Δ° *and* $[a,b]\times[c,d] \subseteq \Delta^\circ$ *with* $a<b$ *and* $c<d$. *Further let* $h : [a,b]\times[c,d] \to [0,\infty)$ *is a twice partially differentiable mapping. If* $f_{ts} \in L([a,b]\times[c,d])$ *and* $|f_{ts}|^q$ *is GA-convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q>1$, *then the following inequality holds:*

$$\begin{aligned}
&|\Phi(a,b,c,d;f,h)| \leq \frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty}{16}\left(\frac{1}{q+1}\right)^{2/q} \\
&\quad\times\left\{\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_1(a,b,c,d;q)\right. \\
&\quad+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_2(a,b,c,d;q) \\
&\quad+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_3(a,b,c,d;q) \\
&\left.+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_4(a,b,c,d;q)\right\}, \quad (2.13)
\end{aligned}$$

where $\|h\|_{\infty} = \sup_{(x,y)\in[a,b]\times[c,d]} h(x,y)$, $\zeta(u,v;k,\eta)$ *is defined in Lemma 2.*

Proof. From Lemma 1, we have

$$
\begin{aligned}
|\Phi(a,b,c,d;f,h)| \leq{} & \frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_{\infty}}{4} \\
& \left[\int_0^1\int_0^1 L_1(t)\,L_2(s)\,|f_{ts}(L_1(t),L_2(s))|\,dsdt\right. \\
& +\int_0^1\int_0^1 U_1(t)\,L_2(s)\,|f_{ts}(U_1(t),L_2(s))|\,dsdt \\
& +\int_0^1\int_0^1 L_1(t)\,U_2(s)\,|f_{ts}(L_1(t),U_2(s))|\,dsdt \\
& \left.+\int_0^1\int_0^1 U_1(t)\,U_2(s)\,|f_{ts}(U_1(t),U_2(s))|\,dsdt\right]. \qquad (2.14)
\end{aligned}
$$

Now by using the GA-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$, Lemma 2 together with the Hölder's inequality for double integrals, we have

$$
\begin{aligned}
&\int_0^1\int_0^1 L_1(t)\,L_2(s)\,|f_{ts}(L_1(t),L_2(s))|\,dsdt \\
&\quad\leq \int_0^1\int_0^1 (L_1(t)\,L_2(s))\left[\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(a,c)|+\left(\frac{1+t}{2}\right)\right. \\
&\left.\left(\frac{1-s}{2}\right)|f_{ts}(a,d)|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(b,c)|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)|f_{ts}(b,d)|\right] \\
&\leq \left[\int_0^1\int_0^1 (L_1(t)\,L_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left\{\left[\int_0^1\int_0^1\left(\frac{1+t}{2}\right)^q\left(\frac{1+s}{2}\right)^q dsdt\right]^{\frac{1}{q}}|f_{ts}(a,c)|\right. \\
&+\left[\int_0^1\int_0^1\left(\frac{1+t}{2}\right)^q\left(\frac{1-s}{2}\right)^q dsdt\right]^{\frac{1}{q}}|f_{ts}(a,d)|+\left[\int_0^1\int_0^1\left(\frac{1-t}{2}\right)^q\left(\frac{1+s}{2}\right)^q dsdt\right]^{\frac{1}{q}} \\
&\qquad\left.\times|f_{ts}(b,c)|+\left[\int_0^1\int_0^1\left(\frac{1-t}{2}\right)^q\left(\frac{1-s}{2}\right)^q dsdt\right]^{\frac{1}{q}}|f_{ts}(b,d)|\right\} \\
&=\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\frac{1}{2^q(q+1)}\right]^{2/q}\left[\left(2^{q+1}-1\right)^{2/q}\right. \\
&\left.\times|f_{ts}(a,c)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(a,d)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q\right].
\end{aligned}
$$

Likewise, we have

$$
\begin{aligned}
&\int_0^1\int_0^1 U_1(t)\,L_2(s)\,|f_{ts}(U_1(t),L_2(s))|\,dsdt \\
&\qquad\leq\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \\
&\quad\times\left[\frac{1}{2^q(q+1)}\right]^{2/q}\left[\left(2^{q+1}-1\right)^{1/q}|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q\right. \\
&\qquad\qquad\left.+\left(2^{q+1}-1\right)^{2/q}|f_{ts}(b,c)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(b,d)|^q\right],
\end{aligned}
$$

$$\begin{aligned}
&\int_0^1\int_0^1 L_1(t)\,U_2(s)\left|f_{ts}\left(L_1(t),U_2(s)\right)\right|dsdt \\
&\qquad\le\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \\
&\quad\times\left[\frac{1}{2^q(q+1)}\right]^{2/q}\left[\left(2^{q+1}-1\right)^{1/q}\left|f_{ts}(a,c)\right|^q+\left(2^{q+1}-1\right)^{1/q}\left|f_{ts}(a,d)\right|^q\right. \\
&\qquad\qquad\left.+\left|f_{ts}(b,c)\right|^q+\left(2^{q+1}-1\right)^{1/q}\left|f_{ts}(b,d)\right|^q\right],
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 U_1(t)\,U_2(s)\left|f_{ts}\left(U_1(t),U_2(s)\right)\right|dsdt \\
&\qquad\le\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \\
&\quad\times\left[\frac{1}{2^q(q+1)}\right]^{2/q}\left[\left(2^{q+1}-1\right)^{1/q}\left|f_{ts}(a,c)\right|^q+\left(2^{q+1}-1\right)^{1/q}\left|f_{ts}(a,d)\right|^q\right. \\
&\qquad\qquad\left.+\left|f_{ts}(b,c)\right|^q+\left(2^{q+1}-1\right)^{1/q}\left|f_{ts}(b,d)\right|^q\right].
\end{aligned}$$

Further employing the above four inequalities in (2.14) and after simplification, we built up the required inequality (2.13). □

Corollary 4. *If we take* $h(x,y)=\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}$, $(x,y)\in[a,b]\times[c,d]$ *in Theorem 3,then*

$$\begin{aligned}
&\left|\Phi\left(a,b,c,d;f,\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\right)\right|\le\frac{1}{16}\left(\frac{1}{q+1}\right)^{2/q} \\
&\qquad\left\{\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_1(a,b,c,d;q)\right. \\
&\qquad+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_2(a,b,c,d;q) \\
&\qquad+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_3(a,b,c,d;q) \\
&\quad\left.+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\Delta_3(a,b,c,d;q)\right\}, \qquad (2.15)
\end{aligned}$$

where $\zeta(u,v;k,\eta)$ *is defined in Lemma 2 and* $\theta(q)=2^{q+1}-1$.

Theorem 4. *Let* $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a twice partially differentiable mapping on* Δ° *and* $[a,b]\times[c,d]\subseteq\Delta^\circ$ *with* $a<b$ *and* $c<d$. *Further let* $h:[a,b]\times[c,d]\to[0,\infty)$ *is a twice partially differentiable mapping. If* $f_{ts}\in L([a,b]\times[c,d])$ *and* $|f_{ts}|^q$ *is GA-convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q>1$ *and* $q\ge r\ge 0$,

then we attain the following inequality:

$$
\begin{aligned}
&|\Phi(a,b,c,d;f,h)| \leq \left(\frac{1}{4}\right)^{\frac{1}{q}+1} (\ln b - \ln a)(\ln d - \ln c)\|h\|_\infty \\
&\times \left\{ \left[\zeta\left(a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}; 0, \frac{1}{2}\right) \zeta\left(c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}; 0, \frac{1}{2}\right)\right]^{1-\frac{1}{q}} \sigma_1(a^r, b^r, c^r, d^r; q) \right. \\
&+ \left[\zeta\left(a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}; 0, -\frac{1}{2}\right) \zeta\left(c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}; 0, \frac{1}{2}\right)\right]^{1-\frac{1}{q}} \sigma_2(a^r, b^r, c^r, d^r; q) \\
&+ \left[\zeta\left(a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}; 0, \frac{1}{2}\right) \zeta\left(c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}; 0, -\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \sigma_3(a^r, b^r, c^r, d^r; q) \\
&\left. + \left[\zeta\left(a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}; 0, -\frac{1}{2}\right) \zeta\left(c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}; 0, -\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \sigma_4(a^r, b^r, c^r, d^r; q) \right\},
\end{aligned} \tag{2.16}
$$

where $\|h\|_\infty = \sup_{(x,y)\in[a,b]\times[c,d]} h(x,y)$ *and* $\zeta(u,v;k,\eta)$ *is defined in Lemma 2.*

Proof. From Lemma 1, it follows that

$$
\begin{aligned}
&|\Phi(a,b,c,d;f,h)| \\
&\leq \frac{(\ln b - \ln a)(\ln d - \ln c)\|h\|_\infty}{4} \left[\int_0^1 \int_0^1 L_1(t) L_2(s) |f_{ts}(L_1(t), L_2(s))| \, dsdt \right. \\
&\qquad + \int_0^1 \int_0^1 U_1(t) L_2(s) |f_{ts}(U_1(t), L_2(s))| \, dsdt \\
&\qquad + \int_0^1 \int_0^1 L_1(t) U_2(s) |f_{ts}(L_1(t), U_2(s))| \, dsdt \\
&\qquad \left. + \int_0^1 \int_0^1 U_1(t) U_2(s) |f_{ts}(U_1(t), U_2(s))| \, dsdt \right].
\end{aligned} \tag{2.17}
$$

Now by virtue of GA-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$, Lemma 2 and by the Hölder's inequality for double integrals, we have in hand

$$
\begin{aligned}
&\int_0^1 \int_0^1 L_1(t) L_2(s) |f_{ts}(L_1(t), L_2(s))| \, dsdt \leq \left(\int_0^1 \int_0^1 (L_1(t) L_2(s))^{\frac{q-r}{q-1}} \, dsdt \right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\int_0^1 \int_0^1 (L_1(t) L_2(s))^r |f_{ts}(L_1(t), L_2(s))|^q \, dsdt \right)^{\frac{1}{q}} \\
&\leq \left(\frac{1}{4}\right)^{\frac{1}{q}} \left[\zeta\left(a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}; 0, \frac{1}{2}\right) \zeta\left(c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}; 0, \frac{1}{2}\right)\right]^{1-\frac{1}{q}} \sigma_1(a^r, b^r, c^r, d^r; q)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\int_0^1 \int_0^1 U_1(t) L_2(s) |f_{ts}(U_1(t), L_2(s))| \, dsdt \leq \left(\int_0^1 \int_0^1 (U_1(t) L_2(s))^{\frac{q-r}{q-1}} \, dsdt \right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\int_0^1 \int_0^1 (U_1(t) L_2(s))^r |f_{ts}(U_1(t), L_2(s))|^q \, dsdt \right)^{\frac{1}{q}} \\
&\leq \left(\frac{1}{4}\right)^{\frac{1}{q}} \left[\zeta\left(a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}; 0, -\frac{1}{2}\right) \zeta\left(c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}; 0, \frac{1}{2}\right)\right]^{1-\frac{1}{q}} \sigma_2(a^r, b^r, c^r, d^r; q)
\end{aligned}
$$

$$\int_0^1 \int_0^1 L_1(t) U_2(s) |f_{ts}(L_1(t), U_2(s))| \, dsdt \leq \left(\int_0^1 \int_0^1 (L_1(t) U_2(s))^{\frac{q-r}{q-q}} dsdt \right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 \int_0^1 (L_1(t) U_2(s))^r |f_{ts}(L_1(t), U_2(s))|^q \, dsdt \right)^{\frac{1}{q}}$$
$$\leq \left(\frac{1}{4}\right)^{\frac{1}{q}} \left[\zeta\left(a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}; 0, \frac{1}{2}\right) \zeta\left(c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}; 0, -\frac{1}{2}\right) \right]^{1-\frac{1}{q}} \sigma_3(a^r, b^r, c^r, d^r; q)$$

and

$$\int_0^1 \int_0^1 U_1(t) U_2(s) |f_{ts}(U_1(t), U_2(s))| \, dsdt \leq \left(\int_0^1 \int_0^1 (U_1(t) U_2(s))^{\frac{q-r}{q-1}} dsdt \right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 \int_0^1 (U_1(t) U_2(s))^r |f_{ts}(U_1(t), U_2(s))|^q \, dsdt \right)^{\frac{1}{q}}$$
$$\leq \left(\frac{1}{4}\right)^{\frac{1}{q}} \left[\zeta\left(a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}; 0, -\frac{1}{2}\right) \zeta\left(c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}; 0, -\frac{1}{2}\right) \right]^{1-\frac{1}{q}} \sigma_4(a^r, b^r, c^r, d^r; q)$$

Using the above four inequalities in (2.17) and simplifying, we obtained the required inequality (2.16). □

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Effect of RTI drug efficacy on the HIV dynamics with two cocirculating target cells

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Abstract

In this paper, we propose and analyze an HIV dynamics model. The model can be seen as a generalization of many HIV dynamics models presented in the literature since it incorporates (i) two classes of target cells, $CD4^+$ T cells and macrophages, (ii) two types of infected cells, short-lived infected cells and the long-lived chronically infected cells, (iii) intracellular discrete delays, (iv) reverse transcriptase inhibitors (RTIs) drugs with different drug efficacies on $CD4^+$T cells and macrophages. The incidence rate of infection is represented by a general function. A bifurcation parameter, known as the basic reproduction number, R_0 is derived. We established a set of conditions on the general function which are sufficient to determine the global dynamics of the model. Using Lyapunov functionals and LaSalle's invariance principle, the global asymptotic stability of the two equilibria of the model is obtained. An example is presented and some numerical simulations are conducted in order to illustrate the dynamical behavior.

Keywords: Delayed-HIV models; Chronically infected cells; Cocirculating target cells; Immune responses; Lyapunov method.

1 Introduction

Human immunodeficiency virus (HIV) is one of the most dangerous human viruses that destroys the immune system and causes acquired immunodeficiency syndrome (AIDS). During the past decades, several HIV mathematical models have been presented and analyzed (see e.g. [1]-[25]). Global stability of equilibria has become one of the most important features which help us to better understanding of the HIV dynamics. Thus, several researchers have devoted extensive efforts to study the global stability of HIV infection models (see e.g. [7], [8], [9], [11], [25], [14], [15], [16], [17], [22], [23], [19] and [24]). Some of these works assume that HIV infects only the $CD4^+$ T cells ([7], [8], [9], [11], [25], [22], [23], [19] and [24]), while, others assume that HIV infects two types of immune cells, $CD4^+$ T cells and macrophages ([14], [15], [18], [16] and [17]). Callaway and Perelson [3] pointed out that there are two types of infected cells, short-lived infected cells (which produce the most amounts of viruses) and the long-lived chronically infected cells. Moreover, the model presented in [3] incroporates reverse transcriptase inhibitors (RTIs) drugs with different drug efficacies on $CD4^+$T cells and macrophages.

Actually, there exists a time lag between the time the HIV contacts $CD4^+$ T cells or macrophages and the time the production of new infectious HIV particles. Intracellular time delay was first introduced into viral infection model by Herz et al. [5]. Since then, several delayed HIV models have been investigated (see e.g. [6], [7], [8], [9], [11], [25], [14], [17], [18], [22] and [19]). In a very recent work, Elaiw and Almualem [17] have

presented the following delayed HIV model:

$$\begin{aligned}
\dot{x}_1(t) &= \lambda_1 - d_1x_1 - (1-\varepsilon)\bar{\beta}_1x_1v, \qquad (1)\\
\dot{x}_2(t) &= \lambda_2 - d_2x_2 - (1-\chi\varepsilon)\bar{\beta}_2x_2v,\\
\dot{y}_1(t) &= (1-q_1)(1-\varepsilon)\bar{\beta}_1x_1(t-\tau_1)v(t-\tau_1) - \delta_1y_1,\\
\dot{y}_2(t) &= (1-q_2)(1-\chi\varepsilon)\bar{\beta}_2x_2(t-\tau_2)v(t-\tau_2) - \delta_2y_2,\\
\dot{z}_1(t) &= q_1(1-\varepsilon)\bar{\beta}_1x_1(t-\tau_1)v(t-\tau_1) - a_1z_1,\\
\dot{z}_2(t) &= q_2(1-\chi\varepsilon)\bar{\beta}_2x_1(t-\tau_1)v(t-\tau_1) - a_2z_2,\\
\dot{v}(t) &= \sum_{i=1}^{2}\left(N_i\delta_ie^{-n_i\kappa_i}y_i(t-\kappa_i) + M_ia_ie^{-h_i\omega_i}z_i(t-\omega_i)\right) - uv(t) \qquad (2)
\end{aligned}$$

where x_i, y_i, z_i, and v represent the concentrations of uninfected cells, short-lived infected cells, long-lived chronically infected cells and free HIV particles, respectively, where $i = 1$, for the CD4$^+$ T cells and $i = 2$, for the macrophages. The birth and death rates of uninfected cells are given by λ_i and d_ix_i, respectively. Parameter $\bar{\beta}_i$ denotes the infection rate constant. Parameters δ_i and a_i are the death rate constants of the two types of infected cells, and u is the clearance rate of HIV. The uninfected target cells become short-lived infected and long-lived chronically infected cells with fractions $(1 - q_i)$ and q_i, respectively, where $q_i.\in (0, 1)$. The average number of free viruses produced in the lifetime of the two types of infected cells are given by N_i and M_i, respectively. Parameter τ_i represents for the time between viral contact with an uninfected cell of class i, until it becomes infected but not yet producer cells. The loss of the cells during the delay period $[t - \tau_i, t]$ is given by $e^{-m_i\tau_i}$, where $m_i > 0$. The parameters κ_i and ω_i represent the time necessary for producing new infectious viruses from the short-lived and long-lived chronically infected cells, respectively. The factors $e^{-n_i\kappa_i}$ and $e^{-h_i\omega_i}$ represent the loss of the two types of infected cells during the delay periods $[t - \kappa_i, t]$ and $[t - \omega_i, t]$, where $n_i > 0$ and $h_i > 0$.

The immune system has two main responses to viral infections. The first is based on the Cytotoxic T Lymphocyte (CTL) cells which are responsible to attack and kill the infected cells. The second immune response is based on the antibodies that are produced by the B cells. The function of the antibodies is to attack the viruses [1]. In some infections such as in malaria, the CTL immune response is less effective than the antibody immune response [26]. Several mathematical models have been proposed to consider the antibody immune response into the viral infection models (see [27]-[33])).

All the models presented in [27]-[33] are based on the assumption that, the virus attacks one class of target cells. Moreover, model (1)-(2) did not consider the immune response. Therefore, our aim in this paper is to propose an HIV dynamics model with humoral immunity. Our model generalize model (1)-(2) by taking into account the humoral immune response. We use Lyapunov functionals and LaSalle's invariance principle to prove the global stability of all the equilibria of the models.

2 The model

In this section, we propose and analyze the following HIV model:

$$\dot{x}_i(t) = \lambda_i - d_ix_i(t) - \phi_i(x_i(t), v(t)), \qquad i = 1, 2, \tag{3}$$

$$\dot{y}_i(t) = (1-q_i)e^{-m_i\tau_i}\phi_i((t-\tau_i), v(t-\tau_i)) - \delta_iy_i(t), \qquad i = 1, 2, \tag{4}$$

$$\dot{z}_i(t) = q_ie^{-m_i\tau_i}\phi_i((t-\tau_i), v(t-\tau_i)) - a_iz_i(t), \qquad i = 1, 2, \tag{5}$$

$$\dot{v}(t) = \sum_{i=1}^{2}\left(N_{y_i}\delta_ie^{-n_i\kappa_i}y_i(t-\kappa_i) + M_{z_i}a_ie^{-r_i\omega_i}z_i(t-\omega_i)\right) - uv(t) - bv(t)f(w(t)), \tag{6}$$

$$\dot{w}(t) = cv(t) - pw(t). \tag{7}$$

The incidence rate of infection is given by a general function $\phi_i(x_i, v)$, where $\phi_1(x_1, v) = (1-\varepsilon)\bar{\phi}_1(x_1, v)$, and $\phi_2(x_2, v) = (1-\chi\varepsilon)\bar{\phi}_2(x_2, v)$. In addition, the neutralize rate of viruses is given by a general nonlinear function $f(w)$. Parameter b is the B cells neutralize rate, the antibody response is induced at a rate proportional to the concentration of free viruses. Parameters c and p are the recruited rate and death rate constants of B cells, respectively. All the parameters and variables of the model have the same meanings as given in (1)-(2).

2.1 Initial conditions

The initial conditions for system (3)-(7) take the form

$$\begin{aligned}
&x_1(\theta) = \varphi_1(\theta),\ y_1(\theta) = \varphi_3(\theta),\ z_1(\theta) = \varphi_5(\theta),\\
&x_2(\theta) = \varphi_2(\theta),\ y_2(\theta) = \varphi_4(\theta),\ z_2(\theta) = \varphi_6(\theta),\\
&v(\theta) = \varphi_7(\theta),\ w(\theta) = \varphi_8(\theta)\\
&\varphi_j(\theta) \geq 0,\ \theta \in [-\varrho, 0),\ \varphi_j(0) > 0,\ j = 1, ..., 8,
\end{aligned} \tag{8}$$

where $\varrho = \max\{\tau_1, \tau_2, \kappa_1, \kappa_2, \omega_1, \omega_2\}$ and $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_8(\theta)) \in C([-\varrho, 0], \mathbb{R}^8_{\geq 0})$, where C is the Banach space of continuous functions mapping the interval $[-\varrho, 0]$ into $\mathbb{R}^8_{\geq 0}$. By the fundamental theory of functional differential equations [35], system (3)-(7) has a unique solution satisfying the initial conditions (8).

Assumption A1 Function ϕ_i, is continuously differentiable and satisfies the following:

(i) $\phi_i(x_i, v) > 0$, $\phi_i(x_i, 0) = \phi_i(0, v) = 0$, for all $x_i > 0$, $v > 0$, $i = 1, 2$,

(ii) $\frac{\partial \phi_i(x_i,v)}{\partial v} > 0$, $\frac{\partial \phi_i(x_i,v)}{\partial x_i} > 0$, for any $x_i > 0$, $v > 0$. Furthermore, $\frac{\partial \phi_i(x_i,0)}{\partial v} > 0$ for any $x_i > 0$, $i = 1, 2$.

Assumption A2 The function $f(\theta)$ is locally Lipschitz on $[0, \infty)$, and satisfies $f(\theta) > 0$ for all $\theta > 0$ and $f(0) = 0$, and $f(\theta)$ is strictly increasing in $[0, \infty)$.

2.2 Non-negativity and boundedness of solutions

In the following, we establish the non-negativity and boundedness of solutions of system (3)-(7) with initial conditions (8).

Proposition 1. Let $(x_1(t), x_2(t), y_1(t), y_2(t), z_1(t), z_2(t), v(t), w(t))$ be any solution of (3)-(7) satisfying the initial conditions (8), then $x_i(t), y_i(t), z_i(t), i = 1, 2, v(t)$ and $w(t)$ are all non-negative for $t \geq 0$ and ultimately bounded.

Proof. First, we prove that $x_i(t) > 0$, $i = 1, 2$, for all $t \geq 0$. Assume that $x_i(t)$ lose its positivity on some local existence interval $[0, l]$ for some constant l and let $t_i^* \in [0, l]$ be such that $x_i(t_i^*) = 0$. From Eq. (3) we have $\dot{x}_i(t_i^*) = \lambda_i > 0$. Hence $x_i(t) > 0$ for some $t \in (t_i^*, t_i^* + \epsilon)$, where $\epsilon > 0$ is sufficiently small. This leads to a contradiction and hence $x_i(t) > 0$, for all $t \geq 0$. Furthermore, from Eqs. (4)-(7) we have

$$\begin{aligned}
y_i(t) &= y_i(0)\, e^{-\delta_i t} + (1-q_i)e^{-m_i\tau_i} \int_0^t e^{-\delta_i(t-\theta)} \phi(x_i(\theta - \tau_i), v(\theta - \tau_i))d\theta, \qquad i = 1, 2,\\
z_i(t) &= z_i(0)\, e^{-a_i t} + q_i e^{-m_i\tau_i} \int_0^t e^{-a_i(t-\theta)} \phi(x_i(\theta - \tau_i), v(\theta - \tau_i))d\theta, \qquad i = 1, 2,\\
v(t) &= v(0)\, e^{-\int_0^t (u+bf(w(\zeta)))d\zeta} + \int_0^t e^{-\int_\theta^t (u+bf(w(\zeta)))d\zeta} \sum_{i=1}^{2} (N_{y_i}\delta_i e^{-n_i\kappa_i} y_i(\theta - \kappa_i) + M_{z_i} a_i e^{-r_i\omega_i} z_i(\theta - \omega_i))d\theta,\\
w(t) &= w(0)e^{-pt} + c\int_0^t e^{-p(t-\theta)} v(\theta)d\theta,
\end{aligned}$$

then $y_i(t) \geq 0$, $z_i(t) \geq 0$, $i = 1, 2$, $v(t) \geq 0$ and $w(t) \geq 0$, for all $t \in [0, \varrho]$. By a recursive argument, we obtain $y_i(t) \geq 0$, $z_i(t) \geq 0$, $v(t) \geq 0$ and $w(t) \geq 0$, $i = 1, 2$, for all $t \geq 0$.

Next we show the boundedness of the solutions. From Eq. (3) we have $\dot{x}_i(t) \le \lambda_i - d_i x_i(t)$, $i = 1, 2$. This implies that $\limsup_{t\to\infty} x_i(t) \le \frac{\lambda_i}{d_i}$, $i = 1, 2$. Let $T_i(t) = e^{-m_i\tau_i} x_i(t - \tau_i) + y_i(t) + z_i(t)$, $i = 1, 2$ then

$$\begin{aligned}\dot{T}_i(t) &= e^{-m_i\tau_i}\lambda_i - e^{-m_i\tau_i} d_i x_i(t-\tau_i) - \delta_i y_i(t) - a_i z_i(t)\\ &\le e^{-m_i\tau_i}\lambda_i - \sigma_i\left(e^{-m_i\tau_i} x_i(t-\tau_i) + y_i(t) + z_i(t)\right) \le \lambda_i - \sigma_i T_i(t),\end{aligned}$$

where $\sigma_i = \min\{d_i, \delta_i, a_i\}$. Hence, $\limsup_{t\to\infty} T_i(t) \le L_i$, where $L_i = \dfrac{\lambda_i}{\sigma_i}$. Since $x_i(t)$, $y_i(t)$ and $z_i(t)$ are all non-negative, then $\limsup_{t\to\infty} y_i(t) \le L_i$, and $\limsup_{t\to\infty} z_i(t) \le L_i$ for all $t \ge 0$. Moreover,

$$\begin{aligned}\dot{v} &= \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i} y_i(t-\kappa_i) + M_{z_i} a_i e^{-r_i\omega_i} z_i(t-\omega_i)\right) - uv - bv(t)f(w(t))\\ &\le \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i} + M_{z_i} a_i e^{-r_i\omega_i}\right) L_i - uv.\end{aligned}$$

Then $\limsup_{t\to\infty} v(t) \le L_3$, for all $t \ge 0$, where $L_3 = \sum_{i=1}^{2}\frac{\left(N_{y_i}\delta_i e^{-n_i\kappa_i} + M_{z_i} a_i e^{-r_i\omega_i}\right)L_i}{u}$. Furthermore, $\dot{w} = cv - pw \le cL_3 - pw$, then $\limsup_{t\to\infty} w(t) \le L_4$, for all $t \ge 0$, where $L_4 = \frac{cL_3}{p}$.Therefore, $x_i(t), y_i(t), z_i(t), v(t)$ and $w(t)$ are ultimately bounded.

2.3 Equilibria

Let Assumptions A1 (i) and A2 be satisfied, then system (3)-(7) has a disease-free equilibrium $E_0 = (x_1^0, x_2^0, 0, 0, 0, 0, 0, 0)$, where $x_i^0 = \frac{\lambda_i}{d_i}$, $i = 1, 2$. The system can also has another positive equilibrium $E_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2, \tilde{z}_1, \tilde{z}_2, \tilde{v}, \tilde{w})$ which is called endemic equilibrium. The coordinates of the endemic equilibrium, if it exists satisfy the equalities:

$$\begin{aligned}&\lambda_i = d_i\tilde{x}_i + \phi_i(\tilde{x}_i, \tilde{v}), \quad \delta_i\tilde{y}_i = (1-q_i)e^{-m_i\tau_i}\phi_i(\tilde{x}_i, \tilde{v}), \quad a_i\tilde{z}_i = q_i e^{-m_i\tau_i}\phi_i(\tilde{x}_i, \tilde{v}),\\ &u\tilde{v} = \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i}\tilde{y}_i + M_{z_i} a_i e^{-r_i\omega_i}\tilde{z}_i\right) - b\tilde{v}f(\tilde{w}), \qquad \tilde{w} = \frac{c}{p}\tilde{v}.\end{aligned}$$

Then the basic infection reproduction number for system (3)-(7) is

$$R_0 = \sum_{i=1}^{2} R_{0i} = \sum_{i=1}^{2}\frac{((1-q_i)N_{y_i}e^{-n_i\kappa_i} + q_i M_{z_i} e^{-r_i\omega_i})e^{-m_i\tau_i}}{u}\frac{\partial\phi_i(x_i^0, 0)}{\partial v}.$$

The term $\partial\phi_i(x_i^0, 0)/\partial v$ represents the maximal average number of target cells of class i that infects by viruses, and R_{01} denotes the basic infection reproduction number of the HIV dynamics with CD4$^+$ T cells (in the absence of macrophages) and R_{02} denotes the basic infection reproduction number of the HIV dynamics with macrophages (in the absence of CD4$^+$T cells), respectively. The parameter R_0 determines whether the infection can be established.

2.4 Global stability analysis

In this subsection, we establish a set of conditions which are sufficient for the global stability of the two equilibria of system (3)-(7) employing Lyapunov method and LaSalle's invariance principle. The following function will be used throughout the paper $H(s) = s - 1 - \ln s$.

Assumption A3 The function ϕ_i, $i = 1, 2$ satisfies:
(i) $\left(\frac{\partial\phi_i(x_i,0)}{\partial v} - \frac{\partial\phi_i(x_i^0,0)}{\partial v}\right)\left(x_i^0 - x_i\right) \le 0$, for $x_i > 0$,
(ii) $\phi_i(x_i, v) \le v\frac{\partial\phi_i(x_i,0)}{\partial v}$, for all $x_i, v > 0$.

Theorem 1. Let Assumptions A1-A3 be satisfied and $R_0 \le 1$, then the disease-free equilibrium E_0 of system (3)-(7) is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$\begin{aligned}W_0 &= \sum_{i=1}^{2}\gamma_i\left[x_i - x_i^0 - \int_{x_i^0}^{x_i}\lim_{v\to 0^+}\frac{\phi_i(x_i^0,v)}{\phi_i(s,v)}ds + \frac{N_{y_i}e^{-n_i\kappa_i}}{\gamma_i}y_i + \frac{M_{z_i}e^{-r_i\omega_i}}{\gamma_i}z_i\right.\\ &\left.+\int_0^{\tau_i}\phi_i(x_i(t-\theta),v(t-\theta))d\theta + \frac{e^{-n_i\kappa_i}N_{y_i}\delta_i}{\gamma_i}\int_0^{\kappa_i}y_i(t-\theta)d\theta + \frac{e^{-r_i\omega_i}M_{z_i}a_i}{\gamma_i}\int_0^{\omega_i}z_i(t-\theta)d\theta\right]\\ &+v + \frac{b}{c}\int_0^{w}f(\theta)d\theta,\end{aligned}$$

where $\gamma_i = e^{-m_i\tau_i}((1-q_i)e^{-n_i\kappa_i}N_{y_i} + q_ie^{-r_i\omega_i}M_{z_i})$, $i = 1, 2$. We calculate $\frac{dW_0}{dt}$ along the trajectories of system (3)-(7) as:

$$\begin{aligned}\frac{dW_0}{dt} &= \sum_{i=1}^{2}\gamma_i\left[\left(1-\lim_{v\to 0^+}\frac{\phi_i(x_i^0,v)}{\phi_i(x_i,v)}\right)(\lambda_i - d_ix_i - \phi_i(x_i,v))\right.\\ &+\frac{N_{y_i}e^{-n_i\kappa_i}}{\gamma_i}((1-q_i)e^{-m_i\tau_i}\phi_i(x_i(t-\tau_i),v(t-\tau_i)) - \delta_iy)\\ &+\frac{M_{z_i}e^{-r_i\omega_i}}{\gamma_i}(q_ie^{-m_i\tau_i}\phi_i(x_i(t-\tau_i),v(t-\tau_i)) - a_iz_i)\\ &+\phi_i(x_i,v) - \phi_i(x_i(t-\tau_i),v(t-\tau_i))\\ &\left.+\frac{e^{-n_i\kappa_i}N_{y_i}\delta_i}{\gamma_i}(y_i - y_i(t-\kappa_i)) + \frac{e^{-r_i\omega_i}M_{z_i}a_i}{\gamma_i}(z_i - z_i(t-\omega_i))\right]\\ &+\sum_{i=1}^{2}\left(N_{y_i}\delta_ie^{-n_i\kappa_i}y_i(t-\kappa_i) + M_{z_i}a_ie^{-r_i\omega_i}z_i(t-\omega_i)\right) - uv - bvf(w) + \frac{b}{c}f(w)(cv-pw). \qquad (9)\end{aligned}$$

Collecting terms of Eq. (9) we get

$$\begin{aligned}\frac{dW_0}{dt} &= \sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\partial\phi_i(x_i^0,0)/\partial v}{\partial\phi_i(x_i,0)/\partial v}\right)(\lambda_i - d_ix_i) + \phi_i(x_i,v)\frac{\partial\phi_i(x_i^0,0)/\partial v}{\partial\phi_i(x_i,0)/\partial v}\right] - uv - \frac{bp}{c}wf(w)\\ &= \sum_{i=1}^{2}\gamma_i\left[\lambda_i\left(1-\frac{x_i}{x_i^0}\right)\left(1-\frac{\partial\phi_i(x_i^0,0)/\partial v}{\partial\phi_i(x_i,0)/\partial v}\right) + \phi_i(x_i,v)\frac{\partial\phi_i(x_i^0,0)/\partial v}{\partial\phi_i(x_i,0)/\partial v}\right] - uv - \frac{bp}{c}wf(w)\\ &= \sum_{i=1}^{2}\gamma_i\lambda_i\left(1-\frac{x_i}{x_i^0}\right)\left(1-\frac{\partial\phi_i(x_i^0,0)/\partial v}{\partial\phi_i(x_i,0)/\partial v}\right) + \sum_{i=1}^{2}\gamma_i\phi_i(x_i,v)\frac{\partial\phi_i(x_i^0,0)/\partial v}{\partial\phi_i(x_i,0)/\partial v} - uv - \frac{bp}{c}wf(w). \qquad (10)\end{aligned}$$

Using A3 we get

$$\begin{aligned}\frac{dW_0}{dt} &\le \sum_{i=1}^{2}\gamma_i\lambda_i\left(1-\frac{x_i}{x_i^0}\right)\left(1-\frac{\partial\phi_i(x_i^0,0)/\partial v}{\partial\phi_i(x_i,0)/\partial v}\right) + \sum_{i=1}^{2}\gamma_iv\frac{\partial\phi_i(x_i^0,0)}{\partial v} - uv - \frac{bp}{c}wf(w)\\ &= \sum_{i=1}^{2}\gamma_i\lambda_i\left(1-\frac{x_i}{x_i^0}\right)\left(1-\frac{\partial\phi_i(x_i^0,0)/\partial v}{\partial\phi_i(x_i,0)/\partial v}\right) + (R_0-1)uv - \frac{bp}{c}wf(w). \qquad (11)\end{aligned}$$

By using Assumption A2, the last term is less than or equal zero. Therefore, If $R_0 \le 1$ then $\frac{dW_0}{dt} \le 0$ for all $x_1, x_2, v, w > 0$. We note that, the solutions of the system (3)-(7) converge to Γ, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$. From Eq. (11) we have $\frac{dW_0}{dt} = 0$ iff $x_i = x_i^0$, $i = 1, 2$, $v = 0$ and $w = 0$. The set Γ is invariant and for any element belongs to Γ satisfies $w = 0$, $v = 0$ then $\dot{v} = 0$. We can see from Eq. (19) that

$$\sum_{i=1}^{2}\left(N_{y_i}\delta_ie^{-n_i\kappa_i}y_i(t-\kappa_i) + M_{z_i}a_ie^{-r_i\omega_i}z_i(t-\omega_i)\right) = 0.$$

Since y_i and z_i are non-negative for $i = 1, 2$, then $y_1 = y_2 = 0$ and $z_1 = z_2 = 0$. It follows that, $\frac{dW_0}{dt} = 0$ iff $x_i = x_i^0$, $y_i = z_i = v = w = 0$, $i = 1,2$. From LaSalle's invariance principle, E_0 is GAS.

To establish the global stability of the endemic equilibrium, we need the following condition.

Assumption A4 Function $\phi_i(x_i, v)$ satisfies the following:

$$\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})} - \frac{v}{\tilde{v}}\right)\left(1 - \frac{\phi_i(x_i,\tilde{v})}{\phi_i(x_i,v)}\right) \le 0, \quad x_i, v > 0$$

Theorem 2. Let Assumptions A1, A2 and A4 hold true and the endemic equilibrium E_1 of system (3)-(7) exists, then E_1 is GAS.

Proof. We consider the Lyapunov functional W_1 as:

$$\begin{aligned}
W_1 = &\sum_{i=1}^{2}\gamma_i \left[x_i - \tilde{x}_i - \int_{\tilde{x}_i}^{x_i}\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(s,\tilde{v})}ds + \frac{N_{y_i}e^{-n_i\kappa_i}}{\gamma_i}\tilde{y}_i H\left(\frac{y_i}{\tilde{y}_i}\right)\right.\\
&+ \frac{M_{z_i}e^{-r_i\omega_i}}{\gamma_i}\tilde{z}_i H\left(\frac{z_i}{\tilde{z}_i}\right) + \phi_i(\tilde{x}_i,\tilde{v})\int_0^{\tau_i} H\left(\frac{\phi_i(x_i(t-\theta),v(t-\theta))}{\phi_i(\tilde{x}_i,\tilde{v})}\right)d\theta\\
&\left.+ \frac{e^{-n_i\kappa_i}N_{y_i}\delta_i\tilde{y}_i}{\gamma_i}\int_0^{\kappa_i} H\left(\frac{y_i(t-\theta)}{\tilde{y}_i}\right)d\theta + \frac{e^{-r_i\omega_i}M_{z_i}a_i\tilde{z}_i}{\gamma_i}\int_0^{\omega_i} H\left(\frac{z_i(t-\theta)}{\tilde{z}_i}\right)d\theta\right] + \tilde{v}H\left(\frac{v}{\tilde{v}}\right)\\
&+ \frac{b}{c}\int_{\tilde{w}}^{w}(f(\theta) - f(\tilde{w}))d\theta.
\end{aligned}$$

Calculating $\frac{dW_1}{dt}$ along the solutions of system (3)-(7) we obtain

$$\begin{aligned}
\frac{dW_1}{dt} = &\sum_{i=1}^{2}\gamma_i \left[\left(1 - \frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)(\lambda_i - d_i x_i - \phi_i(x_i,v))\right.\\
&+ \frac{N_{y_i}e^{-n_i\kappa_i}}{\gamma_i}\left(1 - \frac{\tilde{y}_i}{y_i}\right)((1-q_i)e^{-m_i\tau_i}\phi_i(x_i(t-\tau_i),v(t-\tau_i)) - \delta_i y_i)\\
&+ \frac{M_{z_i}e^{-r_i\omega_i}}{\gamma_i}\left(1 - \frac{\tilde{z}_i}{z_i}\right)(q_i e^{-m_i\tau_i}\phi_i(x_i(t-\tau_i),v(t-\tau_i)) - a_i z_i)\\
&+ \phi_i(x_i,v) - \phi_i(x_i(t-\tau_i),v(t-\tau_i)) + \phi_i(\tilde{x}_i,\tilde{v})\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right)\\
&+ \frac{e^{-n_i\kappa_i}N_{y_i}\delta_i\tilde{y}_i}{\gamma_i}\left(\frac{y_i}{\tilde{y}_i} - \frac{y_i(t-\kappa_i)}{\tilde{y}_i} + \ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right)\right)\\
&\left.+\frac{e^{-r_i\omega_i}M_{z_i}a_i\tilde{z}_i}{\gamma_i}\left(\frac{z_i}{\tilde{z}_i} - \frac{z_i(t-\omega_i)}{\tilde{z}_i} + \ln\left(\frac{z_i(t-\omega_i)}{z_i}\right)\right)\right]\\
&+ \left(1 - \frac{\tilde{v}}{v}\right)\left(\sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i}y_i(t-\kappa_i) + M_{z_i}a_i e^{-r_i\omega_i}z_i(t-\omega_i)\right) - uv - bvf(w)\right)\\
&+ \frac{b}{c}(f(w) - f(\tilde{w}))(cv - pw).
\end{aligned} \tag{12}$$

Collecting terms of Eq. (12) we get

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\gamma_i \Bigg[\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)(\lambda_i - d_i x_i) + \phi_i(x_i,v)\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})} + \frac{N_{y_i}e^{-n_i\kappa_i}\delta_i}{\gamma_i}\tilde{y}_i + \frac{M_{z_i}e^{-r_i\omega_i}a_i}{\gamma_i}\tilde{z}_i \\
&- \frac{(1-q_i)e^{-m_i\tau_i}N_{y_i}e^{-n_i\kappa_i}}{\gamma_i}\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i} - \frac{q_ie^{-m_i\tau_i}M_{z_i}e^{-r_i\omega_i}}{\gamma_i}\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i} \\
&+ \phi_i(\tilde{x}_i,\tilde{v})\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right) \\
&+ \frac{e^{-n_i\kappa_i}N_{y_i}\delta_i\tilde{y}_i}{\gamma_i}\ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right) + \frac{e^{-r_i\omega_i}M_{z_i}a_i\tilde{z}_i}{\gamma_i}\ln\left(\frac{z_i(t-\omega_i)}{z_i}\right)\Bigg] \\
&- \sum_{i=1}^{2}N_{y_i}\delta_i e^{-n_i\kappa_i}\frac{\tilde{v}y_i(t-\kappa_i)}{v} - \sum_{i=1}^{2}M_{z_i}a_ie^{-r_i\omega_i}\frac{\tilde{v}z_i(t-\omega_i)}{v} - uv + u\tilde{v} \\
&+ b\tilde{v}f(w) - \frac{bp}{c}wf(w) - bvf(\tilde{w}) + \frac{bp}{c}wf(\tilde{w}).
\end{aligned}$$

Using the equilibrium conditions for E_1:

$$\begin{aligned}
&\lambda_i = d_i\tilde{x}_i + \phi_i(\tilde{x}_i,\tilde{v}), \quad (1-q_i)e^{-m_i\tau_i}\phi_i(\tilde{x}_i,\tilde{v}) = \delta_i\tilde{y}_i, \quad q_ie^{-m_i\tau_i}\phi_i(\tilde{x}_i,\tilde{v}) = a_i\tilde{z}_i, \\
&u\tilde{v} = \sum_{i=1}^{2}\left(N_{y_i}\delta_ie^{-n_i\kappa_i}\tilde{y}_i + M_{z_i}a_ie^{-r_i\omega_i}\tilde{z}_i\right) - b\tilde{v}f(\tilde{w}), \quad \tilde{w} = \frac{c}{p}\tilde{v}
\end{aligned}$$

and the following equality

$$uv = u\tilde{v}\frac{v}{\tilde{v}} = \frac{v}{\tilde{v}}\left(\sum_{i=1}^{2}(N_{y_i}\delta_ie^{-n_i\kappa_i}\tilde{y}_i + M_{z_i}a_ie^{-r_i\omega_i}\tilde{z}_i) - b\tilde{v}f(\tilde{w})\right) = \frac{v}{\tilde{v}}\sum_{i=1}^{2}\gamma_i\phi_i(\tilde{x}_i,\tilde{v}) - bvf(\tilde{w}),$$

we obtain

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\gamma_i\Bigg[d_i\tilde{x}_i\left(1-\frac{x_i}{\tilde{x}_i}\right)\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right) + \phi_i(\tilde{x}_i,\tilde{v})\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right) \\
&+ \phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})} - \frac{v}{\tilde{v}}\right) + \frac{2N_{y_i}e^{-n_i\kappa_i}\delta_i}{\gamma_i}\tilde{y}_i + \frac{2M_{z_i}e^{-r_i\omega_i}a_i}{\gamma_i}\tilde{z}_i \\
&- \frac{N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i}{\gamma_i}\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})} + \frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right) \\
&- \frac{M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i}{\gamma_i}\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})} + \frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right) \\
&+ \frac{N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i}{\gamma_i}\left(\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right) + \ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right)\right) \\
&+ \frac{M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i}{\gamma_i}\left(\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right) + \ln\left(\frac{z_i(t-\omega_i)}{z_i}\right)\right)\Bigg] \\
&- b\tilde{v}f(\tilde{w}) + b\tilde{v}f(w) - \frac{bp}{c}wf(w) + \frac{bp}{c}wf(\tilde{w}). \qquad (13)
\end{aligned}$$

Using the following equalities

$$\begin{aligned}
\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right) &= \ln\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\ln\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\\
&\quad+\ln\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)+\ln\left(\frac{\tilde{v}y_i}{v\tilde{y}_i}\right),\\
\ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right) &= \ln\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right)+\ln\left(\frac{v\tilde{y}_i}{\tilde{v}y_i}\right),\\
\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right) &= \ln\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\ln\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\\
&\quad+\ln\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)+\ln\left(\frac{\tilde{v}z_i}{v\tilde{z}_i}\right),\\
\ln\left(\frac{z_i(t-\omega_i)}{z_i}\right) &= \ln\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right)+\ln\left(\frac{v\tilde{z}_i}{\tilde{v}z_i}\right).
\end{aligned}$$

Eq. (13) can be rewritten as

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\Bigg[\gamma_i d_i\tilde{x}_i\left(1-\frac{x_i}{\tilde{x}_i}\right)\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}-1+\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)\\
&-\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}-1-\ln\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)\right)-\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}-1-\ln\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)\right)\\
&-N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}-1-\ln\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\right)\\
&-N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}-1-\ln\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right)\right)\\
&-M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}-1-\ln\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\right)\\
&-M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}-1-\ln\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right)\right)\Bigg]-\frac{bp}{c}(w-\tilde{w})(f(w)-f(\tilde{w})).
\end{aligned}\tag{14}$$

Then Eq. (14) becomes,

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\Bigg[\gamma_i d_i\tilde{x}_i\left(1-\frac{x_i}{\tilde{x}_i}\right)\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}\right)\left(1-\frac{\phi_i(x_i,\tilde{v})}{\phi_i(x_i,v)}\right)\\
&-\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left\{H\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+H\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)\right\}\\
&-N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i\left\{H\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}\right)+H\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right)\right\}\\
&-M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i\left\{H\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}\right)+H\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right)\right\}\Bigg]-\frac{bp}{c}(w-\tilde{w})(f(w)-f(\tilde{w})).
\end{aligned}$$

By using Assumption A2, the last term is less than or equal zero. It is easy to see that, if $\tilde{x}_1,\tilde{x}_2,\tilde{y}_1,\tilde{y}_2,\tilde{z}_1,\tilde{z}_2,\tilde{v}$ and $\tilde{w}>0$, then $\frac{dW_1}{dt}\le 0$ for all $x_1,x_2,y_1,y_2,z_1,z_2,v$ and $w>0$. The solutions of the system limit to Γ, the largest invariant subset of $\{\frac{dW_1}{dt}=0\}$. It can be seen that $\frac{dW_1}{dt}=0$ if and only if $x_i=\tilde{x}_i$, $v=\tilde{v}$, $w=\tilde{w}$ and $H=0$ i.e.

$$\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}=\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}=1 \tag{15}$$

From Eq. (15), we have $y_i=\tilde{y}_i$ and $z_i=\tilde{z}_i$. It follows that $\frac{dW_1}{dt}$ equal to zero at E_1. LaSalle's invariance principle implies the global stability of E_1.

3 Example and numerical simulations

We introduce the following example:

$$\dot{x}_i(t) = \lambda_i - d_i x_i(t) - \frac{\beta_i x_i^{k_i}(t) v(t)}{(x_i^{k_i}(t) + \rho_i)(v(t) + \varsigma_i)}, \qquad i = 1, 2, \tag{16}$$

$$\dot{y}_i(t) = (1 - q_i) e^{-m_i \tau_i} \frac{\beta_i x_i^{k_i}(t - \tau_i) v(t - \tau_i)}{(x_i^{k_i}(t - \tau_i) + \rho_i)(v(t - \tau_i) + \varsigma_i)} - \delta_i y_i(t), \qquad i = 1, 2, \tag{17}$$

$$\dot{z}_i(t) = q_i e^{-m_i \tau_i} \frac{\beta_i x_i^{k_i}(t - \tau_i) v(t - \tau_i)}{(x_i^{k_i}(t - \tau_i) + \rho_i)(v(t - \tau_i) + \varsigma_i)} - a_i z_i(t), \qquad i = 1, 2, \tag{18}$$

$$\dot{v}(t) = \sum_{i=1}^{2} \left(N_{y_i} \delta_i e^{-n_i \kappa_i} y_i(t - \kappa_i) + M_{z_i} a_i e^{-r_i \omega_i} z_i(t - \omega_i) \right) - uv(t) - bv(t)w(t), \tag{19}$$

$$\dot{w}(t) = cv(t) - pw(t). \tag{20}$$

For this example we have

$$\phi_i(x_i, v) = \frac{\beta_i x_i^{k_i} v}{(x_i^{k_i} + \rho_i)(v + \varsigma_i)}, \qquad f(w) = w \tag{21}$$

where $k_i, \rho_i, \varsigma_i > 0$, $i = 1, 2$. Function ϕ_i satisfies the following:

$$\begin{aligned}
\frac{\partial \phi_i(x_i, v)}{\partial x_i} &= \frac{k_i \rho_i \beta_i x_i^{k_i - 1} v}{(x_i^{k_i} + \rho_i)^2 (v + \varsigma_i)} > 0, \text{for all } x_i > 0,\ v > 0, \\
\frac{\partial \phi_i(x_i, v)}{\partial v} &= \frac{\varsigma_i \beta_i x_i^{k_i}}{(x_i^{k_i} + \rho_i)(v + \varsigma_i)^2} > 0, \text{for all } x_i > 0, \\
\frac{\partial \phi_i(x_i, 0)}{\partial v} &= \frac{\beta_i x_i^{k_i}}{\varsigma_i (x_i^{k_i} + \rho_i)} > 0, \text{for all } x_i > 0,\ v > 0, \\
\phi_i(x_i, v) &= \frac{\beta_i x_i^{k_i} v}{(x_i^{k_i} + \rho_i)(v + \varsigma_i)} \leq \frac{\beta_i x_i^{k_i} v}{\varsigma_i (x_i^{k_i} + \rho_i)} = v \frac{\partial \phi_i(x_i, 0)}{\partial v}, \text{for all } x_i > 0,\ v > 0, \\
\left(\frac{\phi_i(x_i, v)}{\phi_i(x_i, \tilde{v})} - \frac{v}{\tilde{v}} \right) \left(1 - \frac{\phi_i(x_i, \tilde{v})}{\phi_i(x_i, v)} \right) &= \frac{-\varsigma_i (v - \tilde{v})^2}{\tilde{v}(\tilde{v} + \varsigma_i)(v + \varsigma_i)} \leq 0, \text{for all } x_i,\ v > 0.
\end{aligned}$$

Thus Assumption A1-A4 hold true and Theorems 1 and 2 are applicable. The basic reproduction number in this case is given by

$$R_0 = \sum_{i=1}^{2} R_{0i} = \sum_{i=1}^{2} \frac{((1 - q_i) e^{-n_i \kappa_i} N_{y_i} + q_i e^{-r_i \omega_i} M_{z_i}) e^{-m_i \tau_i}}{u} \frac{\beta_i (x_i^0)^{k_i}}{\varsigma_i ((x_i^0)^{k_i} + \rho_i)}.$$

Without loss of generality we let, $\tau_e = \tau_1 = \tau_2 = \kappa_1 = \kappa_2 = \omega_1 = \omega_2$. In Table 1, we present the values of some parameters of model (16)-(20). The effect of the drug efficacy ε and time delay τ_e on the qualitative behavior of the system will be studied below in details. All computations are carried out by MATLAB.

3.1 Evolution of the system state with different initial conditions

We have chosen three different initial conditions as follows:

IC1: $\varphi_1(\theta) = 600$, $\varphi_2(\theta) = 200$, $\varphi_3(\theta) = 1$, $\varphi_4(\theta) = 0.5$, $\varphi_5(\theta) = 1$, $\varphi_6(\theta) = 2$, $\varphi_7(\theta) = 1$, $\varphi_8(\theta) = 0.02$,

IC2: $\varphi_1(\theta) = 700$, $\varphi_2(\theta) = 350$, $\varphi_3(\theta) = 2$, $\varphi_4(\theta) = 2$, $\varphi_5(\theta) = 3$, $\varphi_6(\theta) = 5$, $\varphi_7(\theta) = 6$, $\varphi_8(\theta) = 1$

IC3: $\varphi_1(\theta) = 800$, $\varphi_2(\theta) = 500$, $\varphi_3(\theta) = 3.5$, $\varphi_4(\theta) = 3.5$, $\varphi_5(\theta) = 6$, $\varphi_6(\theta) = 8$, $\varphi_7(\theta) = 10$, $\varphi_8(\theta) = 1.4$,

where $\theta \in [-\varrho, 0)$. We will fix the delay parameter $\tau_e = 0.01$ day^{-1}, and using two sets of the parameter ε to get the following two cases.

Case (I): In this case, we choose $\varepsilon = 0.8$ then we get $R_0 = 0.79 < 1$. Figure 1 shows that, the state of the system eventually approach to the infection-free equilibrium $E_0 = (1000, 600, 0, 0, 0, 0, 0, 0)$ for the three initial conditions IC1-IC3. This supports the results of Theorem 1 that the infection-free equilibrium E_0 is GAS. In

Table 1: The values of the parameters of model (16)-(20).

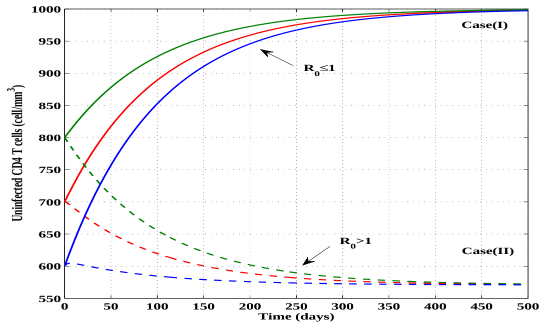
Parameter	Value	Parameter	Value
λ_1	10 cells mm^{-3}day^{-1}	λ_2	6 cells mm^{-3}day^{-1}
$\bar{\beta}_1$	8 cells mm^{-3}day^{-1}	$\bar{\beta}_2$	5 cells mm^{-3}day^{-1}
d_1	0.01 day^{-1}	d_2	0.01 day^{-1}
δ_1	0.5 day^{-1}	δ_2	0.3 day^{-1}
a_1	0.3 day^{-1}	a_2	0.1 day^{-1}
q_1	0.5	q_2	0.5
ς_1	10 virus mm^{-3}	ς_2	10 virus mm^{-3}
k_1	2	k_2	2
N_{y_1}	9 virus cells^{-1}	N_{y_2}	4 virus cells^{-1}
M_{z_1}	4 virus cells^{-1}	M_{z_2}	1 virus cells^{-1}
ρ_1	0.1 cells^{k_1} mm$^{-3k_1}$	ρ_2	0.1 cells^{k_1} mm$^{-3k_1}$
m_1	1 day^{-1}	m_2	1 day^{-1}
n_1	1 day^{-1}	n_2	1 day^{-1}
r_1	1 day^{-1}	r_2	1 day^{-1}
χ	0.5	u	1 day^{-1}
b	1 cells mm^{-3}day^{-1}	p	6 day^{-1}
c	1 day^{-1}	ε	Varied
τ_e	Varied		

this case, the virus particles will be cleared from the body.

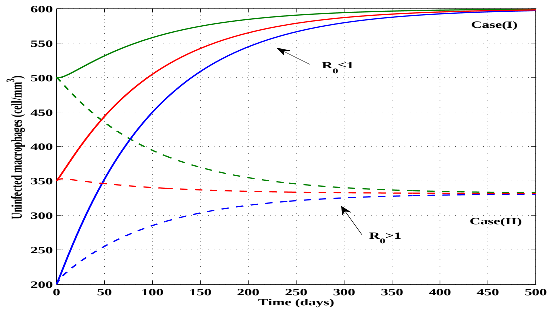
Case (II): In this case, we choose $\varepsilon = 0$ then we calculate $R_0 = 2.13 > 1$. Consequently, the system has two equilibria E_0 and E_1, and based on Theorem 2, E_1 is GAS. From Figure 1 we can see that, our simulation results are consistent with the theoretical results of Theorem 2. We observe that, the state of the system converge the endemic equilibrium $E_1 = (571.06, 332.13, 4.25, 4.43, 7.08, 13.28, 11.58, 1.93)$. for the three initial conditions IC1-IC3. In this case, the infection becomes chronic.

3.2 Effect of the drug efficacy on the dynamical behavior of the system

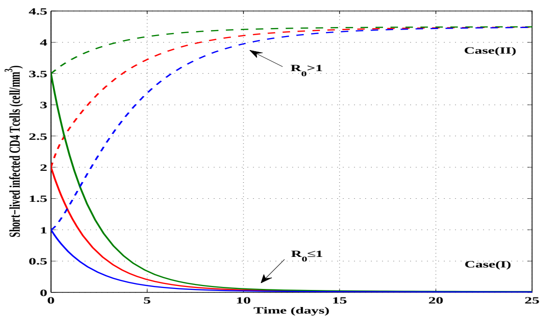
In this case, we will fix the delay parameter $\tau_e = 0.01$ day^{-1}. Figures 2 shows the effect of the parameter ε on the evolution of the uninfected CD4$^+$T cells and macrophages, short-lived infected cells, long-lived chronically infected cells, free virus particles and B cells. When there is no treatment i.e. $\varepsilon = 0$, the trajectory of the system tends to the endemic equilibrium $E_1 = (571.06, 332.13, 4.25, 4.43, 7.08, 13.28, 11.58, 1.93)$. Since E_1 exists, then according to Theorem 2, E_1 is GAS. We can see from the figures that, our simulation results are consistent with the theoretical results of Theorem 2. We observe that, as the drug efficacy is increased from $\varepsilon = 0$ to $\varepsilon = 0.8$, E_1 is still exists and is GAS, moreover, the concentrations of the uninfected CD4$^+$T cells and macrophages are increasing, while the concentrations of the short-lived infected cells, long-lived chronically infected cells, free virus particles and B cells are decreasing. When $\varepsilon = 0.98$, the basic reproduction number is given by $R_0 = 0.73 < 1$, then according to Theorem 1, the disease-free equilibrium E_0 is GAS. We can see that, the concentrations of uninfected CD4$^+$T cells and macrophages are increasing and converge to their normal values $\frac{\lambda_1}{d_1} = 1000$ cells mm^{-3}, $\frac{\lambda_2}{d_2} = 600$ cells mm^{-3}, respectively, while the concentrations of short-lived infected cells, long-lived chronically infected cells, free viruses and B cells are decaying and tend to zero. It means that, the numerical results are also compatible with the results of Theorem 1. In this case, the treatment with such drug



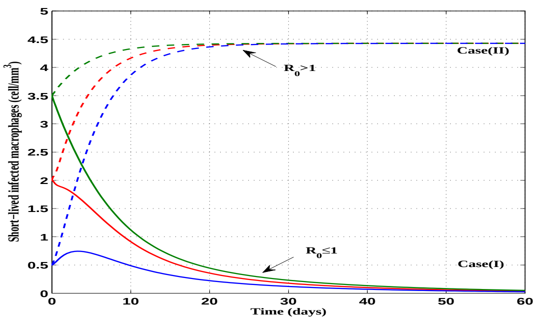
(a) Uninfected CD4$^+$T cells



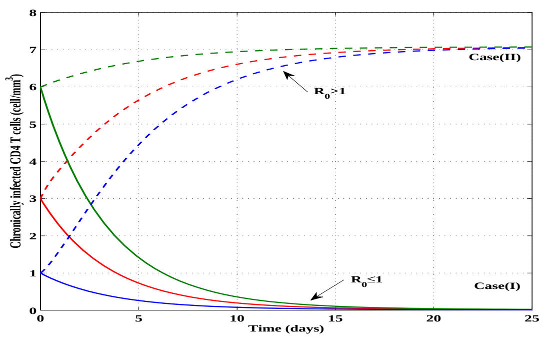
(b) Uninfected macrophages



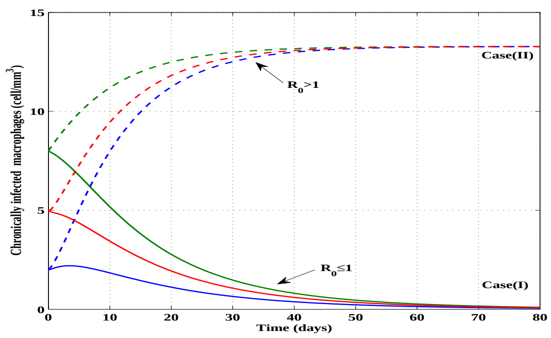
(c) Short-lived infected CD4$^+$T cells



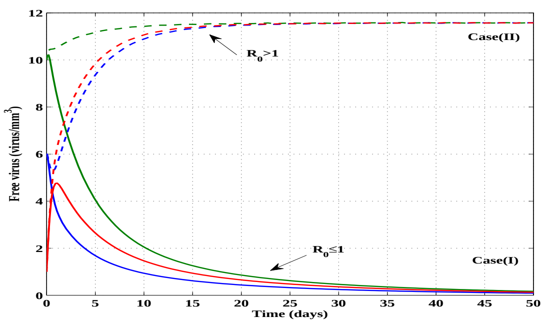
(d) Short-lived infected macrophages



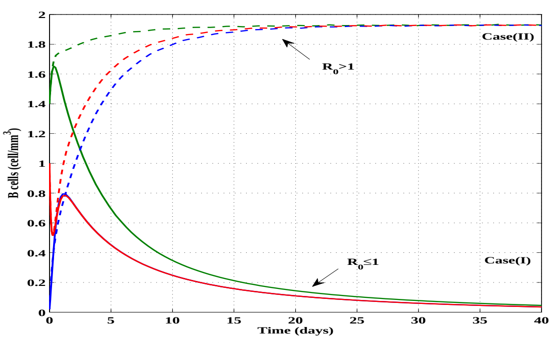
(e) Chronically infected CD4$^+$T cells



(f) Chronically infected macrophages



(g) Free virus



(h) B cells

Figure 1: The evolution of the system state in different initial conditions for model (16) -(20).

efficacy succeeded to eliminate the viruses from the blood.

3.3 Effect of the time delay on the dynamical behavior of the system

In this case, we will fix the drug efficacy $\varepsilon = 0.2$. Figure 3 shows the effect of the parameter τ_e on the evolution of the state variables of the system. When $\tau_e = 0.01$, the trajectory of the system tends to the endemic equilibrium $E_1 = (684.2, 378.23, 3.13, 3.66, 5.2, 10.9, 9.75, 1.62)$. Then E_1 exists and according to Theorem 2 E_1 is GAS. It means that, both the numerical and theoretical results of Theorem 2 are consistent. One can see that, as the time delay is increased from $\tau_e = 0.01$ to $\tau_e = 0.7$, E_1 is still exists and is GAS, in addition, the concentrations of the uninfected CD4$^+$T cells and macrophages are increased, while the concentrations of the short-lived infected cells, long-lived chronically infected cells, free virus particles and B cells are decreased. When $\tau_e = 1$, the basic reproduction number is given by $R_0 = 0.71 < 1$, then according to Theorem 1, E_0 is GAS. We can see that, the concentrations of uninfected CD4$^+$T cells and macrophages are increasing and converge to their normal values $\frac{\lambda_1}{d_1} = 1000$ cells mm^{-3}, $\frac{\lambda_2}{d_2} = 600$ cells mm^{-3}, respectively, while the concentrations of short-lived infected cells, long-lived chronically infected cells, free viruses and B cells are decaying and tend to zero. Figure 3 shows that the numerical results are also compatible with the results of Theorem 1. This shows the effect of time delay on preventing the disease from development.

3.4 Effects of the drug efficacy and the delay on the basic reproduction number:

Figure 4 shows the effect of the parameters ε and τ_e on the basic reproduction number R_0. We note that, $R_0 > 1$ for small values of ε or τ_e, and the endemic equilibrium exists and is GAS, while the disease-free equilibrium is unstable. When $R_0 = 1$ (which is a bifurcation point), both disease-free equilibrium and endemic equilibrium coincide and it is GAS. Moreover, as ε or τ_e is increasing, R_0 is decreasing until it becomes less than one, which makes the endemic equilibrium does not exists and the disease-free equilibrium is GAS. From a biological point of view, the intracellular delay plays a similar role as antiviral treatment in eliminating the virus. We observe that, even if there is no treatment i.e. $\varepsilon = 0$, sufficiently large delay suppress viral replication and clear the virus. This give us some suggestions on new drugs to prolong the increase the intracellular delay period.

3.5 Effects of two types of target cells on the dynamics and controls of HIV infection

In this subsection, we show the effects of two types of target cells on the dynamics and controls of HIV infection. We note that if $R_0 < 1$, then it is sure that $R_{01} < 1$ and $R_{02} < 1$. But if one neglect the presence of the macrophages in the HIV dynamics model, then the HIV model system (16) -(20) will become

$$\dot{x}_1(t) = \lambda_1 - d_1 x_1(t) - \frac{(1-\varepsilon)\bar{\beta}_1 x_1^{k_1}(t) v(t)}{(x_1^{k_1}(t) + \rho_1)(v(t) + \varsigma_1)}, \tag{22}$$

$$\dot{y}_1(t) = (1-q_1)e^{-m_1\tau_1} \frac{(1-\varepsilon)\bar{\beta}_1 x_1^{k_1}(t-\tau_1) v(t-\tau_1)}{(x_1^{k_1}(t-\tau_1) + \rho_1)(v(t-\tau_1) + \varsigma_1)} - \delta_1 y_1(t), \tag{23}$$

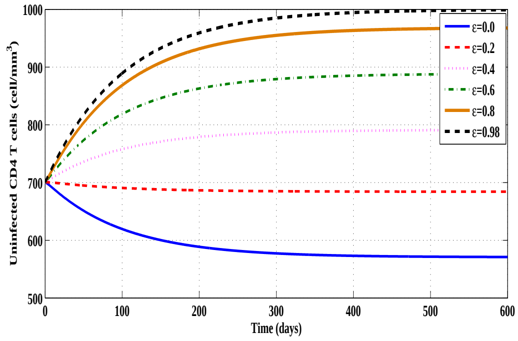
$$\dot{z}_1(t) = q_1 e^{-m_1\tau_1} \frac{(1-\varepsilon)\bar{\beta}_1 x_1^{k_1}(t-\tau_1) v(t-\tau_1)}{(x_1^{k_1}(t-\tau_1) + \rho_1)(v(t-\tau_1) + \varsigma_1)} - a_1 z_1(t), \tag{24}$$

$$\dot{v}(t) = N_{y_1}\delta_1 e^{-n_1\kappa_1} y_1(t-\kappa_1) + M_{z_1} a_1 e^{-r_1\omega_1} z_1(t-\omega_1) - uv(t) - bv(t)w(t), \tag{25}$$

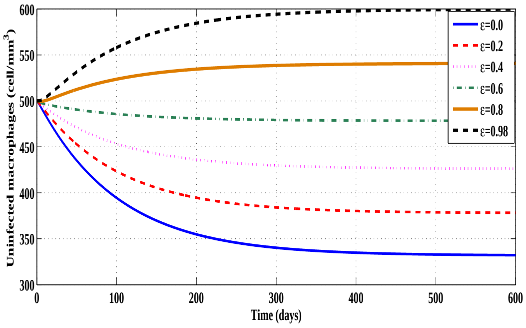
$$\dot{w}(t) = cv(t) - pw(t). \tag{26}$$

The basic reproduction number of model (22)-(26) is given by

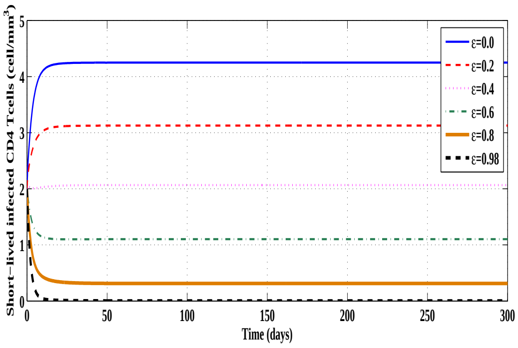
$$R_{01} = \frac{((1-q_1)e^{-n_1\kappa_1} N_{y_1} + q_1 e^{-r_1\omega_1} M_{z_1})}{u} \frac{e^{-m_1\tau_1}(1-\varepsilon)\bar{\beta}_1 (x_1^0)^{k_1}}{\varsigma_1((x_1^0)^{k_1} + \rho_1)}.$$



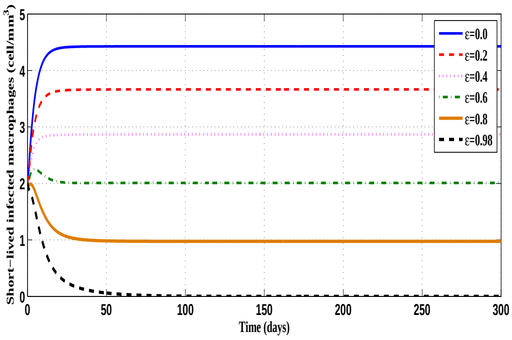
(a) Uninfected CD4$^+$T cells



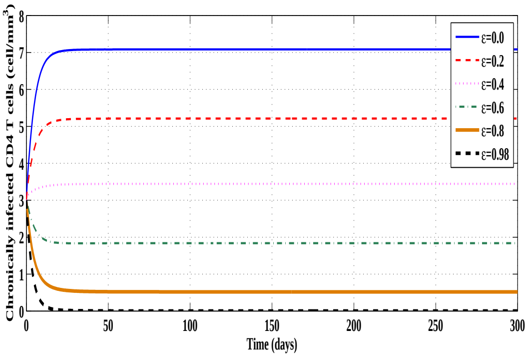
(b) Uninfected macrophages



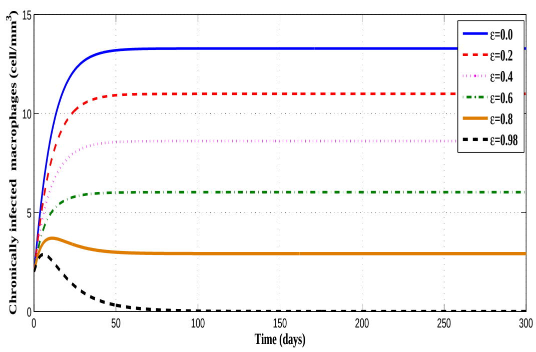
(c) Short-lived infected CD4$^+$T cells



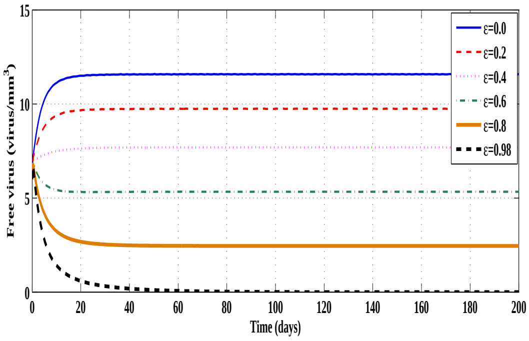
(d) Short-lived infected macrophages



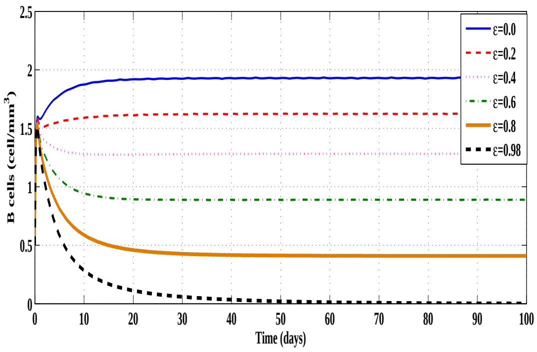
(e) Chronically infected CD4$^+$T cells



(f) Chronically infected macrophages

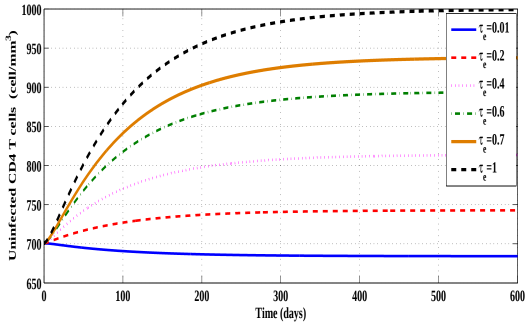


(g) Free virus

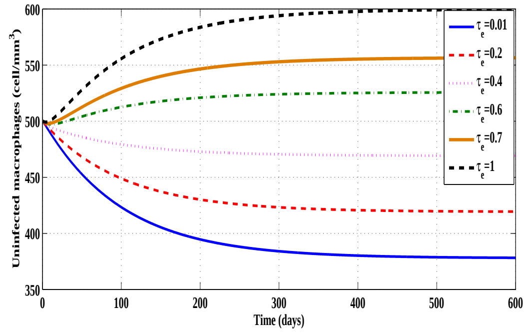


(h) B cells

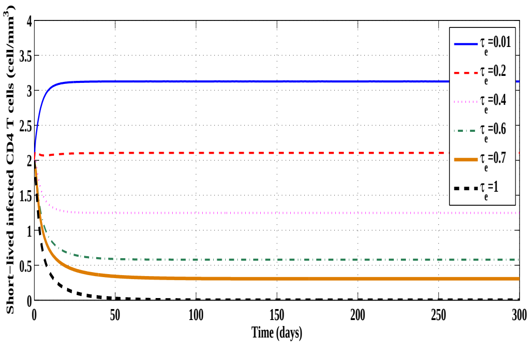
Figure 2: The evolution of the system state with different values of drug efficacy for model (16) -(20).



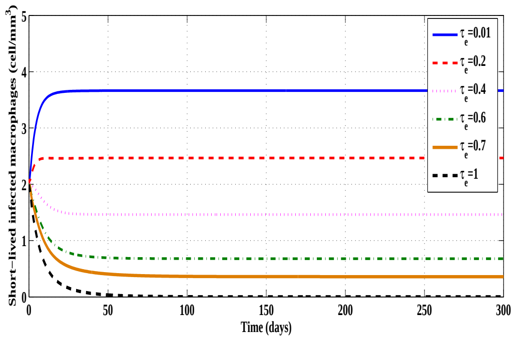
(a) Uninfected CD4$^+$T cells



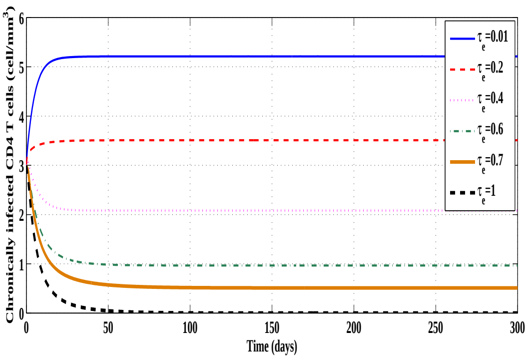
(b) Uninfected macrophages



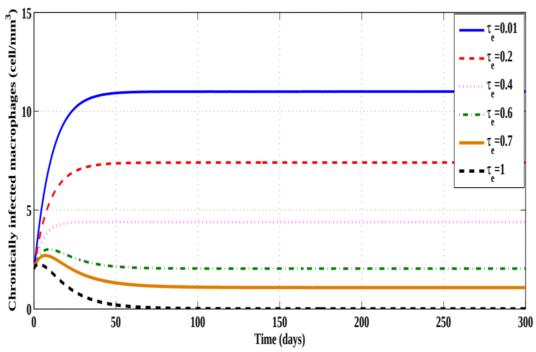
(c) Short-lived infected CD4$^+$T cells



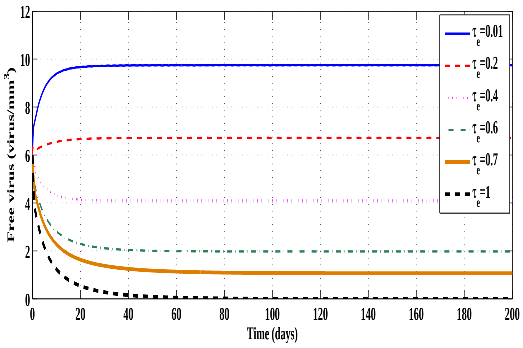
(d) Short-lived infected macrophages



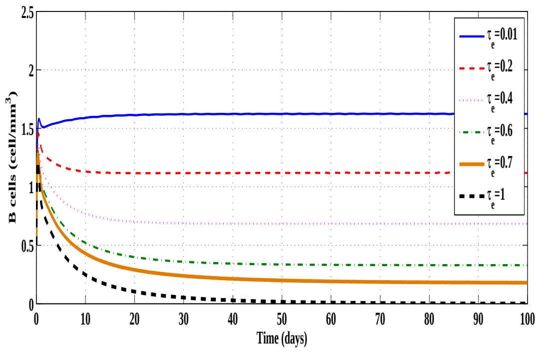
(e) Chronically infected CD4$^+$T cells



(f) Chronically infected macrophages



(g) Free virus



(h) B cells

Figure 3: The evolution of the system state with different values of delayed for model (16) -(20).

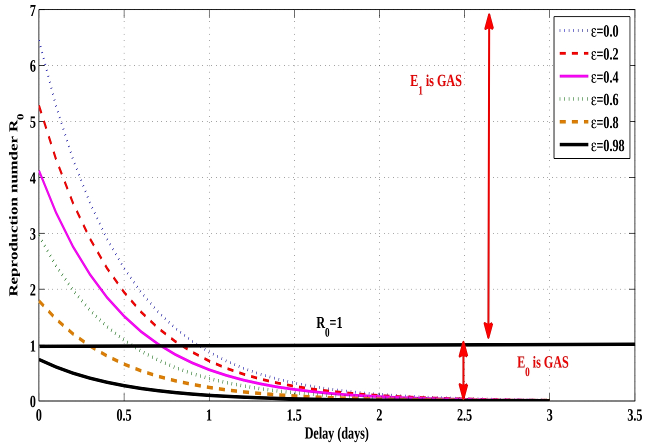


Figure 4: Effects of the drug efficacy and delays on the basis reproduction number of model (3)-(7)

Now we show that there is a number of parameter values for which $R_{01} \le 1$, but $R_0 > 1$, and in such cases the solutions of system (22)-(26) tend to E_0 (in $\mathbb{R}^5_{\ge 0}$) as $t \to \infty$, while those of (16) -(20) tend to E_1 (in $\mathbb{R}^8_{\ge 0}$) as $t \to \infty$. We calculate the critical drug efficacy for system (16) -(20), E_0 is GAS when $R_0 \le 1$ i.e.

$$\varepsilon_1^{crit} \le \varepsilon < 1, \qquad \varepsilon_1^{crit} = \max\left\{0, \frac{\overline{R}_0 - 1}{\overline{R}_{01} + \chi \overline{R}_{02}}\right\},$$

where $\overline{R}_0 = R_0 \mid_{\varepsilon=0}$ and $\overline{R}_{0i} = R_{0i} \mid_{\varepsilon=0}$, $i = 1, 2$.
For system (22)-(26), E_0 is GAS when $R_{01} \le 1$ i.e.

$$\varepsilon_2^{crit} \le \varepsilon < 1, \quad \varepsilon_2^{crit} = \max\left\{0, \frac{\overline{R}_{01} - 1}{\overline{R}_{01}}\right\}.$$

Clearly, $\varepsilon_1^{crit} > \varepsilon_2^{crit}$. Then, if one design treatment with drug efficacy $\varepsilon_2^{crit} \le \varepsilon \le \varepsilon_1^{crit}$, then E_0 is GAS for system (22)-(26) but unstable for system (16) -(20). Using the data in Table 1 and $\tau_e = 0.01$, we have $\varepsilon_1^{crit} = 0.93$ and $\varepsilon_2^{crit} = 0.80$. Let us choose $\varepsilon = 0.88$, then $R_{01} \mid_{\varepsilon=0.88} = 0.62 < 1$, but $R_0 \mid_{\varepsilon=0.88} = 1.31 > 1$. Therefore, more accurate treatment can be designed using the model (16) -(20) than those designed using model (22)-(26). Figure 5 shows the effect of two target cells on dynamics and control of HIV infection. We observe that, if we choose $\varepsilon = 0.88$, then the trajectory of model (16) -(20 tends to the infection-free equilibrium $E_0 = (1000, 0, 0, 0, 0, 0)$, while the trajectory of model (16) -(20) tends to the endemic equilibrium $E_1 = (990.54, 573.24, 0.1, 0.4, 0.15, 1.31, 1.04, 0.17)$.

3.6 Effect of long-lived chronically infected cells on the dynamics and controls of HIV infection

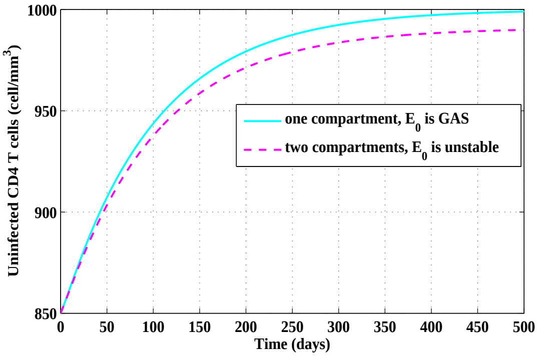
To show the effect of the presence of long-lived chronically infected cells on the dynamics and controls of HIV infection, we write the HIV model without long-lived chronically infected cells as:

$$\dot{x}_i(t) = \lambda_i - d_i x_i(t) - \frac{\beta_i x_i^{k_i}(t) v(t)}{(x_i^{k_i}(t) + \rho_i)(v(t) + \varsigma_i)}, \tag{27}$$

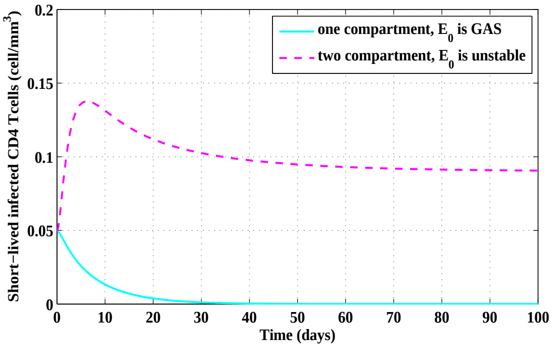
$$\dot{y}_i(t) = \frac{e^{-m_i \tau_i} \beta_i x_i^{k_i}(t - \tau_i) v(t - \tau_i)}{(x_i^{k_i}(t - \tau_i) + \rho_i)(v(t - \tau_i) + \varsigma_i)} - \delta_i y_i(t), \tag{28}$$

$$\dot{v}(t) = \sum_{i=1}^{2} N_{y_i} \delta_i e^{-n_i \kappa_i} y_i(t - \kappa_i) - uv(t) - bv(t)w(t), \tag{29}$$

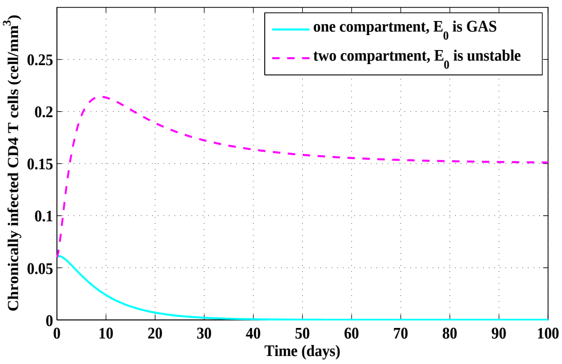
$$\dot{w}(t) = cv(t) - pw(t). \tag{30}$$



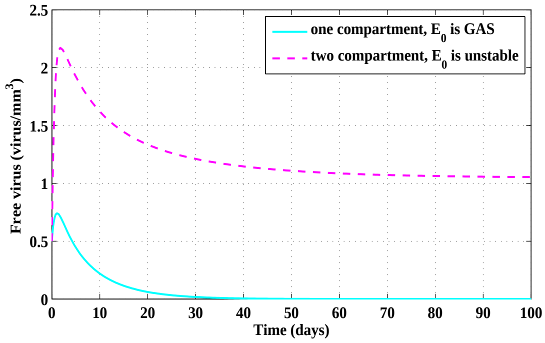
(a) Uninfected CD4$^+$T cells for model (16)-(20) and model (22)-(26).



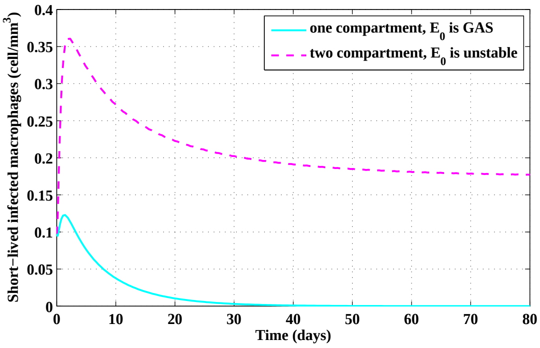
(b) Short-lived CD4$^+$T cells for model (16) -(20) and ((22)-(26).



(c) Chronically infected CD4$^+$T cells for model (16) -(20) and (22)-(26).



(d) Free virus for model (16) -(20)) and (22)-(26).



(e) B cells for model (16) -(20) and (22)-(26).

Figure 5: Effect of two types of target cells on the dynamics and controls of HIV infection

The basic reproduction number for system (27)-(30) is given by

$$\widetilde{R}_0 = \sum_{i=1}^{2} \widetilde{R}_{0i} = \sum_{i=1}^{2} \frac{e^{-n_i\kappa_i} e^{-m_i\tau_i} N_{y_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)},$$

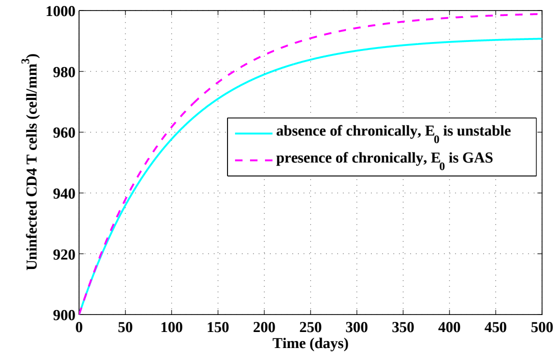
where $\widetilde{R}_0 = R_0 \mid_{q_1=q_2=0}$. Since $e^{-n_i\kappa_i} N_{y_i} > e^{-r_i\omega_i} M_{z_i}$, $i = 1, 2$, then we have

$$\begin{aligned}
R_0 &= \sum_{i=1}^{2} \frac{((1-q_i)e^{-n_i\kappa_i} N_{y_i} + q_i e^{-r_i\omega_i} M_{z_i})}{u} \frac{e^{-m_i\tau_i} \beta_i (x_i^0)^{k_i}}{\varsigma_i((x_i^0)^{k_i} + \rho_i)} \\
&= \sum_{i=1}^{2} \frac{e^{-n_i\kappa_i} e^{-m_i\tau_i} N_{y_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)} - \sum_{i=1}^{2} \frac{(e^{-n_i\kappa_i} N_{y_i} - e^{-r_i\omega_i} M_{z_i}) q_i e^{-m_i\tau_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)} \\
&= \widetilde{R}_0 - \sum_{i=1}^{2} \frac{(e^{-n_i\kappa_i} N_{y_i} - e^{-r_i\omega_i} M_{z_i}) q_i e^{-m_i\tau_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)} < \widetilde{R}_0.
\end{aligned}$$

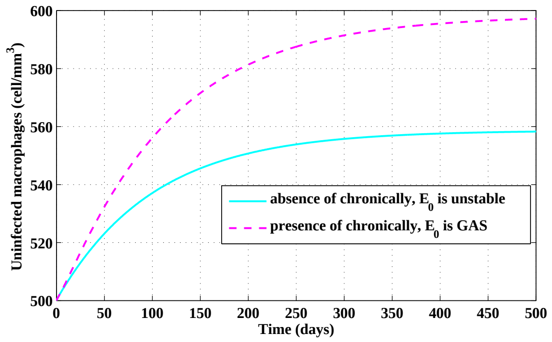
Therefore even without the incorporation of treatment, the long-lived infected cell population decreases the basic reproduction number of the system. Now, we calculate the critical drug efficacy needed in order stabilize the system around the infection-free equilibrium. The critical drug efficacy for systems (16) -(20) and (27)-(30) is given by ε_1^{crit} and ε_3^{crit}, respectively, where,

$$\varepsilon_3^{crit} = \max\left\{0, \frac{\widehat{R}_0 - 1}{\widehat{R}_0}\right\}$$

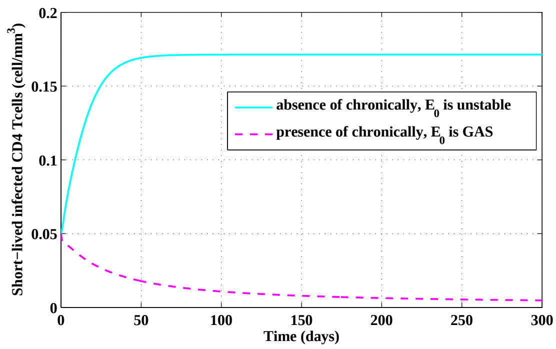
where $\widehat{R}_0 = \widetilde{R}_0 \mid_{\varepsilon=0} = R_0 \mid_{\varepsilon=q_1=q_2=0}$. Using the data given in Table 1 with $\tau_e = 0.01$, we have $\varepsilon_1^{crit} = 0.93$ and $\varepsilon_3^{crit} = 0.99$. Therefore the drug efficacy necessary to drive the system to the infection-free equilibrium is actually less for system (16) -(20) than that for system (27)-(30). Figure 6 shows the effect of chronically infected cells on dynamic and control of HIV infection. We observed that, if we choose $\varepsilon = 0.93$, then the trajectory of model (16)-(20) tends to infection-free equilibrium $E_0 = (1000, 600, 0, 0, 0, 0, 0, 0)$, while in the model (27)-(30), $\widetilde{R}_0 = 1.54 > 1$ and the trajectory tends to the endemic equilibrium with humoral immunity $E_1 = (990.99, 558.53, 0.17, 1.36, 1.83, 0.3)$.



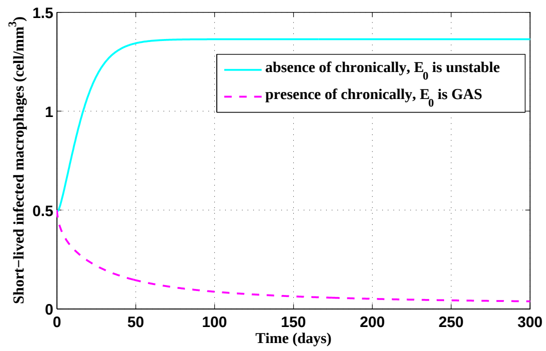
(a) Uninfected CD4$^+$T cells for model (16)-(20) and (27)-(30).



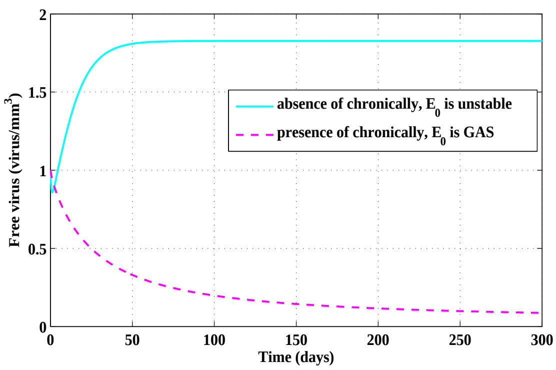
(b) Uninfected macrophages for model (16) -(20) and (27)-(30).



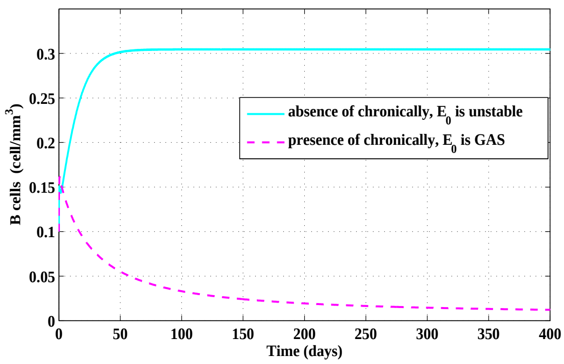
(c) Short-lived CD4$^+$T cells for model (16) -(20) and (27)-(30).



(d) Short-lived macrophages for model (16) -(20) and (27)-(30).



(e) Free virus for model (16) -(20) and (27)-(30).



(f) B cells for model (16) -(20) and (27)-(30).

Figure 6: Effect of long-lived chronically infected cells on the dynamics and controls of HIV infection

4 Acknowledgment

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COMPOSITION OPERATORS ON DIRICHLET-TYPE SPACES

LIU YANG, YECHENG SHI*

ABSTRACT. In this note, motivated by [8], under the conditions of weighted function in [10], we characterize bounded and compact composition operator on Dirichlet-type spaces D_K. We also give an equavalent charaterization of composition operator on D_K, if the composition operator on D_K spaces is Hilbert-Schmidt.

Keywords: D_K spaces; composition operators; Hilbert-Schmidt

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. Let $K : [0, \infty) \to [0, \infty)$ be a right-continuous and nondecreasing function. The Dirichlet-type spaces D_K, consists of those functions $f \in H(\mathbb{D})$, such that

$$\|f\|^2_{D_K} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 K(1 - |z|^2) dA(z) < \infty.$$

When $K(t) = t^\alpha$, $0 < \alpha < 1$, it give the classical Dirichlet-type space D_α. For more informations on D_α and D_K spaces, we refer to [1], [3], [12], [19], [25].

Let φ be a holomorphic self-map of $\mathbb{D}$. The composition operator C_φ on D_K is defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in D_K.$$

There are many papers study composition operator, we refer to [4], [13], [14], [15], [17], [20], [21], [22], [24], [26]. Recently, Kellay and Lefèvre using Nevanlinna counting function, characterize bounded and compact composition operator on Dirichlet-type space D_K under certain conditions in [13]. Later, Pau and Pèrez studied the essential norm and closed ranged of composition operator on D_α in [17].

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In this paper, motivated by [8], we generalize Theorem 2.2 of [8] to D_K spaces. We also give a characterizations of boundedness and compactness of composition operator C_φ on D_K spaces by φ^n. Furthermore, equivalent characterizations of composition operator on D_K spaces belong to Hilbert-Schmidt was gave.

Throughout this paper, suppose that $K : [0,\infty) \to [0,\infty)$ is a right-continuous and nondecreasing function. Satisfying

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1.1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{1.2}$$

where

$$\varphi_K(s) = \sup_{0\le t\le 1} K(st)/K(t), \quad 0 < s < \infty.$$

To learn more about weight function K, we refer to [2], [3], [9], [10] and [16].

Throughout this paper, for two functions f and g, $f \asymp g$ means that $g \lesssim f \lesssim g$, that is, there are positive constants C_1 and C_2 depend on K and index s, α, such that $C_1 g \le f \le C_2 g$.

2. Auxiliary results

Before to proof, we need to know some results. The following lemma can be found in Lemma 2.1 of [2].

Lemma 1. *Let (1.1) and (1.2) hold for K. If $2 - \frac{\alpha}{2} < s < 1 + c$, then*

$$\int_{\mathbb{D}} \frac{K\left(1 - |\sigma_a(w)|^2\right)}{(1-|w|^2)^s |1-\overline{w}z|^\alpha} dA(w) \lesssim \frac{K\left(1-|\sigma_a(z)|^2\right)}{(1-|z|^2)^{s+\alpha-2}}$$

for all $a, z \in \mathbb{D}$, where $\sigma_a(z) = \frac{z-a}{1-\bar{a}z}$.

Lemma 2. *Suppose that K satisfies (1.1) and (1.2). Then*

$$1 + \sum_{n=1}^\infty \frac{n+1}{K(\frac{1}{n+1})} t^n \asymp \frac{1}{(1-t)^2 K(1-t)}$$

for all $0 \le t < 1$.

Proof. Without loss of generality, we can assume $1/3 < t < 1$, otherwise, it obvious. Make change of variables $y = \frac{1}{x}$, an easy computation gives

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n+1}{K(\frac{1}{n+1})}t^n &\asymp \sum_{n=1}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}}\frac{t^{\frac{1}{x}}}{x^3K(x)}dx\\ &\asymp \int_0^1\frac{t^{\frac{1}{x}}}{x^3K(x)}dx \asymp \int_1^{\infty}\frac{yt^y}{K(\frac{1}{y})}dy.\end{aligned}$$

Let $y = \frac{\gamma}{-\ln t}$. We can deduce that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n+1}{K(\frac{1}{n+1})}t^n &\asymp \frac{1}{(\ln\frac{1}{t})^2}\int_{-\ln t}^{\infty}\frac{\gamma e^{-\gamma}}{K(\frac{1}{\gamma}\ln\frac{1}{t})}d\gamma\\ &= \frac{1}{(\ln\frac{1}{t})^2K(\ln\frac{1}{t})}\int_{-\ln t}^{\infty}\frac{\gamma e^{-\gamma}K(\ln\frac{1}{t})}{K(\frac{1}{\gamma}\ln\frac{1}{t})}d\gamma\\ &\lesssim \frac{1}{(1-t)^2K(1-t)}\int_{-\ln t}^{\infty}\gamma e^{-\gamma}\varphi_K(\gamma)d\gamma.\end{aligned}$$

By [10], under conditions (1.1) and (1.2), there exists an enough small $c > 0$ only depending on K such that

$$\varphi_K(s) \lesssim s^c,\ 0 < s \le 1$$

and

$$\varphi_K(s) \lesssim s^{1-c},\ s \ge 1.$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n+1}{K(\frac{1}{n+1})}t^n &\lesssim \frac{1}{(1-t)^2K(1-t)}\int_{-\ln t}^{\infty}\gamma e^{-\gamma}\varphi_K(\gamma)d\gamma\\ &\lesssim \frac{1}{(1-t)^2K(1-t)}\left(\int_0^{\infty}e^{-\gamma}\gamma^{2-c}d\gamma + \int_0^{\infty}e^{-\gamma}\gamma^{1+c}d\gamma\right)\\ &\asymp \frac{1}{(1-t)^2K(1-t)}\left(\Gamma(3-c)+\Gamma(2+c)\right),\end{aligned}$$

where $\Gamma(.)$ is the Gamma function. It follows that

$$1+\sum_{n=1}^{\infty}\frac{n+1}{K(\frac{1}{n+1})}t^n \lesssim \frac{1}{(1-t)^2K(1-t)}.$$

Conversely, since K is nondecreasing, we deduce that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n+1}{K(\frac{1}{n+1})}t^n &\asymp \frac{1}{(\ln\frac{1}{t})^2K(\ln\frac{1}{t})}\int_{-\ln t}^{\infty}\frac{\gamma e^{-\gamma}K(\ln\frac{1}{t})}{K(\frac{1}{\gamma}\ln\frac{1}{t})}d\gamma\\ &\gtrsim \frac{1}{(1-t)^2K(1-t)}\int_{\ln 2}^{\infty}\frac{\gamma e^{-\gamma}K(\ln\frac{1}{t})}{K(\frac{1}{\gamma}\ln\frac{1}{t})}d\gamma\\ &\gtrsim \frac{1}{(1-t)^2K(1-t)}\int_{\ln 2}^{\infty}\gamma e^{-\gamma}d\gamma\\ &\asymp \frac{1}{(1-t)^2K(1-t)}.\end{aligned}$$

The proof is completed. □

The next lemma can be found in Theorem 5 of [23].

Lemma 3. *Let (1.2) hold for K. Then for any $\alpha > 0$ and $0 \le \beta < 1$, we have*

$$\int_0^1 r^{\alpha-1}(\log\frac{1}{r})^{-\beta}K(\log\frac{1}{r})dr \asymp \left(\frac{1-\beta}{\alpha}\right)^{1-\beta}K\left(\frac{1-\beta}{\alpha}\right).$$

3. Boundedness and compactness

In this section, motivated by [8], we discuss the boundedness and compactness of compostion operators by a general computation.

Theorem 1. *Suppose that (1.1) and (1.2) hold for K, $s \ge 0$. Suppose $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi \in D_K$. Then C_φ is bounded on D_K if and only if*

$$\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}}K(1-|z|^2)dA(z) < \infty.$$

Proof. Let

$$F_a(z) = \frac{(1-|a|^2)^{1+s}}{\sqrt{K(1-|a|^2)}}\frac{1}{(1-\overline{a}z)^{1+s}}, \quad s \ge 0.$$

Using Lemma 1, it is easy to check that $F_a \in D_K$. If C_φ is bounded on D_K, then $\|C_\varphi(F_a)\|_{D_K} < \infty$, that is,

$$\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}}K(1-|z|^2)dA(z) < \infty.$$

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On the other hand, we know that for any pseudohyperbolic discs $D(z,r)$, we have $1-|w| \asymp 1-|z| \asymp |1-\overline{w}z|$, for any $w \in D(z,r)$ (see [27, page 69]). Let $f \in D_K$. Applying sub-mean-property to $|f'|^2$, we have

$$\begin{aligned}|f'(z)|^2 &\leq \int_{D(z,r)} \frac{|f'(w)|^2}{|1-\overline{w}z|^2} dA(w)\\ &\asymp \int_{D(z,r)} \frac{|f'(w)|^2(1-|w|^2)^{2+2s}}{|1-\overline{w}z|^{4+2s}} dA(w)\\ &\lesssim \int_{\mathbb{D}} \frac{|f'(w)|^2(1-|w|^2)^{2+2s}}{|1-\overline{w}z|^{4+2s}} dA(w).\end{aligned}$$

Therefore, we get

$$\int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 K(1-|z|^2) dA(z)$$

$$\begin{aligned}&\leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{|f'(w)|^2}{|1-\overline{w}\varphi(z)|^{4+2s}} (1-|w|^2)^{2+2s} dA(w) \right) |\varphi'(z)|^2 K(1-|z|^2) dA(z)\\ &\leq \left(\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^{2+2s}}{K(1-|w|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{w}\varphi(z)|^{4+2s}} K(1-|z|^2) dA(z) \right)\\ &\qquad \times \int_{\mathbb{D}} |f'(w)|^2 K(1-|w|^2) dA(w) < \infty.\end{aligned}$$

The proof is completed. □

Theorem 2. *Suppose that (1.1) and (1.2) hold for K, $s \geq 0$. Suppose $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi \in D_K$. Then C_φ is compact on D_K if and only if*

$$\lim_{|a|\to 1} \frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}} K(1-|z|^2) dA(z) = 0.$$

Proof. Let

$$G(w) = \frac{(1-|w|^2)^{2+2s}}{K(1-|w|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{w}\varphi(z)|^{4+2s}} K(1-|z|^2) dA(z).$$

Let $\{f_k\}_{k=1}^\infty$ be a bounded sequence of D_K such that $f_k \to 0$ weakly. Therefore, $f_k' \to 0$ uniformly on compact sets. From the proof of Theorem 1 and

dominated convergence theorem, when $k \to \infty$, and $r \to 1$, it follows that

$$\begin{aligned}
&\|C_\varphi(f_k)\|_{D_K}^2 - |f_k(\varphi(0))|^2 \\
\le &\int_{\mathbb{D}} |f_k'(w)|^2 G(w) K(1-|w|^2) dA(w) \\
\le &\int_{r\mathbb{D}} |f_k'(w)|^2 G(w) K(1-|w|^2) dA(w) \\
&\qquad + \int_{\mathbb{D}\setminus r\mathbb{D}} |f_k'(w)|^2 G(w) K(1-|w|^2) dA(w) \to 0.
\end{aligned}$$

Thus, C_φ is compact.

Conversely, if C_φ is compact, let $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{D}$, $|a_k| \to 1$,

$$F_{a_k}(z) = \frac{(1-|a_k|^2)^{1+s}}{\sqrt{K(1-|a_k|^2)}} \frac{1}{(1-\overline{a_k}z)^{1+s}}.$$

Then, it is easy to verify that $F_{a_k} \to 0$ uniformly on compact sets. Thus, $\|C_\varphi(F_{a_k})\|_{D_K} \to 0$ as $k \to \infty$. The proof is completed. □

4. φ^n-TYPE CHARACTERIZATIONS

In [24], Wulan, Zheng and Zhu gave an interesting characterizations of compostion operators C_φ by φ^n. In this section, we are going to give an analogy results on D_K spaces.

Theorem 3. *Let (1.1) and (1.2) hold for K. Suppose $\varphi \in D_K$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $C_\varphi : D_K \to D_K$. Then*

(1) *If*

$$\sup_n \frac{1}{K(\frac{1}{n})} \|\varphi^n\|_{D_K}^2 < \infty,$$

then C_φ is bounded;

(2) *If C_φ is bounded, then*

$$\sup_n \frac{1}{nK(\frac{1}{n})} \|\varphi^n\|_{D_K}^2 < \infty.$$

Proof. (1). Let $a, z \in \mathbb{D}$ and $s > 0$. Since

$$|1 - \bar{a}\varphi(z)| \ge 1 - |a||\varphi(z)|$$

and

$$\frac{1}{(|1-|a||\varphi(z)|)^{4+2s}} \asymp \frac{1}{(|1-|a|^2|\varphi(z)|^2)^{4+2s}}.$$

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Note that

$$\frac{1}{(|1-|a|^2|\varphi(z)|^2)^{4+2s}} \asymp \sum_{n=0}^{\infty} \frac{\Gamma(n+4+2s)}{n!\Gamma(4+2s)}|a|^{2n}|\varphi(z)|^{2n},$$

it follows that

$$\begin{aligned}
&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}}K(1-|z|^2)dA(z)\\
\lesssim&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{(1-|a||\varphi(z)|)^{4+2s}}K(1-|z|^2)dA(z)\\
\asymp&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\sum_{n=0}^{\infty}\frac{\Gamma(n+4+2s)}{n!\Gamma(4+2s)}|a|^{2n}|\varphi(z)|^{2n}|\varphi'(z)|^2K(1-|z|^2)dA(z).
\end{aligned}$$

By Stirling formula, we get

$$\frac{\Gamma(n+4+2s)}{n!\Gamma(4+2s)} \sim n^{3+2s},\ n \to \infty.$$

Therefore,

$$\begin{aligned}
&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}}K(1-|z|^2)dA(z)\\
\lesssim&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\sum_{n=0}^{\infty}n^{3+2s}|a|^{2n}\int_{\mathbb{D}}|\varphi(z)|^{2n}|\varphi'(z)|^2K(1-|z|^2)dA(z)\\
\leq&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\sum_{n=0}^{\infty}(n+1)^{3+2s}|a|^{2n}\int_{\mathbb{D}}|\varphi(z)|^{2n}|\varphi'(z)|^2K(1-|z|^2)dA(z)\\
\leq&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\sum_{n=0}^{\infty}(n+1)^{1+2s}|a|^{2n}\|\varphi^n\|^2_{D_K}\\
\lesssim&\sup_n\frac{1}{K(\frac{1}{n})}\|\varphi^n\|^2_{D_K}\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\sum_{n=0}^{\infty}(n+1)^{1+2s}K(\frac{1}{n})|a|^{2n}.
\end{aligned}$$

Following the proof of Lemma 2, we have

$$\sum_{n=0}^{\infty}(n+1)^{1+2s}K(\frac{1}{n})|a|^{2n} \asymp \frac{K(1-|a|^2)}{(1-|a|^2)^{2+2s}}.$$

Thus,

$$\begin{aligned}
&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}}K(1-|z|^2)dA(z)\\
\lesssim&\sup_n\frac{1}{K(\frac{1}{n})}\|\varphi^n\|^2_{D_K}.
\end{aligned}$$

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Hence, by Theorem 1, we prove (1).

(2). Suppose that C_φ is bounded on D_K. Let $f_n(z) = z^n/\|z^n\|^2_{D_K}$. Then, we have $\|f_n\|^2_{D_K} = 1$. An easy computation gives,

$$\infty > \|C_\varphi f_n\|^2_{D_K} = \frac{\|\varphi^n\|^2_{D_K}}{\|z^n\|^2_{D_K}} \gtrsim \frac{1}{nK(\frac{1}{n})}\|\varphi^n\|^2_{D_K}.$$

The last inequality is deduced by Lemma 3. The proof is completed. □

Theorem 4. *Let (1.1) and (1.2) hold for K. Suppose $\varphi \in D_K$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $C_\varphi : D_K \to D_K$. Then*

(1) *If*

$$\lim_{n\to\infty} \frac{1}{K(\frac{1}{n})}\|\varphi^n\|^2_{D_K} = 0,$$

then C_φ is compact;

(2) *If C_φ is compact, then*

$$\lim_{n\to\infty} \frac{1}{nK(\frac{1}{n})}\|\varphi^n\|^2_{D_K} = 0.$$

Proof. (1). The proof is similar to (1) of Theorem 3.

(2). Let $\{f_n\}$ be a bounded sequence in D_K that convergence to 0 weakly. If C_φ is compact on D_K, then $\|C_\varphi f_n\|_{D_K} \to 0$, as $n \to \infty$. Thus, for any $z \in \mathbb{D}$, we have

$$f_n(\varphi(z)) \to 0, \ \ n \to \infty.$$

Since $\{z^n/\|z^n\|_{D_K}, n \geq 1\}$ is bounded in D_K and it converges to 0 pointwise, the compactness of C_φ on D_K implies that

$$\lim_{n\to\infty} \frac{\|\varphi^n\|^2_{D_K}}{\|z^n\|^2_{D_K}} = \frac{1}{nK(\frac{1}{n})}\|\varphi^n\|^2_{D_K} = 0.$$

The proof is completed. □

5. Hilbert-Schmidt class

Let Hilbert-Schmidt class be the space of all compact operators on Hilbert space with its singular value sequence $\{\lambda_n\} \in l^2$, the 2-summable sequence space (see [27, page 18]). The following theorem give an equavalent charaterizations of composition operator on D_K spaces, when it belong to Hilbert-Schmidt class.

Theorem 5. *Let (1.1) and (1.2) hold for K. Suppose $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi \in D_K$ and C_φ is compact. Then C_φ is Hilbert-Schmidt on D_K if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \frac{K(1-|z|^2)}{K(1-|\varphi(z)|^2)} dA(z) < \infty.$$

Proof. Without loss of generality, we can assume $\{1\} \cup \{\frac{z^n}{\sqrt{n}\sqrt{K(\frac{1}{n})}}\}_{n=1}^{\infty}$ is an orthonormal basis in D_K and $\varphi(0) = 0$. From Theorem 1.22 of [27], C_φ is Hilbert-Schmidt on D_K if and only if

$$\sum_{n=1}^{\infty} \frac{D_K(\varphi^n)}{nK(\frac{1}{n})} < \infty.$$

Applying Lemma 2, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{D_K(\varphi^n)}{nK(\frac{1}{n})} &= \sum_{n=1}^{\infty} \frac{n}{K(\frac{1}{n})} \int_{\mathbb{D}} |\varphi^2(z)|^{n-1} |\varphi'(z)|^2 K(1-|z|^2) dA(z) \\ &= \sum_{n=0}^{\infty} \frac{n+1}{K(\frac{1}{n+1})} \int_{\mathbb{D}} |\varphi^2(z)|^{n} |\varphi'(z)|^2 K(1-|z|^2) dA(z) \\ &\asymp \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \frac{K(1-|z|^2)}{K(1-|\varphi(z)|^2)} dA(z).\end{aligned}$$

The proof is completed. $\square$

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On left multidimensional Riemann-Liouville fractional integral

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Abstract

Here we study some important properties of left multidimensional Riemann-Liouville fractional integral operator, such as of continuity and boundedness.

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1 Motivation

From [1], p. 388 we have

Theorem 1 *Let $r > 0$, $F \in L_{\infty}(a,b)$, and*

$$G(s) = \int_a^s (s-t)^{r-1} F(t)\,dt,$$

all $s \in [a,b]$. Then $G \in AC([a,b])$ (absolutely continuous functions) for $r \geq 1$, and $G \in C([a,b])$, only for $r \in (0,1)$.

2 Main Results

We give

Theorem 2 *Let $f \in L_{\infty}([a,b] \times [c,d])$, $\alpha_1, \alpha_2 > 0$. Consider the function*

$$F(x_1,x_2) = \int_{a_1}^{x_1} \int_{a_2}^{x_2} (x_1-t_1)^{\alpha_1-1} (x_2-t_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1 dt_2, \quad (1)$$

1

where $a_1, x_1 \in [a,b]$, $a_2, x_2 \in [c,d] : a_1 \leq x_1,\ a_2 \leq x_2$.

Then F *is continuous on* $[a_1, b] \times [a_2, d]$.

Proof. (I) Let $a_1, b_1, b_1^* \in [a,b]$ with $b_1 > b_1^* > a_1$, and $a_2, b_2, b_2^* \in [c,d]$ with $b_2 > b_2^* > a_2$.

We observe that

$$F(b_1,b_2) - F(b_1^*,b_2^*) =$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} (b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 -$$

$$\int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*} (b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 = \tag{2}$$

$$\int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*} (b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 -$$

$$\int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*} (b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 +$$

$$\int_{b_1^*}^{b_1}\int_{a_2}^{b_2^*} (b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 +$$

$$\int_{a_1}^{b_1^*}\int_{b_2^*}^{b_2} (b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 +$$

$$\int_{b_1^*}^{b_1}\int_{b_2^*}^{b_2} (b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2.$$

Call

$$I(b_1^*,b_2^*) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*} \left|(b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1} - (b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}\right| dt_1dt_2. \tag{3}$$

Thus

$$|F(b_1,b_2) - F(b_1^*,b_2^*)| \leq$$

$$\left\{ I(b_1^*,b_2^*) + \frac{(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\left[\frac{(b_2-a_2)^{\alpha_2} - (b_2-b_2^*)^{\alpha_2}}{\alpha_2}\right] + \right.$$

$$\left. \left(\frac{(b_1-a_1)^{\alpha_1} - (b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)\frac{(b_2-b_2^*)^{\alpha_2}}{\alpha_2} + \frac{(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\frac{(b_2-b_2^*)^{\alpha_2}}{\alpha_2} \right\} \|f\|_\infty. \tag{4}$$

Hence, by (4), it holds

$$\delta := \lim_{\substack{(b_1^*,b_2^*)\to(b_1,b_2) \\ \text{or} \\ (b_1,b_2)\to(b_1^*,b_2^*)}} |F(b_1,b_2) - F(b_1^*,b_2^*)| \leq \Big(\lim_{\substack{(b_1^*,b_2^*)\to(b_1,b_2) \\ \text{or} \\ (b_1,b_2)\to(b_1^*,b_2^*)}} I(b_1^*,b_2^*)\Big)\|f\|_\infty =: \rho. \tag{5}$$

If $\alpha_1 = \alpha_2 = 1$, then $\rho = 0$, proving $\delta = 0$.

If $\alpha_1 = 1$, $\alpha_2 > 0$ we get

$$I\left(b_1^*, b_2^*\right) = \left(b_1^* - a_1\right)\left(\int_{a_2}^{b_2^*} \left|\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1}\right| dt_2\right). \tag{6}$$

Assume $\alpha_2 > 1$, then $\alpha_2 - 1 > 0$. Hence by $b_2 > b_2^*$, then $b_2 - t_2 > b_2^* - t_2 \geq 0$, and $\left(b_2 - t_2\right)^{\alpha_2 - 1} > \left(b_2^* - t_2\right)^{\alpha_2 - 1}$ and $\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1} > 0$.

That is

$$I\left(b_1^*, b_2^*\right) = \left(b_1^* - a_1\right)\left[\left.\frac{\left(b_2 - t_2\right)^{\alpha_2}}{\alpha_2}\right|_{b_2^*}^{a_2} - \frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}\right]$$

$$= \left(b_1^* - a_1\right)\left[\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(b_2 - b_2^*\right)^{\alpha_2} - \left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}\right]. \tag{7}$$

Clearly, then

$$\lim_{\substack{b_2^* \to b_2 \\ \text{or} \\ b_2 \to b_2^*}} I\left(b_1^*, b_2^*\right) = 0. \tag{8}$$

Similarly and symmetrically, we obtain that

$$\lim_{\substack{b_1^* \to b_1 \\ \text{or} \\ b_1 \to b_1^*}} I\left(b_1^*, b_2^*\right) = 0, \tag{9}$$

for the case of $\alpha_2 = 1$, $\alpha_1 > 1$.

If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I\left(b_1^*, b_2^*\right) = \left(b_1^* - a_1\right)\left(\int_{a_2}^{b_2^*} \left(\left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_2 - t_2\right)^{\alpha_2 - 1}\right) dt_2\right) =$$

$$\left(b_1^* - a_1\right)\left[\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2 - a_2\right)^{\alpha_2} + \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2}\right]. \tag{10}$$

Clearly, then

$$\lim_{\substack{b_2^* \to b_2 \\ \text{or} \\ b_2 \to b_2^*}} I\left(b_1^*, b_2^*\right) = 0. \tag{11}$$

Similarly and symmetrically, we derive that

$$\lim_{\substack{b_1^* \to b_1 \\ \text{or} \\ b_1 \to b_1^*}} I\left(b_1^*, b_2^*\right) = 0, \tag{12}$$

for the case of $\alpha_2 = 1$, $0 < \alpha_1 < 1$.

Case now of $\alpha_1, \alpha_2 > 1$, then

$$I\left(b_1^*, b_2^*\right) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left[\left(b_1 - t_1\right)^{\alpha_1 - 1} \left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \right] dt_1 dt_2 =$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1} \right) \left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2} \right) - \frac{\left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}. \tag{13}$$

That is

$$\lim_{\substack{(b_1, b_2) \to \left(b_1^*, b_2^*\right) \\ \text{or} \\ \left(b_1^*, b_2^*\right) \to (b_1, b_2)}} I\left(b_1^*, b_2^*\right) = 0. \tag{14}$$

Case now of $0 < \alpha_1, \alpha_2 < 1$, then

$$I\left(b_1^*, b_2^*\right) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left[\left(b_1^* - t_1\right)^{\alpha_1 - 1} \left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(b_2 - t_2\right)^{\alpha_2 - 1} \right] dt_1 dt_2 =$$

$$\frac{\left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2} - \left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1} \right) \left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2} \right). \tag{15}$$

That is again, when $0 < \alpha_1, \alpha_2 < 1$,

$$\lim_{\substack{(b_1, b_2) \to \left(b_1^*, b_2^*\right) \\ \text{or} \\ \left(b_1^*, b_2^*\right) \to (b_1, b_2)}} I\left(b_1^*, b_2^*\right) = 0. \tag{16}$$

Next we treat the case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$.

We observe that

$$I\left(b_1^*, b_2^*\right) \leq I^*\left(b_1^*, b_2^*\right) := \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left| \left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1} \right| dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \left| \left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1} \right| dt_1 dt_2. \tag{17}$$

Therefore it holds

$$I^*\left(b_1^*, b_2^*\right) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(\left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_2 - t_2\right)^{\alpha_2 - 1} \right) dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \left(\left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1} \right) dt_1 dt_2 = \tag{18}$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1} \right) \left[\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2 - a_2\right)^{\alpha_2} + \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2} \right] +$$

$$\frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \left[\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1} - \left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right]. \tag{19}$$

So, in case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$, we proved that

$$\lim_{\substack{(b_1,b_2)\to(b_1^*,b_2^*) \\ \text{or} \\ (b_1^*,b_2^*)\to(b_1,b_2)}} I\left(b_1^*,b_2^*\right) = 0. \tag{20}$$

Finally, we prove the case of $\alpha_2 > 1$ and $0 < \alpha_1 < 1$. We have that

$$I^*\left(b_1^*,b_2^*\right) \overset{(17)}{=} \int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*} (b_1-t_1)^{\alpha_1-1}\left[(b_2-t_2)^{\alpha_2-1} - (b_2^*-t_2)^{\alpha_2-1}\right] dt_1 dt_2$$

$$+\int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*} (b_2^*-t_2)^{\alpha_2-1}\left((b_1^*-t_1)^{\alpha_1-1} - (b_1-t_1)^{\alpha_1-1}\right) dt_1 dt_2 = \tag{21}$$

$$\left(\frac{(b_1-a_1)^{\alpha_1} - (b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)\left[\frac{(b_2-a_2)^{\alpha_2} - (b_2-b_2^*)^{\alpha_2} - (b_2^*-a_2)^{\alpha_2}}{\alpha_2}\right] +$$

$$\frac{(b_2^*-a_2)^{\alpha_2}}{\alpha_2}\left[\frac{(b_1^*-a_1)^{\alpha_1} - (b_1-a_1)^{\alpha_1} + (b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right]. \tag{22}$$

Hence again it holds

$$\lim_{\substack{(b_1,b_2)\to(b_1^*,b_2^*) \\ \text{or} \\ (b_1^*,b_2^*)\to(b_1,b_2)}} I\left(b_1^*,b_2^*\right) = 0. \tag{23}$$

We proved $\rho = 0$, and $\delta = 0$ in all cases of this section.

The case of $b_1^* > b_1$ and $b_2^* > b_2$, as symmetric to $b_1 > b_1^*$ and $b_2 > b_2^*$ we treated, it is omitted, a totally similar treatment.

(II) The remaining cases are: let $a_1, b_1, b_1^* \in [a,b]$; $a_2, b_2, b_2^* \in [c,d]$, we can have

(II_1) $b_1 > b_1^*$ and $b_2 < b_2^*$,

or

(II_2) $b_1 < b_1^*$ and $b_2 > b_2^*$.

Notice that (II_1) and (II_2) cases are symmetric, and treated the same way. As such we treat only the case (II_1).

We observe again that

$$F(b_1,b_2) - F\left(b_1^*,b_2^*\right) =$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} (b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1} f(t_1,t_2)\, dt_1 dt_2 -$$

$$\int_{a_1}^{b_1^*}\int_{a_2}^{b_2^*} (b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1} f(t_1,t_2)\, dt_1 dt_2 = \tag{24}$$

$$\int_{a_1}^{b_1^*}\int_{a_2}^{b_2} (b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1} f(t_1,t_2)\, dt_1 dt_2 +$$

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$$\int_{b_1^*}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2-$$
$$\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2-$$
$$\int_{a_1}^{b_1^*}\int_{b_2}^{b_2^*}(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2=$$
$$\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left((b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}-(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}\right)f(t_1,t_2)\,dt_1dt_2$$
$$+\int_{b_1^*}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2-$$
$$\int_{a_1}^{b_1^*}\int_{b_2}^{b_2^*}(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2. \tag{25}$$

We call

$$I(b_1^*,b_2):=\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left|(b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}-(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}\right|dt_1dt_2. \tag{26}$$

Hence, we have

$$|F(b_1,b_2)-F(b_1^*,b_2^*)|\le$$
$$\left\{I(b_1^*,b_2)+\frac{(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}+\frac{(b_1^*-a_1)^{\alpha_1}}{\alpha_1}\frac{(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right\}\|f\|_\infty. \tag{27}$$

Therefore it holds

$$\delta:=\lim_{\substack{|b_1-b_1^*|\to0,\\ |b_2-b_2^*|\to0}}|F(b_1,b_2)-F(b_1^*,b_2^*)|\le\left(\lim_{\substack{|b_1-b_1^*|\to0,\\ |b_2-b_2^*|\to0}}I(b_1^*,b_2)\right)\|f\|_\infty=:\theta. \tag{28}$$

We will prove that $\theta=0$, hence $\delta=0$, in all possible cases.

If $\alpha_1=\alpha_2=1$, then $I(b_1^*,b_2)=0$, hence $\theta=0$.

If $\alpha_1=1$, $\alpha_2>0$ we get

$$I(b_1^*,b_2)=(b_1^*-a_1)\left(\int_{a_2}^{b_2}\left|(b_2-t_2)^{\alpha_2-1}-(b_2^*-t_2)^{\alpha_2-1}\right|dt_2\right). \tag{29}$$

Assume $\alpha_2>1$, then $\alpha_2-1>0$. Hence

$$I(b_1^*,b_2)=(b_1^*-a_1)\left(\int_{a_2}^{b_2}\left((b_2^*-t_2)^{\alpha_2-1}-(b_2-t_2)^{\alpha_2-1}\right)dt_2\right)$$
$$=(b_1^*-a_1)\left[\frac{(b_2^*-t_2)^{\alpha_2}\big|_{b_2}^{a_2}-(b_2-a_2)^{\alpha_2}}{\alpha_2}\right]$$

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$$= (b_1^* - a_1) \left[\frac{(b_2^* - a_2)^{\alpha_2} - (b_2^* - b_2)^{\alpha_2} - (b_2 - a_2)^{\alpha_2}}{\alpha_2} \right]. \tag{30}$$

Clearly, then

$$\lim_{\substack{|b_1 - b_1^*| \to 0, \\ |b_2 - b_2^*| \to 0}} I(b_1^*, b_2) = 0, \tag{31}$$

hence $\theta = 0$.

Let the case now of $\alpha_2 = 1$, $\alpha_1 > 1$. Then

$$I(b_1^*, b_2) = (b_2 - a_2) \left(\int_{a_1}^{b_1^*} \left| (b_1 - t_1)^{\alpha_1 - 1} - (b_1^* - t_1)^{\alpha_1 - 1} \right| dt_1 \right)$$

$$= (b_2 - a_2) \left[\frac{(b_1 - a_1)^{\alpha_1} - (b_1 - b_1^*)^{\alpha_1} - (b_1^* - a_1)^{\alpha_1}}{\alpha_1} \right]. \tag{32}$$

Then $\theta = 0$.

If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I(b_1^*, b_2) = (b_1^* - a_1) \int_{a_2}^{b_2} \left| (b_2 - t_2)^{\alpha_2 - 1} - (b_2^* - t_2)^{\alpha_2 - 1} \right| dt_2 =$$

$$(b_1^* - a_1) \int_{a_2}^{b_2} \left((b_2 - t_2)^{\alpha_2 - 1} - (b_2^* - t_2)^{\alpha_2 - 1} \right) dt_2 =$$

$$(b_1^* - a_1) \left[\frac{(b_2 - a_2)^{\alpha_2} - (b_2^* - a_2)^{\alpha_2} + (b_2^* - b_2)^{\alpha_2}}{\alpha_2} \right]. \tag{33}$$

Hence $\theta = 0$.

Let now $\alpha_2 = 1$, $0 < \alpha_1 < 1$. Then

$$I(b_1^*, b_2) = (b_2 - a_2) \int_{a_1}^{b_1^*} \left| (b_1 - t_1)^{\alpha_1 - 1} - (b_1^* - t_1)^{\alpha_1 - 1} \right| dt_1$$

$$= (b_2 - a_2) \int_{a_1}^{b_1^*} \left((b_1^* - t_1)^{\alpha_1 - 1} - (b_1 - t_1)^{\alpha_1 - 1} \right) dt_1$$

$$= (b_2 - a_2) \left[\frac{(b_1^* - a_1)^{\alpha_1} - (b_1 - a_1)^{\alpha_1} + (b_1 - b_1^*)^{\alpha_1}}{\alpha_1} \right]. \tag{34}$$

Hence $\theta = 0$.

We observe that:

$$I(b_1^*, b_2) \leq \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left| (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} - (b_1 - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} \right| dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left| (b_1 - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} - (b_1^* - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} \right| dt_1 dt_2 =: J(b_1^*, b_2), \tag{35}$$

i.e.

$$I\left(b_1^*, b_2\right) \leq J\left(b_1^*, b_2\right).$$

Hence it holds

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left(b_1-t_1\right)^{\alpha_1-1}\left|\left(b_2-t_2\right)^{\alpha_2-1}-\left(b_2^*-t_2\right)^{\alpha_2-1}\right| dt_1dt_2 \quad (36)$$

$$+\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left(b_2^*-t_2\right)^{\alpha_2-1}\left|\left(b_1-t_1\right)^{\alpha_1-1}-\left(b_1^*-t_1\right)^{\alpha_1-1}\right| dt_1dt_2.$$

Case of $\alpha_1, \alpha_2 > 1$. Then

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left(b_1-t_1\right)^{\alpha_1-1}\left(\left(b_2^*-t_2\right)^{\alpha_2-1}-\left(b_2-t_2\right)^{\alpha_2-1}\right) dt_1dt_2$$

$$+\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left(b_2^*-t_2\right)^{\alpha_2-1}\left(\left(b_1-t_1\right)^{\alpha_1-1}-\left(b_1^*-t_1\right)^{\alpha_1-1}\right) dt_1dt_2 =$$

$$\left(\frac{\left(b_1-a_1\right)^{\alpha_1}-\left(b_1-b_1^*\right)^{\alpha_1}}{\alpha_1}\right)\left[\left(\frac{\left(b_2^*-a_2\right)^{\alpha_2}-\left(b_2^*-b_2\right)^{\alpha_2}}{\alpha_2}\right)-\frac{\left(b_2-a_2\right)^{\alpha_2}}{\alpha_2}\right]$$

$$+\left(\frac{\left(b_2^*-a_2\right)^{\alpha_2}-\left(b_2^*-b_2\right)^{\alpha_2}}{\alpha_2}\right)\left[\left(\frac{\left(b_1-a_1\right)^{\alpha_1}-\left(b_1-b_1^*\right)^{\alpha_1}}{\alpha_1}\right)-\frac{\left(b_1^*-a_1\right)^{\alpha_1}}{\alpha_1}\right]. \quad (37)$$

So that $\theta = 0$.

Case of $0 < \alpha_1, \alpha_2 < 1$, then

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left(b_1-t_1\right)^{\alpha_1-1}\left(\left(b_2-t_2\right)^{\alpha_2-1}-\left(b_2^*-t_2\right)^{\alpha_2-1}\right) dt_1dt_2$$

$$+\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left(b_2^*-t_2\right)^{\alpha_2-1}\left(\left(b_1^*-t_1\right)^{\alpha_1-1}-\left(b_1-t_1\right)^{\alpha_1-1}\right) dt_1dt_2 =$$

$$\left(\frac{\left(b_1-a_1\right)^{\alpha_1}-\left(b_1-b_1^*\right)^{\alpha_1}}{\alpha_1}\right)\left[\frac{\left(b_2-a_2\right)^{\alpha_2}}{\alpha_2}-\left(\frac{\left(b_2^*-a_2\right)^{\alpha_2}-\left(b_2^*-b_2\right)^{\alpha_2}}{\alpha_2}\right)\right] \quad (38)$$

$$+\left(\frac{\left(b_2^*-a_2\right)^{\alpha_2}-\left(b_2^*-b_2\right)^{\alpha_2}}{\alpha_2}\right)\left[\frac{\left(b_1^*-a_1\right)^{\alpha_1}}{\alpha_1}-\left(\frac{\left(b_1-a_1\right)^{\alpha_1}-\left(b_1-b_1^*\right)^{\alpha_1}}{\alpha_1}\right)\right].$$

One more time $\theta = 0$.

Next case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$. We observe that

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left(b_1-t_1\right)^{\alpha_1-1}\left(\left(b_2-t_2\right)^{\alpha_2-1}-\left(b_2^*-t_2\right)^{\alpha_2-1}\right) dt_1dt_2 \quad (39)$$

$$+\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left(b_2^*-t_2\right)^{\alpha_2-1}\left(\left(b_1-t_1\right)^{\alpha_1-1}-\left(b_1^*-t_1\right)^{\alpha_1-1}\right) dt_1dt_2 =$$

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$$\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)\left[\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}-\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\right]$$
$$+\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\left[\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)-\frac{(b_1^*-a_1)^{\alpha_1}}{\alpha_1}\right]. \tag{40}$$

Hence $\theta=0$.

Finally, we prove the case of $\alpha_2>1$ and $0<\alpha_1<1$. In that case it holds

$$J(b_1^*,b_2)=\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha_1-1}\left((b_2^*-t_2)^{\alpha_2-1}-(b_2-t_2)^{\alpha_2-1}\right)dt_1dt_2 \tag{41}$$

$$+\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}(b_2^*-t_2)^{\alpha_2-1}\left((b_1^*-t_1)^{\alpha_1-1}-(b_1-t_1)^{\alpha_1-1}\right)dt_1dt_2=$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)\left[-\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}+\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\right]$$
$$+\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\left[-\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)+\frac{(b_1^*-a_1)^{\alpha_1}}{\alpha_1}\right]. \tag{42}$$

Hence again $\theta=0$.

We have proved that $\delta=0$, in all possible subcases of (II_1).

We have proved that F is a continuous function over $[a_1,b]\times[a_2,d]$. ■

Now we can state:

Theorem 3 *Let $f\in L_\infty\left(\prod_{i=1}^k[a_i,b_i]\right)$, $\alpha_i>0$, $i=1,...,k\in\mathbb{N}$. Consider the function*

$$F(x_1,...,x_k)=\int_{a_1^*}^{x_1}...\int_{a_k^*}^{x_k}\prod_{i=1}^k(x_i-t_i)^{\alpha_i-1}f(t_1,...,t_k)\,dt_1...dt_k, \tag{43}$$

where $a_i^*,x_i\in[a_i,b_i]$, $a_i^*\le x_i$, $i=1,...,k$.

Then F is continuous on $\prod_{i=1}^k[a_i^*,b_i]$.

Remark 4 *In the setting of Theorem 3: Consider the left multidimensional Riemann-Liouville fractional integral of order $\alpha=(\alpha_1,...,\alpha_k)$:*

$$\left(I_{a_+^*}^{\alpha}f\right)(x)=\frac{1}{\prod_{i=1}^k\Gamma(\alpha_i)}\int_{a_1^*}^{x_1}...\int_{a_k^*}^{x_k}\prod_{i=1}^k(x_i-t_i)^{\alpha_i-1}f(t_1,...,t_k)\,dt_1...dt_k, \tag{44}$$

where $a^*=(a_1^*,...,a_k^*)$, $x=(x_1,...,x_k)$, $a_i^*\le x_i$, $i=1,...,k$. Here Γ denotes the gamma function.

By Theorem 3 we get that $\left(I_{a_+^*}^{\alpha}f\right)(x)$ is a continuous function for every $x\in\prod_{i=1}^k[a_i^*,b_i]$.

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We notice that

$$\left|\left(I_{a_+^*}^{\alpha} f\right)(x)\right| \leq \frac{1}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \left(\int_{a_1^*}^{x_1} ... \int_{a_k^*}^{x_k} \prod_{i=1}^{k} (x_i - t_i)^{\alpha_i - 1} dt_1 ... dt_k \right) \|f\|_{\infty}$$

$$= \frac{\|f\|_{\infty}}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} \left(\int_{a_i^*}^{x_i} (x_i - t_i)^{\alpha_i - 1} dt_i \right) = \frac{\|f\|_{\infty}}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} \frac{(x_i - a_i^*)^{\alpha_i}}{\alpha_i} \quad (45)$$

$$= \|f\|_{\infty} \left(\prod_{i=1}^{k} \frac{(x_i - a_i^*)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right).$$

That is

$$\left|\left(I_{a_+^*}^{\alpha} f\right)(x)\right| \leq \left(\prod_{i=1}^{k} \frac{(x_i - a_i^*)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \|f\|_{\infty}. \quad (46)$$

In particular we get that

$$\left(I_{a_+^*}^{\alpha} f\right)(a^*) = 0, \quad (47)$$

and

$$\left\|I_{a_+^*}^{\alpha} f\right\|_{\infty} \leq \left(\prod_{i=1}^{k} \frac{(b_i - a_i^*)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \|f\|_{\infty}. \quad (48)$$

That is $I_{a_+^*}^{\alpha} f$ is a bounded linear operator, which here is also a positive operator.

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Weak closure operations on ideals of BCK-algebras

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Abstract. Weak closure operation, which is more general form than closure operation, on ideals of BCK-algebras is introduced, and related properties are investigated. Regarding weak closure operation, finite type and (strong) quasi-primeness are considered. Also positive implicative (resp., commutative and implicative) weak closure operations are discussed.

1. Introduction

Semi-prime closure operations on ideals of BCK-algebras are introduced in the paper [1], and a finite type of closure operations on ideals of BCK-algebras are discussed in [2].

In this paper, we consider more general form than closure operations on ideals of BCK-algebras. We introduce the notion of weak closure operations on ideals of BCK-algebras. Regarding weak closure operation, we define finite type and (strong) quasi-primeness, and investigate related properties. We also discuss positive implicative (resp., commutative and implicative) weak closure operations, and provide several examples to illustrate notions and properties.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a BCI-*algebra* if it satisfies the following conditions:

(I) $(\forall x,y,z\in X)$ $(((x*y)*(x*z))*(z*y)=0)$,

(II) $(\forall x,y\in X)$ $((x*(x*y))*y=0)$,

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(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK*-algebra.* Any BCK/BCI-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X (see [4]) if it satisfies:

$$0 \in A, \tag{2.1}$$

$$(\forall x \in X)\,(\forall y \in A)\,(x * y \in A\ \Rightarrow\ x \in A)\,. \tag{2.2}$$

For any subset A of X, the ideal generated by A is defined to be the intersection of all ideals of X containing A, and it is denoted by $\langle A \rangle$. If A is finite, then we say that $\langle A \rangle$ is *finitely generated ideal* of X (see [4]).

A subset A of a BCK-algebra X is called a *commutative ideal* of X (see [4]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A)\,((x * y) * z \in A\ \Rightarrow\ x * (y * (y * x)) \in A)\,. \tag{2.3}$$

A subset A of a BCK-algebra X is called a *positive implicative ideal* of X (see [4]) if it satisfies (2.1) and

$$(\forall x, y, z \in X)\,((x * y) * z \in A,\ y * z \in A\ \Rightarrow\ x * z \in A)\,. \tag{2.4}$$

A subset A of a BCK-algebra X is called an *implicative ideal* of X (see [4]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A)\,((x * (y * y)) * z \in A\ \Rightarrow\ x \in A)\,. \tag{2.5}$$

Denote by $\mathcal{I}_{pi}(X)$ (resp., $\mathcal{I}_c(X)$ and $\mathcal{I}_m(X)$) the set of all positive implicative (resp., commutative and implicative) ideals of X.

We refer the reader to the books [3, 4] for further information regarding BCK/BCI-algebras.

3. **Weak Closure operations**

In what follows, let X and $\mathcal{I}(X)$ be a BCK-algebra and a set of all ideals of X, respectively, unless otherwise specified .

Definition 3.1. A mapping $c : \mathcal{I}(X) \to \mathcal{I}(X)$ is called a *weak closure operation* on $\mathcal{I}(X)$ if the following conditions are valid.

$$(\forall A \in \mathcal{I}(X))\,(A \subseteq c(A))\,, \tag{3.1}$$

$$(\forall A, B \in \mathcal{I}(X))\,(A \subseteq B \;\Rightarrow\; c(A) \subseteq c(B))\,. \tag{3.2}$$

If a weak closure operation $c : \mathcal{I}(X) \to \mathcal{I}(X)$ satisfies the condition

$$(\forall A \in \mathcal{I}(X))\,(c(c(A)) = c(A))\,, \tag{3.3}$$

then we say that c is a closure operation on $\mathcal{I}(X)$ (see [2]). In what follows, we use A^{cl} instead of $c(A)$.

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

We have 8 ideals of X, and they are $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 4\}$, $A_4 = \{0, 1, 4\}$, $A_5 = \{0, 1, 2, 3\}$, $A_6 = \{0, 2, 4\}$, and $A_7 = X$. Define a mapping $c : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_4$, $A_2^{cl} = A_5$, $c(A_3) = A_6$, and $c(A_4) = c(A_5) = c(A_6) = c(A_7) = A_7$. Then c is a weak closure operation on $\mathcal{I}(X)$. But it is not a closure operation on $\mathcal{I}(X)$ since $c(A_2^{cl}) = c(A_5) = A_7$.

In a BCK-algebra X, let $x \wedge y$ denote the greatest lower bound of x and y. Note that $0 \wedge x = 0$ for all $x \in X$. For any element x of X, consider the following condition

$$(\exists\, y \in X \setminus \{0\})\,(x \wedge y = 0)\,. \tag{3.4}$$

In the following example, we know that there are two kinds of element. One is an element x satisfying the condition (3.4). The other is an element x which does not satisfy the condition (3.4).

Example 3.3. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	4	4	4	0

Then X is a BCK-algebra. We know that 1 and 2 satisfy the condition (3.4), but 3 and 4 do not satisfy the condition (3.4).

On the basis of this consideration, we define the zeromeet element in a BCK-algebra.

Definition 3.4. An element x of X is called a *zeromeet element* of X if the condition (3.4) is valid. Otherwise, x is called a non-zeromeet element of X.

Denote by $Z(X)$ the set of all zeromeet elements of X, that is,

$$Z(X) = \{x \in X \mid x \wedge y = 0 \text{ for some nonzero element } y \in X\}.$$

Obviously, $0 \in Z(X)$. We know that $0, 1, 2 \in Z(X)$ and $3, 4 \notin Z(X)$ in Example 3.3.

Lemma 3.5. *For any $x, y \in X$, if $x, y \notin Z(X)$, then $x \wedge y \notin Z(X)$, that is, the set $X \setminus Z(X)$ is closed under the operation $\wedge$.*

Proof. Let $x, y \in X \setminus Z(X)$ and assume that $x \wedge y \in Z(X)$. Then $x \wedge (y \wedge a) = (x \wedge y) \wedge a = 0$ for some nonzero element $a \in X$. Since $x \notin Z(X)$, it follows that $y \wedge a = 0$ and so that $a = 0$ since $y \notin Z(X)$. This is a contradiction, and thus $x \wedge y \notin Z(X)$. □

For any subsets A and B of X, we define

$$A \wedge B := \langle \{a \wedge b \mid a \in A,\ b \in B\rangle.$$

We use $x \wedge A$ instead of $\{x\} \wedge A$, that is, $x \wedge A := \langle \{x \wedge a \mid a \in A\}\rangle$.

Definition 3.6. A weak closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is said to be *quasi-prime* if it satisfies:

$$(\forall a \in X \setminus Z(X))\,(\forall A \in \mathcal{I}(X))\,\left(a \wedge A^{cl} \subseteq (a \wedge A)^{cl}\right). \tag{3.5}$$

Example 3.7. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

We know that $Z(X) = \{0\}$ and there are four ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$ and $A_3 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0, A_1^{cl} = A_2, A_2^{cl} = A_3$ *and* $A_3^{cl} = A_3$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$. For $1, 2, 3 \in X \setminus Z(X)$, we have

$1 \wedge A_0^{cl} = 1 \wedge A_0 = \langle\{0\}\rangle = A_0 = A_0^{cl} = (1 \wedge A_0)^{cl}$,
$1 \wedge A_1^{cl} = 1 \wedge A_2 = \langle\{0, 1\}\rangle = A_1 \subseteq A_2 = A_1^{cl} = (1 \wedge A_1)^{cl}$,
$1 \wedge A_2^{cl} = 1 \wedge A_3 = \langle\{0, 1\}\rangle = A_1 \subseteq A_2 = A_1^{cl} = (1 \wedge A_2)^{cl}$,
$1 \wedge A_3^{cl} = 1 \wedge A_3 = \langle\{0, 1\}\rangle = A_1 \subseteq A_2 = A_1^{cl} = (1 \wedge A_3)^{cl}$,

$2 \wedge A_0^{cl} = 2 \wedge A_0 = \langle\{0\}\rangle = A_0 = A_0^{cl} = (2 \wedge A_0)^{cl}$,
$2 \wedge A_1^{cl} = 2 \wedge A_2 = \langle\{0,1,2\}\rangle = A_2 = A_1^{cl} = (2 \wedge A_1)^{cl}$,
$2 \wedge A_2^{cl} = 2 \wedge A_3 = \langle\{0,1,2\}\rangle = A_2 \subseteq A_3 = A_2^{cl} = (2 \wedge A_2)^{cl}$,
$2 \wedge A_3^{cl} = 2 \wedge A_3 = \langle\{0,1,2\}\rangle = A_2 \subseteq A_3 = A_2^{cl} = (2 \wedge A_3)^{cl}$,
$3 \wedge A_0^{cl} = 3 \wedge A_0 = \langle\{0\}\rangle = A_0 = A_0^{cl} = (3 \wedge A_0)^{cl}$,
$3 \wedge A_1^{cl} = 3 \wedge A_2 = \langle\{0,1,2\}\rangle = A_2 = A_1^{cl} = (3 \wedge A_1)^{cl}$,
$3 \wedge A_2^{cl} = 3 \wedge A_3 = \langle\{0,1,2,3\}\rangle = A_3 = A_2^{cl} = (3 \wedge A_2)^{cl}$,
$3 \wedge A_3^{cl} = 3 \wedge A_3 = \langle\{0,1,2,3\}\rangle = A_3 = A_3^{cl} = (3 \wedge A_3)^{cl}$,
Therefore "cl" is a quasi-prime weak closure operation on $\mathcal{I}(X)$.

Definition 3.8. A weak closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is said to be *strong quasi-prime* if it satisfies:

$$(\forall a \in X \setminus Z(X))\,(\forall A \in \mathcal{I}(X))\,\left(a \wedge A^{cl} = (a \wedge A)^{cl}\right). \tag{3.6}$$

Example 3.9. Consider a BCK-algebra $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

We know that $Z(X) = \{0,1,2\}$ and there are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0,1\}$, $A_2 = \{0,2\}$, $A_3 = \{0,1,2\}$, $A_4 = \{0,1,2,3\}$ and $A_5 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl} = A_1$, $A_1^{cl} = A_2^{cl} = A_3$, $A_3^{cl} = A_4^{cl} = A_4$ and $A_5^{cl} = A_5$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$. For $3,4 \in X \setminus Z(X)$, we have
$3 \wedge A_0^{cl} = 3 \wedge A_1 = \langle\{0,1\}\rangle = A_1 = A_0^{cl} = (3 \wedge A_0)^{cl}$,
$3 \wedge A_1^{cl} = 3 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_1^{cl} = (3 \wedge A_1)^{cl}$,
$3 \wedge A_2^{cl} = 3 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_2^{cl} = (3 \wedge A_2)^{cl}$,
$3 \wedge A_3^{cl} = 3 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_3^{cl} = (3 \wedge A_3)^{cl}$,
$3 \wedge A_4^{cl} = 3 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (3 \wedge A_4)^{cl}$,
$3 \wedge A_5^{cl} = 3 \wedge A_5 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (3 \wedge A_5)^{cl}$,
$4 \wedge A_0^{cl} = 4 \wedge A_1 = \langle\{0,1\}\rangle = A_1 = A_0^{cl} = (4 \wedge A_0)^{cl}$,
$4 \wedge A_1^{cl} = 4 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_1^{cl} = (4 \wedge A_1)^{cl}$,
$4 \wedge A_2^{cl} = 4 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_2^{cl} = (4 \wedge A_2)^{cl}$,
$4 \wedge A_3^{cl} = 4 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_3^{cl} = (4 \wedge A_3)^{cl}$,
$4 \wedge A_4^{cl} = 4 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (4 \wedge A_4)^{cl}$,
$4 \wedge A_5^{cl} = 4 \wedge A_5 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (4 \wedge A_5)^{cl}$.
Therefore "cl" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$.

Given an ideal A of X and an operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ on $\mathcal{I}(X)$, we consider the following set:

$$K := \cup\{B^{cl} \mid B \subseteq A,\ B \in \mathcal{I}_f(X)\} \tag{3.7}$$

where $\mathcal{I}_f(X)$ is the set of all finitely generated ideals of X. The following example shows that the set K in (3.7) may not be an ideal of X in general.

Example 3.10. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$, $A_3 = \{0, 1, 3\}$ and $A_4 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl} = A_3$, $A_1^{cl} = A_2$, $A_2^{cl} = A_0$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_3$. For the ideal A_2 of X, we have

$$\cup\{B^{cl} \mid B \subseteq A,\ B \in \mathcal{I}_f(X)\} = A_0^{cl} \cup A_1^{cl} \cup A_2^{cl} = \{0, 1, 2, 3\}$$

which is not an ideal of X.

We provide a condition for the set K in (3.7) to be an ideal of X.

Theorem 3.11. *If $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is a weak closure operation on $\mathcal{I}(X)$, then the set K in (3.7) is an ideal of X for any ideal A of X.*

Proof. Obviously, $0 \in K$. Let $x, y \in X$ such that $x * y \in K$ and $y \in K$. Then there exist B_x, $B_y \in \mathcal{I}_f(X)$ such that $B_x \subseteq A$, $B_y \subseteq A$, $x * y \in B_x^{cl}$ and $y \in B_y^{cl}$. Since $B_x, B_y \subseteq B_x + B_y = \langle B_x \cup B_y\rangle$, we have $x * y \in B_x^{cl} \subseteq (B_x + B_y)^{cl}$ and $y \in B_y^{cl} \subseteq (B_x + B_y)^{cl}$, which imply that $x \in (B_x + B_y)^{cl}$. Since B_x, $B_y \in \mathcal{I}_f(X)$, we get $B_x + B_y \in \mathcal{I}_f(X)$ and $B_x + B_y \subseteq A$. Therefore $x \in K$, and K is an ideal of X. $\square$

Corollary 3.12. *If $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is a closure operation on $\mathcal{I}(X)$, then the set K in (3.7) is an ideal of X for any ideal A of X.*

Lemma 3.13 ([4])**.** (Extension property) *Let A and B be ideals of X such that $A \subseteq B$. If A is a positive implicative (resp., commutative and implicative) ideal, then so is B.*

Using Lemma 3.13 and (3.1), we have the following theorem.

Theorem 3.14. *Let "cl" be a weak closure operation on $\mathcal{I}(X)$. If A is a positive implicative (resp., commutative and implicative) ideal of X, then so is A^{cl}.*

The following example shows that the converse of Theorem 3.14 is not true in general.

Example 3.15. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$ and $A_4 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $(A_0)^{cl} = A_0$, $(A_1)^{cl} = (A_2)^{cl} = A_3$, and $(A_3)^{cl} = (A_4)^{cl} = A_4$. Then "$cl$"" is a weak closure operation on $\mathcal{I}(X)$. The ideal $A_2 = \{0, 2\}$ is not positive implicative (resp., commutative and implicative) ideal, but $(A_2)^{cl} = A_3 = \{0, 1, 2\}$ is a positive implicative (resp., commutative and implicative) ideal of X.

Theorem 3.16. *An operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ on $\mathcal{I}(X)$ defined by*

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl} = \cap\{I_\lambda \mid I_\lambda \in \mathcal{I}_\Gamma(X),\, A \subseteq I_\lambda, \lambda \in \Lambda\}\right) \tag{3.8}$$

is a weak closure operation on $\mathcal{I}(X)$ where $\mathcal{I}_\Gamma(X) \in \{\mathcal{I}_{pi}(X), \mathcal{I}_c(X), \mathcal{I}_m(X)\}$ and Λ is any index set.

Proof. Obviously, $A \subseteq A^{cl}$ for every $A \in \mathcal{I}(X)$. Let $A, B \in \mathcal{I}(X)$ be such that $A \subseteq B$. Then

$$\begin{aligned} B^{cl} &= \cap\{I_\lambda \mid I_\lambda \in \mathcal{I}_\Gamma(X),\, B \subseteq I_\lambda, \lambda \in \Lambda\} \\ &\supseteq \cap\{I_\lambda \mid I_\lambda \in \mathcal{I}_\Gamma(X),\, A \subseteq I_\lambda, \lambda \in \Lambda\} \\ &= A^{cl}, \end{aligned}$$

and so "cl" is a weak closure operation on $\mathcal{I}(X)$. □

The following example illustrates Theorem 3.16.

Example 3.17. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1, 3\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 2, 4\}$ and $A_5 = X$.

(1) Define a mapping $cl_1 : \mathcal{I}(X) \to \mathcal{I}(X)$ by

$$A^{cl_1} = \cap\{B \mid A \subseteq B \text{ and } B \in \mathcal{I}_{pi}(X)\}.$$

Then we have
$A_0^{cl_1} = A_1 \cap A_3 \cap A_5 = A_1$, $A_1^{cl_1} = A_1 \cap A_3 \cap A_5 = A_1$,
$A_2^{cl_1} = A_3 \cap A_5 = A_3$, $A_3^{cl_1} = A_3 \cap A_5 = A_3$, $A_4^{cl_1} = A_5 = A_5^{cl_1}$.
We can check that "cl_1" is a weak closure operation on $\mathcal{I}(X)$.

(2) We define an operation "cl_2" on $\mathcal{I}(X)$ by

$$A^{cl_2} = \cap\{B \mid A \subseteq B \text{ and } B \in \mathcal{I}_c(X)\}.$$

Then we have
$A_0^{cl_2} = A_2 \cap A_3 \cap A_4 \cap A_5 = A_2$, $A_1^{cl_2} = A_3 \cap A_5 = A_3$,
$A_2^{cl_2} = A_2 \cap A_3 \cap A_4 \cap A_5 = A_2$, $A_3^{cl_2} = A_3 \cap A_5 = A_3$,
$A_4^{cl_2} = A_4 \cap A_5 = A_4$, $A_5^{cl_2} = A_5$.
It is routine to verify that "cl_2" is a weak closure operation on $\mathcal{I}(X)$.

(3) We define an operation "cl_3" on $\mathcal{I}(X)$ by

$$A^{cl_3} = \cap\{B \mid A \subseteq B \text{ and } B \in \mathcal{I}_m(X)\}.$$

Then we have
$A_0^{cl_3} = A_3 \cap A_5 = A_3$, $A_1^{cl_3} = A_3 \cap A_5 = A_3$,
$A_2^{cl_3} = A_3 \cap A_5 = A_3$, $A_3^{cl_3} = A_3 \cap A_5 = A_3$,
$A_4^{cl_3} = A_5$, $A_5^{cl_3} = A_5$.
It is easy to show that "cl_3" is weak closure operation on $\mathcal{I}(X)$.

Let $\{cl_\lambda \mid \lambda \in \Lambda\}$ be a collection of operations on $\mathcal{I}(X)$. We define the intersection of cl_λ's, denoted by $\underset{\lambda\in\Lambda}{\cap} cl_\lambda$, as follows:

$$\underset{\lambda\in\Lambda}{\cap} cl_\lambda : \mathcal{I}(X) \to \mathcal{I}(X),\ A \mapsto \underset{\lambda\in\Lambda}{\cap} A^{cl_\lambda}.$$

Note that if cl_λ is a weak closure operation on $\mathcal{I}(X)$ for all $\lambda \in \Lambda$, then $\underset{\lambda\in\Lambda}{\cap} cl_\lambda$ is a weak closure operation on $\mathcal{I}(X)$ (see [2]). But the following example shows that the union of weak closure operations may not be a weak closure operation.

Example 3.18. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

There are four ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1, 4\}$, $A_2 = \{0, 2\}$ and $A_3 = X$. Define a mapping $cl_1 : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl_1} = A_1$, $A_1^{cl_1} = A_3$, $A_2^{cl_1} = A_3$, $A_3^{cl_1} = A_3$. Then "cl_1" is a weak closure operation on $\mathcal{I}(X)$. Also, define a mapping $cl_2 : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl_2} = A_2$, $A_1^{cl_2} = A_3$, $A_2^{cl_2} = A_3$, $A_3^{cl_2} = A_3$. Then "cl_2" is a weak closure operation on $\mathcal{I}(X)$.

Now if we define "cl_3" by $A^{cl_3} = A^{cl_1} \cup A^{cl_2}$, then "$cl_3$" is not a weak closure operation on $\mathcal{I}(X)$ because for an ideal A_0 of X, we have

$$A_0^{cl_3} = A_0^{cl_1} \cup A_0^{cl_2} = A_1 \cup A_2 = \{0, 1, 2, 4\}$$

which is not an ideal of X. Thus "cl_3" is not a weak closure operation on $\mathcal{I}(X)$.

Definition 3.19. Given a (weak) closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ on $\mathcal{I}(X)$, we define a new operation $cl_f : \mathcal{I}(X) \to \mathcal{I}(X)$ by

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A,\ B \in \mathcal{I}_f(X)\} \right), \tag{3.9}$$

where $\mathcal{I}_f(X)$ is the set of all finitely generated ideals of X.

Definition 3.20. A (weak) closure operation cl on $\mathcal{I}(X)$ is said to be of *finite type* if the following assertion is valid.

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl} = A^{cl_f} \right). \tag{3.10}$$

Note that every weak closure operation on a finite BCK-algebra is of finite type.

Example 3.21. Let X be a BCK-algebra of infinite order. Define an operation "cl'' on $\mathcal{I}(X)$ as follows:

$$A^{cl} = \begin{cases} X & \text{if } A \text{ is a maximal ideal or } A = X, \\ M & \text{otherwise,} \end{cases} \tag{3.11}$$

where M is a maximal ideal of X containing A. We can easily check that "cl" is a weak closure operation. Now let A be a maximal ideal of X which is not finitely generated. Then

$$A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A \text{ and } B \in \mathcal{I}_f(X)\} \subseteq M \subsetneq X = A^{cl},$$

and thus "cl" is a weak closure operation which is not of finite type.

For two operations "cl_1" and "cl_2" on $\mathcal{I}(X)$, we say that "cl_1" is *weaker* than "cl_2", denoted by $cl_1 \leq cl_2$, if $A^{cl_1} \subseteq A^{cl_2}$ for every $A \in \mathcal{I}(X)$.

Theorem 3.22. *Given an operation "cl" on $\mathcal{I}(X)$, we have*

(i) *If "cl" is a weak closure operation of finite type, then so is "cl_f", and it is largest in the set of weak closure operations which are weaker than "cl".*
(ii) *If "cl" is a (strong) quasi-prime weak closure operation, then so is "cl_f".*

Proof. (i) Let "cl" be a weak closure operation of finite type. Then "cl_f" is a weak closure operation on $\mathcal{I}(X)$ (see [2]). To prove that "cl_f" is of finite type, we should prove that $A^{cl_f} = A^{(cl_f)_f}$ for every ideal A of X. Clearly, we have $A^{cl_f} \subseteq A^{(cl_f)_f}$. Suppose that $x \in A^{(cl_f)_f}$. Then there exists a finitely generated ideal B such that $B \subseteq A$ and $x \in B^{cl_f}$. Since "cl" is a weak closure operation of finite type, we have $B^{cl} = B^{cl_f}$. Thus $x \in B^{cl}$, $B \subseteq A$ and B is finitely generated ideal. Therefore $x \in A^{cl_f}$ and $A^{cl_f} = A^{(cl_f)_f}$ which means that "cl_f" is a weak closure

operation on $\mathcal{I}(X)$ of finite type. Now let c be a weak closure operation on $\mathcal{I}(X)$ of finite type which is weaker than "cl". Let A be an ideal of X and $a \in A^c$. Then there exists a finitely generated ideal B of X such that $B \subseteq A$ and $a \in B^c$. It follows from $c \leq cl$ that $a \in B^{cl}$. Therefore $a \in A^{cl_f}$, and so $c \leq cl_f$.

(ii) Suppose that "cl" be a quasi prime weak closure operation on $\mathcal{I}(X)$. To prove that "cl_f" is a quasi prime weak closure operation, it is enough to show that $a \wedge A^{cl_f} \subseteq (a \wedge A)^{cl_f}$. Now let $x \in a \wedge A^{cl_f} = \langle \{a \wedge \alpha \mid \alpha \in A^{cl_f}\} \rangle$. Then there exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in A^{cl_f}$ such that

$$(\cdots((x \wedge (a \wedge \alpha_1)) * (a \wedge \alpha_2)) * \cdots) * (a \wedge \alpha_n) = 0.$$

Since $\alpha_i \in A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A \text{ and } B \in \mathcal{I}_f(X)\}$ for each $1 \leq i \leq n$, we have $\alpha_i \in A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A \textit{ and } B \in \mathcal{I}_f(X)\}$, and so there exists a finitely generated ideal B such that $\alpha_i \in B^{cl}$ and $B \subseteq A$. Since $\alpha_i \in B^{cl}$, we have

$$a \wedge \alpha_i \in \{a \wedge \beta \mid \beta \in B\} \subseteq \langle \{a \wedge \beta \mid \beta \in B\} \rangle = a \wedge B^{cl},$$

which implies that $a \wedge \alpha_i \in a \wedge B$ and

$$(\cdots((x \wedge (a \wedge \alpha_1)) * (a \wedge \alpha_2)) * \cdots) * (a \wedge \alpha_n) = 0.$$

This means that $x \in a \wedge B^{cl}$. Since "cl" is a quasi prime weak closure operation on $\mathcal{I}(X)$, it follows that

$$x \in a \wedge B^{cl} \subseteq (a \wedge B)^{cl} \subseteq (a \wedge A)^{cl} \subseteq (a \wedge A)^{cl_f}.$$

Therefore $x \in (a \wedge A)^{cl_f}$ and "cl_f" is a quasi-prime weak closure operation on $\mathcal{I}(X)$. Similarly, we can check that if "cl" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_f" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$. □

Definition 3.23. An operation $\alpha : \mathcal{I}(X) \to \mathcal{I}(X)$ is called a *positive implicative* (resp. *commutative* and *implicative*) *weak closure operation* if the following conditions are valid.

(i) For any $A, B \in \mathcal{I}_{pi}(X)$ (resp. $\mathcal{I}_c(X)$ and $\mathcal{I}_m(X)$),

$$A \subseteq A^{\alpha}, \tag{3.12}$$

$$A \subseteq B \;\Rightarrow\; A^{\alpha} \subseteq B^{\alpha}. \tag{3.13}$$

(ii) $(\forall A \notin \mathcal{I}_{pi}(X)(\text{resp., } \mathcal{I}_c(X) \text{ and } \mathcal{I}_m(X)))\,(A^{\alpha} = A)$.

Obviously, every positive implicative (resp., commutative and implicative) weak closure operation is a weak closure operation, but the converse is not true in general as seen in the following example.

Example 3.24. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0,1,3\}$, $A_2 = \{0,2\}$, $A_3 = \{0,1,2,3\}$, $A_4 = \{0,2,4\}$ and $A_5 = X$. Note that A_1, A_3 and A_5 are positive implicative ideals and A_0, A_2 and A_4 are not positive implicative ideals. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl} = A_0$ $A_1^{cl} = A_3$, $A_2^{cl} = A_2$, $A_3^{cl} = A_5$, $A_4^{cl} = A_4$ and $A_5^{cl} = X$. Then "cl" is a positive implicative weak closure operation on $\mathcal{I}(X)$. Now we define an operation "cl_1" on $\mathcal{I}(X)$ as follows:

$$A_0^{cl_1} = A_1,\ A_1^{cl_1} = A_3,\ A_2^{cl_1} = A_4,\ A_3^{cl_1} = A_5,\ A_4^{cl_1} = A_5 \text{ and } A_5^{cl_1} = X.$$

Then "cl_1" is a weak closure operation on $\mathcal{I}(X)$, but it is not positive implicative because the ideal A_2 is not a positive implicative ideal and $A_2^{cl_1} = A_4 \neq A_2$.

Example 3.25. Consider a BCK-algebra $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0,1\}$, $A_2 = \{0,1,2\}$, $A_3 = \{0,1,2,3\}$ and $A_4 = X$ where A_3 and A_4 are commutative ideals and A_0, A_1 and A_2 are not commutative ideals. Now define "cl" as follows:

$$A_0^{cl} = A_1,\ A_1^{cl} = A_2,\ A_2^{cl} = A_3,\ A_3^{cl} = A_4 \text{ and } A_4^{cl} = X$$

Then "cl" is a weak closure operation on $\mathcal{I}(X)$, but it is not commutative since the ideal A_2 is not a commutative ideal and $A_2^{cl} = A_3 \neq A_2$.

Example 3.26. Let $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

Then X is a BCK-algebra with seven ideals $A_0 = \{0\}$, $A_1 = \{0,1\}$, $A_2 = \{0,1,2\}$, $A_3 = \{0,1,4\}$ $A_4 = \{0,1,2,3\}$, $A_5 = \{0,1,2,4\}$ and $A_6 = X$. Note that A_2, A_4, A_5 and A_6 are implicative

ideals and A_0, A_1 and A_3 are not implicative ideals. Now we define an operation define "*cl*" on $\mathcal{I}(X)$ by

$$A_0^{cl} = A_1,\ A_1^{cl} = A_2,\ A_2^{cl} = A_5,\ A_3^{cl} = A_5,\ A_4^{cl} = A_6,\ A_5^{cl} = A_6 \text{ and } A_6^{cl} = X.$$

Then "*cl*" is a weak closure operation on $\mathcal{I}(X)$, but it is not implicative since the ideal A_3 is not an implicative ideal and $A_3^{cl} = A_5 \neq A_3$.

Given a weak closure operation, we kame a positive implicative weak closure operation.

Theorem 3.27. *Given $A \in \mathcal{I}(X)$, let "cl" be a weak closure operation on $\mathcal{I}(X)$ and "cl_{pi}" be an operation on $\mathcal{I}(X)$ such that $cl \leq cl_{pi}$ and*

(i) $(\forall C \in \mathcal{I}(X))\left(A \subseteq C \ \Rightarrow\ C^{cl_{pi}} = C^{cl}\right)$.
(ii) $(\forall C \in \mathcal{I}(X))\left(C \subsetneq A \ \Rightarrow\ C^{cl_{pi}} = C\right)$.
(iii) *For any $C \in \mathcal{I}(X)$, if A and C have no inclusion relation, then $C^{cl_{pi}} = C$.*

If A is positive implicative (resp., commutative and implicative) ideals of X, then "cl_{pi}" is a positive implicative (resp., commutative and implicative) weak closure operation on $\mathcal{I}(X)$.

Proof. Let A and C be ideals of X such that $A \subseteq C$. Suppose that A is a positive implicative (resp., commutative and implicative) ideal of X. Then C is a positive implicative (resp., commutative and implicative) ideal of X by Lemma 3.13. Let A and C be ideals of X such that $C \subseteq A$. If A is not a positive implicative (resp., commutative and implicative) ideal of X, then C is not a positive implicative (resp., commutative and implicative) ideal of X. Therefore "*cl*" is a positive implicative (resp., commutative and implicative) weak closure operation on $\mathcal{I}(X)$. □

The following examples illustrate Theorem 3.27.

Example 3.28. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table,

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 4\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 1, 4\}$ and $A_5 = X$ in which A_1, A_3, A_4 and A_5 are positive implicative ideals and A_0 and A_2 are not positive implicative ideals. Now define "*cl*" as follows:

$$A_0^{cl} = A_0, A_1^{cl} = A_3, A_2^{cl} = A_4, A_3^{cl} = A_3, A_4^{cl} = A_5 \text{ and } A_5^{cl} = X.$$

Then "*cl*" is a weak closure operation. Now let $A = \{0, 4\} = A_2$ which is not a positive implicative ideal. By using Theorem 3.27 we have "cl_{pi} as follows:

$$A_0^{cl_{pi}} = A_0,\ A_1^{cl_{pi}} = A_1,\ A_2^{cl_{pi}} = A_4,\ A_3^{cl_{pi}} = A_3,\ A_4^{cl_{pi}} = A_5 \text{ and } A_5^{cl_{pi}} = X.$$

Clearly, $cl \leq cl_{pi}$. But, "cl_{pi}" is not a positive implicative weak closure operation because $A_2^{cl_{pi}} = A_4 \neq A_2$. Now let $A = \{0, 1\} = A_1$ which is a positive implicative ideal. By using Theorem 3.27 we have "cl_{pi}" as follows:

$$A_0^{cl_{pi}} = A_0,\ A_1^{cl_{pi}} = A_3,\ A_2^{cl_{pi}} = A_2,\ A_3^{cl_{pi}} = A_3,\ A_4^{cl_{pi}} = A_5 \text{ and } A_5^{cl_{pi}} = X.$$

Clearly, $cl \leq cl_{pi}$ and "cl_{pi}" is a positive implicative weak closure operation.

Example 3.29. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	4	4	4	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2, 3\}$, and $A_4 = X$ in which A_3 and A_4 are commutative ideals and A_0, A_1 and A_2 are not commutative ideals. Now define "cl" as follows:

$$A_0^{cl} = A_1, A_1^{cl} = A_3, A_2^{cl} = A_3, A_3^{cl} = A_4 \text{ and } A_4^{cl} = X.$$

Then "cl" is a weak closure operation. Now let $A = \{0, 1\} = A_1$ which is not a commutative ideal. By using Theorem 3.27 we have "cl_c as follows:

$$A_0^{cl_c} = A_0, A_1^{cl_c} = A_3, A_2^{cl_c} = A_2, A_3^{cl_c} = A_4 \text{ and } A_4^{cl_c} = X.$$

Clearly, $cl \leq cl_c$. But, "cl_c" is not a commutative weak closure operation because $A_1^{cl_c} = A_3 \neq A_1$. Now let $A = \{0, 1, 2, 3\} = A_3$ which is a commutative ideal. By using Theorem 3.27 we have "cl_c" as follows:

$$A_0^{cl_c} = A_0, A_1^{cl_c} = A_1, A_2^{cl_c} = A_2, A_3^{cl_c} = A_4 \text{ and } A_4^{cl_c} = X.$$

Clearly, $cl \leq cl_c$ and "cl_c" is a commutative weak closure operation.

Example 3.30. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 2\}$, $A_2 = \{0, 1\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 1, 4\}$ and $A_5 = X$ in which A_2, A_3, A_4 and A_5 are implicative ideals and A_0 and A_1 are not implicative ideals. Now define "cl" as follows:

$$A_0^{cl} = A_1, A_1^{cl} = A_3, A_2^{cl} = A_4, A_3^{cl} = A_5, A_4^{cl} = A_4 \text{ and } A_5^{cl} = X.$$

Then "cl" is a weak closure operation. Now let $A = \{0, 2\} = A_1$ which is not an implicative ideal. By using Theorem 3.27 we have "cl_m" as follows:

$$A_0^{cl_m} = A_0, A_1^{cl_m} = A_3, A_2^{cl_m} = A_2, A_3^{cl_m} = A_5, A_4^{cl_m} = A_4 \text{ and } A_5^{cl_m} = X.$$

Clearly, $cl \leq cl_m$. But, "cl_m" is not an implicative weak closure operation because $A_1^{cl_m} = A_3 \neq A_1$. Now let $A = \{0, 1\} = A_2$ which is an implicative ideal. By using Theorem 3.27 we have "cl_m" as follows:

$$A_0^{cl_m} = A_0, A_1^{cl_m} = A_1, A_2^{cl_m} = A_4, A_3^{cl_m} = A_5, A_4^{cl_m} = A_4 \text{ and } A_5^{cl_m} = X.$$

Clearly, $cl \leq cl_m$ and "cl_m" is an implicative weak closure operation.

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Communication between relation information systems*

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Abstract: Communication between information systems is considered as an important issue in granular computing. A relation information system is the generalization of an information system. This paper investigates communication between relation information systems and obtain some invariant characterizations of relation information systems under homomorphism.

Keywords: Relation information system; Reduction; Consistent function; Relation mapping; Homomorphism.

1 Introduction

Rough set theory, proposed by Pawlak [17], is an important tool for dealing with fuzzyness and uncertainty of knowledge and has become an active branch of information science. With more than thirty years development, rough set theory has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [13, 14, 15, 16].

Communication between information systems is a very important topic in the field of artificial intelligence. In mathematics, it can be explained as a mapping between information systems. The approximations and reductions in the original system can be regarded as encoding while the image system is seen as an interpretive system. The concept of homomorphisms as a kind of tool to study relationships between information systems with rough sets was introduced by Grzymala-Busse [1, 2]. A homomorphism can be viewed as a special communication between two information systems. As explained in [23], homomorphisms allow one to translate the information contained in one granular world into the

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granularity of another granular world and thus provide a communication mechanism for exchanging information with other granular worlds. Li et al. [5] studied invariant characters of information systems under some homomorphism. Wang et al. [20, 21] introduced the notions of consistent functions, relation mappings and relation information systems which are the generalization of information systems. By using these notions, they proposed the homomorphisms as a mechanism for communicating between relation information systems. Zhu et al. [26] obtained some improved results on communication between relation information systems. Li et al. [12] investigated communication between knowledge bases. It should be pointed out that some other related works investigating information systems through homomorphisms [1, 2, 3, 5, 25] are based on equivalence relations or other particular relations and are quite different from [20, 21, 26].

The purpose of this paper is to investigate some invariant characterizations of relation information systems under homomorphisms.

2 Preliminaries

In this section, we recall some basic concepts on consistent functions, relation mappings and relation information systems.

Throughout this paper, U denotes a non-empty finite set called the universe, 2^U denotes the family of all subsets of U, $2^{U\times U}$ denotes the family of all binary relations on U, All mappings are assumed to be surjective.

For $R \in 2^{U\times U}$, the successor neighborhood of $x \in U$ with respect to R will be denoted by $R_s(x)$, that is, $R_s(x) = \{y \in U : xRy\}$ ([22]). Denote

$$U/R = \{R_s(x) : x \in U\}.$$

If R is an equivalence relation on U, then $\forall\, x \in U$, $R_s(x) = [x]_R$.

For $\mathcal{R} \subseteq 2^{U\times U}$, denote $ind(\mathcal{R}) = \bigcap\limits_{R\in\mathcal{R}} R$.

2.1 Consistent functions

Definition 2.1 ([20, 21]). *Let U and V be two finite nonempty universes, $f\colon U \to V$ a mapping and $R \in 2^{U\times U}$. Define*

$$[x]_f = \{u \in U : f(u) = f(x)\},$$

$$(x)_R = \{u \in U : R_s(u) = R_s(x)\}.$$

Then $\{[x]_f : x \in U\}$ and $\{(x)_R : x \in U\}$ are two partitions on U. If $[x]_f \subseteq R_s(u)$ or $[x]_f \cap R_s(u) \neq \emptyset$ for any $x, u \in U$, then f is called a type-1 consistent function with respect to R on U. If $[x]_f \subseteq (x)_R$ for any $x \in U$, then f is called a type-2 consistent function with respect to R on U.

Remark 2.2. (1) $\forall\, x \in U$, $[x]_f = f^{-1}(f(x))$.

(2) *If R is an equivalence relation on U, then $\forall\, x \in U$, $(x)_R = [x]_R$.*

(3) *If f is type-2 consistent with respect to R on U and $f(u) = f(x)$, then $R_s(u) = R_s(x)$.*

Obviously,

f is type-1 $\iff$ If $[x]_f \cap R_s(y) \neq \emptyset$, then $[x]_f \subseteq R_s(y)$
$\iff$ If $[x]_f \not\subseteq R_s(y)$, then $[x]_f \cap R_s(y) = \emptyset$,
f is type-2 $\iff$ If $f(x_1) = f(x_2)$, then $R_s(x_1) = R_s(x_2)$.

2.2 Relation mappings

Definition 2.3 ([20, 21]). *Let $f : U \to V$ be a mapping. Define*

$$\hat{f} : 2^{U\times U} \to 2^{V\times V},\ R| \to \hat{f}(R) = \bigcup_{x\in U} (\{f(x)\} \times f(R_s(x)));$$

$$\hat{f}^{-1} : 2^{V\times V} \to 2^{U\times U},\ T| \to \hat{f}^{-1}(T) = \bigcup_{y\in V} (\{f^{-1}(y)\} \times f^{-1}(T_s(y))).$$

Then $\hat{f}$ and $\hat{f}^{-1}$ are called the relation mapping and inverse relation mapping induced by f, respectively.

Obviously,

$$y_1\hat{f}(R)y_2 \iff \exists\ x_1, x_2 \in U,\ y_1 = f(x_1),\ y_2 = f(x_1) \text{ and } x_1Rx_2,$$

$$x_1\hat{f}^{-1}(T)x_2 \iff \exists\ y_1, y_2 \in V,\ y_1 = f(x_1), y_2 = f(x_1) \text{ and } y_1Ty_2.$$

For $\mathcal{R} \subseteq 2^{U\times U}$, denote

$$\hat{f}(\mathcal{R}) = \{\hat{f}(R) \mid R \in \mathcal{R}\}.$$

Proposition 2.4 ([20]). *If $f : U \to V$ is both type-1 and type-2 consistent with respect to $R \in 2^{U\times U}$, then*

$$\hat{f}^{-1}(\hat{f}(R)) = R.$$

2.3 Relation information systems

Definition 2.5 ([13]). *An information system is a pair (U, A) of non-empty finite sets U and A, where U is a set of objects and A is a set of attributes; each attribute $a \in A$ is a function $a : U \to V_a$, where V_a is the set of values (called domain) of attribute a.*

If (U, A) is an information system and $B \subseteq A$, then an equivalence relation (or indiscernibility relation) R_B can be defined by

$$(x, y) \in R_B \iff a(x) = a(y),\ \forall\ a \in B.$$

Definition 2.6 ([20]). *A pair $(U, \mathcal{R})$ is called a relation information system, if $\mathcal{R} \subseteq 2^{U\times U}$.*

Definition 2.7. *Let (U, A) be an information system. Put*

$$\mathcal{R} = \{R_{\{a\}} :\ a \in A\}.$$

Then the pair $(U, \mathcal{R})$ is called the relation information system induced by (U, A).

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Definition 2.8 ([20]). *Let $f\colon U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. If f is type-1 (resp. type-2) consistent with respect to R on U for every $R \in \mathcal{R}$, then f is called type-1 (resp. type-2) consistent with respect to $\mathcal{R}$ on U.*

Proposition 2.9 ([20]). *Let $f : U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. If f is both type-1 and type-2 consistent with respect to $\mathcal{R}$, then $\hat{f}(ind(\mathcal{R})) = ind(\hat{f}(\mathcal{R}))$.*

Proposition 2.10 ([20]). *Let $f : U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. If f is both type-1 and type-2 consistent with respect to $\mathcal{R}$, then $\hat{f}^{-1}(\hat{f}(ind(\mathcal{R})) = ind(\mathcal{R})$.*

Definition 2.11 ([20]). *Let $f : U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. Then the pair $(V, \hat{f}(\mathcal{R}))$ is called an f-induced relation information system of $(U, \mathcal{R})$.*

Definition 2.12 ([20]). *Let $(U, \mathcal{R})$ be a relation information system and $(V, \hat{f}(\mathcal{R}))$ an f-induced relation information system of $(U, \mathcal{R})$. If f is both type-1 and type-2 consistent with respect to $\mathcal{R}$ on U, then f is called a homomorphism from $(U, \mathcal{R})$ to $(V, \hat{f}(\mathcal{R}))$. We write*

$$(U, \mathcal{R}) \ \sim_f \ (V, \hat{f}(\mathcal{R})).$$

We often consider reductions in a relation information system by deleting unrelated or unimportant elements with the requirement of keeping the ability of classification.

Definition 2.13 ([20]). *Let $(U, \mathcal{R})$ be a relation information system and $\mathcal{P} \subseteq \mathcal{R}$.*

(1) *$\mathcal{P}$ is called a coordination subfamily of $\mathcal{R}$, if $ind(\mathcal{P}) = ind(\mathcal{R})$.*

(2) *$R \in \mathcal{P}$ is called independent in $\mathcal{P}$, if $ind(\mathcal{P} - \{R\}) \neq ind(\mathcal{P})$; $\mathcal{P}$ is called a independent subfamily of $\mathcal{R}$, if $\forall\ R \in \mathcal{P}$, R is independent in $\mathcal{P}$.*

(3) *$\mathcal{P}$ is called a reductions of $\mathcal{R}$, if $\mathcal{P}$ is both coordination and independent.*

In this paper, the set of all coordination subfamilies (resp., all reductions) of $\mathcal{R}$ is denoted by $co(\mathcal{R})$ (resp., $red(\mathcal{R})$).

Obviously,

$\mathcal{P} \in red(\mathcal{R}) \ \Leftrightarrow \mathcal{P} \in co(\mathcal{R})$ and $\forall\ \mathcal{Q} \subset \mathcal{P}$, $\mathcal{Q} \notin co(\mathcal{R})$.

3 Some results on reductions in relation information systems

Proposition 3.1. *Let $(U, \mathcal{R})$ be a relation information system. Then $red(\mathcal{R}) \neq \emptyset$.*

Proof. Suppose $\forall\ R \in \mathcal{R}$, $\mathcal{R} - \{R\} \notin co(\mathcal{R})$. Then $\mathcal{R} \in red(\mathcal{R})$.

Suppose $\exists\ R_1 \in \mathcal{R}$, $\mathcal{R} - \{R_1\} \in co(\mathcal{R})$. Then, we consider $\mathcal{R} - \{R_1\}$. Again suppose $\forall\ R \in \mathcal{R} - \{R_1\}$, $(\mathcal{R} - \{R_1\}) - \{R\} \notin co(\mathcal{R})$. Then $\mathcal{R} - \{R_1\} \in red(\mathcal{R})$. Again suppose $\exists\ R_2 \in \mathcal{R} - \{R_1\}$, $(\mathcal{R} - \{R_1\}) - \{R_2\} \in co(\mathcal{R})$. Then, we consider

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$\mathcal{R}-\{R_1, R_2\}$. Repeat this process. Since $\mathcal{R}$ is finite, we can find a reductions of $\mathcal{R}$.

Thus $red(\mathcal{R}) \neq \emptyset$. □

Definition 3.2. *Let $(U, \mathcal{R})$ be a relation information system. Put*

$$\mathcal{D}(x, y) = \{R \in \mathcal{R} | (x, y) \notin R\} \ (x, y \in U).$$

Then

(1) $\mathcal{D}(x, y)$ is called is called the discernibility subfamily of $\mathcal{R}$ on x and y.

(2) $\mathfrak{D}(\mathcal{R}) = (d_{ij})_{n\times n}$ is called the discernibility matrix of $\mathcal{R}$ where $U = \{x_1, x_2, \cdots, x_n\}$ and $d_{ij} = \mathcal{D}(x_i, x_j)$ $(1 \leq i, j \leq n)$.

Example 3.3. *Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. We consider the relation information system $(U, \mathcal{R})$ where $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$ and*

$U/R_1 = \{\{x_1, x_2, x_5\}, \{x_3, x_4, x_6\}\}$,
$U/R_2 = \{\{x_1, x_6\}, \{x_2, x_3, x_4, x_5\}\}$,
$U/R_3 = \{\{x_1, x_2, x_5, x_6\}, \{x_3, x_4\}\}$,
$U/R_4 = \{\{x_1, x_2, x_5\}, \{x_3, x_4, x_6\}\}$.

We can obtain the discernibility matrix $\mathfrak{D}(\mathcal{R})$ as follows:

$$\begin{pmatrix} \emptyset & \{R_2\} & \mathcal{R} & \mathcal{R} & \{R_2\} & \{R_1, R_4\} \\ \{R_2\} & \emptyset & \{R_1, R_3, R_4\} & \{R_1, R_3, R_4\} & \emptyset & \{R_1, R_2, R_4\} \\ \mathcal{R} & \{R_1, R_3, R_4\} & \emptyset & \emptyset & \{R_1, R_3, R_4\} & \{R_2, R_3\} \\ \mathcal{R} & \{R_1, R_3, R_4\} & \emptyset & \emptyset & \{R_1, R_3, R_4\} & \{R_2, R_3\} \\ \{R_2\} & \emptyset & \{R_1, R_3, R_4\} & \{R_1, R_3, R_4\} & \emptyset & \{R_1, R_2, R_4\} \\ \{R_1, R_4\} & \{R_1, R_2, R_4\} & \{R_2, R_3\} & \{R_2, R_3\} & \{R_1, R_2, R_4\} & \emptyset \end{pmatrix}$$

Discernibility family can expediently judge coordination families and reductions.

Proposition 3.4. *Let $(U, \mathcal{R})$ be a relation information system. Then*

$$\mathcal{P} \in co(\mathcal{R}) \iff If \ (x, y) \notin ind(\mathcal{R}), \ then \ \mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset.$$

Proof. (1) "$\Longrightarrow$". Let $(x, y) \notin ind(\mathcal{R})$. Since $\mathcal{P} \in co(\mathcal{R})$, we have $ind(\mathcal{P}) = ind(\mathcal{R})$. Then $(x, y) \notin ind(\mathcal{P})$. It follows $(x, y) \notin P \ for \ some \ P \in \mathcal{P}$.

$(x, y) \notin P$ implies $P \in \mathcal{D}(x, y)$. Then $P \in \mathcal{P} \cap \mathcal{D}(x, y)$.

Thus $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$.

"$\Longleftarrow$". Suppose $\mathcal{P} \notin co(\mathcal{R})$. Then $ind(\mathcal{P}) \neq ind(\mathcal{R})$. It follows $ind(\mathcal{P}) - ind(\mathcal{R}) \neq \emptyset$. Pick

$$(x, y) \in ind(\mathcal{P}) - ind(\mathcal{R}).$$

Since $(x, y) \notin ind(\mathcal{R})$, we have $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$.

Note that $(x, y) \in ind(\mathcal{P})$. Then $\forall \ P \in \mathcal{P}$, $(x, y) \in P$. So $P \notin \mathcal{D}(x, y)$. Thus $\mathcal{P} \cap \mathcal{D}(x, y) = \emptyset$. This is a contradiction.

Thus $\mathcal{P} \in co(\mathcal{R})$. □

Theorem 3.5. *Let $(U, \mathcal{R})$ be a relation information system. Then $\mathcal{P} \in red(\mathcal{R})$ $\Longleftrightarrow$ (1) If $(x, y) \notin ind(\mathcal{R})$, then $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$;*
(2) $\forall\ R \in \mathcal{P}$, $\exists\ (x_R, y_R) \in ind(\mathcal{R})$, $(\mathcal{P} - \{R\}) \cap \mathcal{D}(x_R, y_R) = \emptyset$.

Proof. This holds by Proposition 3.4. □

Definition 3.6. *Let $(U, \mathcal{R})$ be a relation information system. Put*

$$core(\mathcal{R}) = \bigcap_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P}.$$

Then $core(\mathcal{R})$ is called the core of $\mathcal{R}$. Moreover,
(1) $R \in \mathcal{R}$ is called necessary, if $R \in core(\mathcal{R})$.
(2) $R \in \mathcal{R}$ is called relatively necessary, if $R \in \bigcup\limits_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P} - core(\mathcal{R})$.
(3) $R \in \mathcal{R}$ is called unnecessary, if $R \in \mathcal{R} - \bigcup\limits_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P}$.

Discernibility family can easily determine the core.

Proposition 3.7. *Let $(U, \mathcal{R})$ be a relation information system. The following are equivalent:*
(1) R is necessary;
(2) R is independent in $\mathcal{R}$;
(3) $\exists\ x, y \in U$, $\mathcal{D}(x, y) = \{R\}$.

Proof. (1) $\Longrightarrow$ (2). Suppose that R is not independent in $\mathcal{R}$. Then

$$ind(\mathcal{R} - \{R\}) = ind(\mathcal{R}).$$

It follows $\mathcal{R} - \{R\} \in co(\mathcal{R})$. Consider $\mathcal{R} - \{R\}$. By Proposition 3.1, $\exists\ \mathcal{P} \subseteq \mathcal{R} - \{R\}$, $\mathcal{P} \in red(\mathcal{R})$.

$\mathcal{P} \subseteq \mathcal{R} - \{R\}$ implies $R \notin \mathcal{P}$. Then R is not necessary. This is a contradiction.

(2) $\Longrightarrow$ (1). Suppose that R is not necessary. Then $\exists\ \mathcal{P} \in red(\mathcal{R})$, $R \notin \mathcal{P}$. So $\mathcal{P} \subseteq \mathcal{R} - \{R\} \subseteq \mathcal{R}$. It follows

$$ind(\mathcal{P}) \supseteq ind(\mathcal{R} - \{R\}) \supseteq ind(\mathcal{R}).$$

By $\mathcal{P} \in red(\mathcal{R})$, $ind(\mathcal{P}) = ind(\mathcal{R})$. Then $ind(\mathcal{R} - \{R\}) = ind(\mathcal{R})$. So R is not independent in $\mathcal{R}$. This is a contradiction.

(2) $\Longrightarrow$ (3). Since R is independent in $\mathcal{R}$, we have $ind(\mathcal{R} - \{R\}) \neq ind(\mathcal{R})$. Then $ind(\mathcal{R} - \{R\}) - ind(\mathcal{R}) \neq \emptyset$. Pick

$$(x, y) \in ind(\mathcal{R} - \{R\}) - ind(\mathcal{R}).$$

Denote $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$. Then $R = R_j$ for some $j \leq n$. So

$$(x, y) \in \bigcap_{1 \leq i \leq n, i \neq j} R_i - \bigcap_{1 \leq i \leq n} R_i.$$

It follows $(x,y) \notin R_j$ and $(x,y) \in R_i$ $(i \neq j)$.

Thus $\mathcal{D}(x,y) = \{R\}$.

(3) $\Longrightarrow$ (2). Since $\exists\ x,y \in U$, $\mathcal{D}(x,y) = \{R\}$, we have

$$(x,y) \notin R,\ (x,y) \in R^{'}\ (R^{'} \neq R).$$

Then $(x,y) \in ind(\mathcal{R} - \{R\})$. But $(x,y) \notin ind(\mathcal{R})$.

Thus $ind(\mathcal{R} - \{R\}) \neq ind(\mathcal{R})$.

Hence R is independent in $\mathcal{R}$. □

Proposition 3.8. *Let $(U,\mathcal{R})$ be a relation information system. Denote*

$$R^\star = \bigcup_{\mathcal{P}\in co(\mathcal{R})} ind(\mathcal{P} - \{R\}).$$

Then the following are equivalent.

(1) *R is unnecessary;*

(2) $\forall\ \mathcal{P} \in co(\mathcal{R})$, $\mathcal{P} - \{R\} \in co(\mathcal{R})$;

(3) *$R^\star = ind(\mathcal{R})$;*

(4) *$R^\star \subseteq R$.*

Proof. (1) $\Longrightarrow$ (2). By Proposition 3.1, $\exists\ \mathcal{Q} \subseteq \mathcal{P}$, $\mathcal{Q} \in red(\mathcal{R})$. Since R is unnecessary, we have $R \notin \mathcal{Q}$. It follows $\mathcal{Q} \subseteq \mathcal{R} - \{R\}$. Then

$$\mathcal{Q} \subseteq \mathcal{P} \cap (\mathcal{R} - \{R\}) = \mathcal{P} - \{R\} \subseteq \mathcal{P}.$$

We have

$$ind(\mathcal{Q}) \supseteq ind(\mathcal{R} - \{R\}) \supseteq ind(\mathcal{P}).$$

Note that $\mathcal{P} \in co(\mathcal{R})$ and $\mathcal{Q} \in red(\mathcal{R})$. Then $ind(\mathcal{P}) = ind(\mathcal{R}) = ind(\mathcal{Q})$.

Thus $ind(\mathcal{P} - \{R\}) = ind(\mathcal{R})$. This shows $\mathcal{P} - \{R\} \in co(\mathcal{R})$.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (1). Suppose $\exists\ \mathcal{P} \in red(\mathcal{R})$, $R \in \mathcal{P}$. Then $\mathcal{P} - \{R\} \subset \mathcal{P}$. Since $\mathcal{P} \in red(\mathcal{R})$, we have $\mathcal{P} - \{R\} \notin co(\mathcal{R})$. Then $ind(\mathcal{P} - \{R\}) - ind(\mathcal{R}) \neq \emptyset$. $\mathcal{P} \in red(\mathcal{R})$ implies $ind(\mathcal{P}) = ind(\mathcal{R})$. Then

$$ind(\mathcal{P} - \{R\}) - ind(\mathcal{P}) \neq \emptyset.$$

Pick $(x,y) \in ind(\mathcal{P} - \{R\}) - ind(\mathcal{P})$. Note that $ind(\mathcal{P}) = ind(\mathcal{P} - \{R\}) \cap R$. Then $(x,y) \notin R$.

Since $\mathcal{P} \in co(\mathcal{R})$ and $R^\star \subseteq R$, we have $ind(\mathcal{P} - \{R\}) \subseteq R$. Then $(x,y) \in R$. This is a contradiction.

Thus R is unnecessary. □

Theorem 3.9. *Let $(U,\mathcal{R})$ be a relation information system. Then*

(1) *R is necessary $\Leftrightarrow \mathcal{R} - \{R\} \notin co(\mathcal{R})$.*

(2) *R is relatively necessary $\Leftrightarrow \mathcal{R} - \{R\} \in co(\mathcal{R})$ and $R^\star \nsubseteq R$.*

(3) *R is unnecessary $\Leftrightarrow R^\star \subseteq R$.*

Proof. This holds by Proposition 3.7 and Proposition 3.8. □

Example 3.10. *In Example 3.3, we have*
(1) R_2 *is necessary.*
(2) R_1 *and* R_4 *are relatively necessary.*
(3) R_3 *is unnecessary.*
(4) $red(\mathcal{R}) = \{\{R_1, R_2\}, \{R_2, R_4\}\}$, $core(\mathcal{R}) = \{R_2\}$.

4 Communication between relation information systems

Proposition 4.1. *Let* $(U, \mathcal{R}) \sim_f (V, \hat{f}(\mathcal{R}))$. *Then*
(1) $\mathcal{P} \in co(\mathcal{R}) \iff \hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.
(2) $co(\hat{f}(\mathcal{R})) = \hat{f}(co(\mathcal{R}))$.

Proof. (1) "$\Longrightarrow$". Since $\mathcal{P} \in co(\mathcal{R})$, we have $ind(\mathcal{P}) = ind(\mathcal{R})$. Then

$$\hat{f}(ind(\mathcal{P})) = \hat{f}(ind(\mathcal{R})).$$

By Proposition 2.6,

$$ind(\hat{f}(\mathcal{P})) = ind(\hat{f}(\mathcal{R})).$$

Thus $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.
"$\Longleftarrow$". Since $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$, we have

$$ind(\hat{f}(\mathcal{P})) = ind(\hat{f}(\mathcal{R})).$$

By Proposition 2.6,

$$\hat{f}(ind(\mathcal{P})) = \hat{f}(ind(\mathcal{R})).$$

Then

$$\hat{f}^{-1}(\hat{f}(ind(\mathcal{P}))) = \hat{f}^{-1}(\hat{f}(ind(\mathcal{R}))).$$

By Proposition 2.7, $ind(\mathcal{P}) = ind(\mathcal{R})$.
Thus $\mathcal{P} \in co(\mathcal{R})$.
(2) By (1),

$$\begin{aligned}\hat{f}(co(\mathcal{R})) &= \{\hat{f}(\mathcal{P}) | \mathcal{P} \in co(\mathcal{R})\} \\ &= \{\hat{f}(\mathcal{P}) | \hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))\} \\ &= co(\hat{f}(\mathcal{R})).\end{aligned}$$

□

Theorem 4.2. *Let* $(U, \mathcal{R}) \sim_f (V, \hat{f}(\mathcal{R}))$. *Then*
(1) $\mathcal{P} \in red(\mathcal{R}) \iff \hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$.
(2) $red(\hat{f}(\mathcal{R})) = \hat{f}(red(\mathcal{R}))$.

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Proof. (1) "$\Longrightarrow$". Since $\mathcal{P} \in red(\mathcal{R})$, we have $\mathcal{P} \in co(\mathcal{R})$. By Proposition 4.1, $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.

$\forall\ \mathcal{T} \subset \hat{f}(\mathcal{P})$. Pick $\mathcal{Q} \subseteq \mathcal{R}$, $\mathcal{T} = \hat{f}(\mathcal{Q})$. Then $\hat{f}(\mathcal{Q}) \subset \hat{f}(\mathcal{P})$. By Proposition 2.4,

$$\mathcal{Q} = \hat{f}^{-1}(\hat{f}(\mathcal{Q})) \subseteq \hat{f}^{-1}(\hat{f}(\mathcal{P})) = \mathcal{P}.$$

Suppose $\mathcal{Q} = \mathcal{P}$. Then $\mathcal{T} = \hat{f}(\mathcal{Q}) = \hat{f}(\mathcal{P})$. This is a contradiction.

Thus $\mathcal{Q} \subset \mathcal{P}$.

Since $\mathcal{P} \in red(\mathcal{R})$, we have $\mathcal{Q} \notin co(\mathcal{R})$. By Proposition 4.1, $\mathcal{T} = \hat{f}(\mathcal{Q}) \notin co(\hat{f}(\mathcal{R}))$.

Hence $\hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$.

"$\Longleftarrow$". Since $\hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$, we have $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$. By Proposition 4.1, $\mathcal{P} \in co(\mathcal{R})$.

$\forall\ \mathcal{Q} \subset \mathcal{P}$, $\hat{f}(\mathcal{Q}) \subseteq \hat{f}(\mathcal{P})$. Suppose $\hat{f}(\mathcal{Q}) = \hat{f}(\mathcal{P})$. By Proposition 2.4,

$$\mathcal{Q} = \hat{f}^{-1}(\hat{f}(\mathcal{Q})) = \hat{f}^{-1}(\hat{f}(\mathcal{P})) = \mathcal{P}.$$

This is a contradiction. Thus $\hat{f}(\mathcal{Q}) \subset \hat{f}(\mathcal{P})$.

Since $\hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$, we have $\hat{f}(\mathcal{Q}) \notin co(\hat{f}(\mathcal{R}))$. By Proposition 4.1, $\mathcal{Q} \notin co(\mathcal{R})$.

Hence $\mathcal{P} \in red(\mathcal{R})$.

(2) By (1),

$$\begin{aligned}\hat{f}(red(\mathcal{R})) &= \{\hat{f}(\mathcal{P}) | \mathcal{P} \in red(\mathcal{R})\} \\ &= \{\hat{f}(\mathcal{P}) | \hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))\} \\ &= red(\hat{f}(\mathcal{R})).\end{aligned}$$

□

Remark 4.3. *Theorem 3.20(1) is Theorem 4.4 in [20]. We just prove this result from another angle.*

Lemma 4.4. *Let* $(U, \mathcal{R}) \ \sim_f \ (V, \hat{f}(\mathcal{R}))$. *Then*

$$\hat{f}(\mathcal{R} - \{R\}) = \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}.$$

Proof. $\forall\ S \in \mathcal{R} - \{R\}$, $S \neq R$. By Proposition 2.4, $\hat{f}(S) \neq \hat{f}(R)$. It follows $\hat{f}(S) \in \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}$. Thus

$$\hat{f}(\mathcal{R} - \{R\}) \subseteq \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}.$$

On the other hand, $\forall\ T \in \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}$, $T = \hat{f}(S)$ for some $S \in \mathcal{R}$. $T \notin \{\hat{f}(R)\}$ implies $\hat{f}(S) \neq \hat{f}(R)$. Then $S \neq R$. So $S \in \mathcal{R} - \{R\}$. It follows $T \in \hat{f}(\mathcal{R} - \{R\})$. Thus

$$\hat{f}(\mathcal{R} - \{R\}) \supseteq \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}.$$

Hence $\hat{f}(\mathcal{R} - \{R\}) = \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}$. □

Theorem 4.5. *Let* $(U,\mathcal{R}) \sim_f (V,\hat{f}(\mathcal{R}))$. *Then*

$$R \in core(\mathcal{R}) \iff \hat{f}(R) \in core(\hat{f}(\mathcal{R})).$$

Proof. This holds by Theorem 3.9(1), Proposition 4.1(1) and Lemma 4.4. □

Theorem 4.6. *Let* $(U,\mathcal{R}) \sim_f (V,\hat{f}(\mathcal{R}))$. *Then*

$$\hat{f}(core(\mathcal{R})) = core(\hat{f}(\mathcal{R})).$$

Proof. By Theorem 3.23,

$$\begin{aligned}\hat{f}(core(\mathcal{R})) &= \{\hat{f}(R) | R \in core(\mathcal{R})\}\\ &= \{\hat{f}(R) | \hat{f}(R) \in core(\hat{f}(\mathcal{R}))\}\\ &= core(\hat{f}(\mathcal{R})).\end{aligned}$$

□

Theorem 4.7. *Let* $(U,\mathcal{R}) \sim_f (V,\hat{f}(\mathcal{R}))$. *Then*

$$R \textit{ is unnecessary} \Leftrightarrow \hat{f}(R) \textit{ is unnecessary.}$$

Proof. "$\Longrightarrow$". $\forall\, \mathcal{T} \in co(\hat{f}(\mathcal{R}))$, pick $\mathcal{P} \subseteq \mathcal{R}$, $\mathcal{T} = \hat{f}(\mathcal{P})$. Then $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$. By Proposition 3.19(1), $\mathcal{P} \in co(\mathcal{R})$.

Since R is unnecessary, by Proposition 3.8, we have $\mathcal{P} - \{R\} \in co(\mathcal{R})$. Then $ind(\mathcal{P} - \{R\}) = ind(\mathcal{R})$. By Proposition 2.6 and Lemma 4.4,

$$ind(\hat{f}(\mathcal{P}) - \{\hat{f}(R)\}) = ind(\hat{f}(\mathcal{P} - \{R\}) = \hat{f}(ind(\mathcal{P} - \{R\})),$$

$$ind(\hat{f}(\mathcal{R})) = \hat{f}(ind(\mathcal{R})).$$

Then $ind(\mathcal{T} - \{\hat{f}(R)\}) = ind(\hat{f}(\mathcal{R}))$. This implies $\mathcal{T} - \{\hat{f}(R)\} \in co(\hat{f}(\mathcal{R}))$.

By Proposition 3.8, $\hat{f}(R)$ is unnecessary.

"$\Longleftarrow$". $\forall\, \mathcal{P} \in co(\mathcal{R})$, by Proposition 4.1(1), $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.

Since $\hat{f}(R)$ is unnecessary, by Proposition 3.8, we have

$$\hat{f}(\mathcal{P}) - \{\hat{f}(R)\} \in co(\hat{f}(\mathcal{R})).$$

Then

$$ind(\hat{f}(\mathcal{P}) - \{\hat{f}(R)\}) = ind(\hat{f}(\mathcal{R})).$$

By Proposition 2.6 and Lemma 4.4,

$$\hat{f}(ind(\mathcal{P} - \{R\})) = ind(\hat{f}(\mathcal{P} - \{R\}) = ind(\hat{f}(\mathcal{P}) - \{\hat{f}(R)\}),$$

$$\hat{f}(ind(\mathcal{R})) = ind(\hat{f}(\mathcal{R})).$$

Then $\hat{f}(ind(\mathcal{P} - \{R\})) = \hat{f}(ind(\mathcal{R}))$.

By Proposition 2.7,

$$ind(\mathcal{P} - \{R\}) = \hat{f}^{-1}(\hat{f}(ind(\mathcal{P} - \{R\}))) = \hat{f}^{-1}(\hat{f}(ind(\mathcal{R}))) = ind(\mathcal{R}).$$

Then $\mathcal{P} - \{R\} \in co(\mathcal{R})$.

By Proposition 3.8, R is unnecessary. □

Corollary 4.8. *Let* $(U,\mathcal{R}) \sim_f (V,\hat{f}(\mathcal{R}))$. *Then*

$$R \text{ is relatively necessary} \iff \hat{f}(R) \text{ is relatively necessary.}$$

Proof. This hods by Theorem 4.5 and Theorem 4.7. □

Example 4.9. *Let* $U = \{x_i | 1 \le i \le 15\}$. *We consider the relation information system* $(U,\mathcal{R})$ *where* $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$,

$U/R_1 = \{\{x_1, x_2, x_4, x_7, x_8, x_9, x_{10}, x_{11}\}, \{x_3, x_5, x_6, x_{12}, x_{13}, x_{14}, x_{15}\}\}$,
$U/R_2 = \{\{x_1, x_4, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\}, \{x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\}\}$,
$U/R_3 = \{\{x_1, x_2, x_4, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\}, \{x_3, x_5, x_6\}\}$,
$U/R_4 = \{\{x_1, x_2, x_4, x_7, x_8, x_9, x_{10}, x_{11}\}, \{x_3, x_5, x_6, x_{12}, x_{13}, x_{14}, x_{15}\}\}$.

Let $V = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. *Define a mapping as follows:*

x_1, x_4, x_{11}	x_2, x_8	x_3, x_6	x_5	x_7, x_9, x_{10}	$x_{12}, x_{13}, x_{14}, x_{15}$
y_1	y_2	y_3	y_4	y_5	y_6

Let $(V,\hat{f}(\mathcal{R}))$ *be the* f*-induced relation information system of* (U,R). *It is very easy to verify that* f *is a homomorphism from* (U,R) *to* $(V,\hat{f}(\mathcal{R}))$.

We have $\hat{f}(\mathcal{R}) = \{\hat{f}(R_1), \hat{f}(R_3), \hat{f}(R_3), \hat{f}(R_4)\}$ *where*

$V/\hat{f}(R_1) = \{\{y_1, y_2, y_5\}, \{y_3, y_4, y_6\}\}$,
$V/\hat{f}(R_2) = \{\{y_1, y_6\}, \{y_2, y_3, y_4, y_5\}\}$,
$V/\hat{f}(R_3) = \{\{y_1, y_2, y_5, y_6\}, \{y_3, y_4\}\}$,
$V/\hat{f}(R_4) = \{\{y_1, y_2, y_5\}, \{y_3, y_4, y_6\}\}$.

By Example 3.10,

$$red(\hat{f}(\mathcal{R})) = \{\{\hat{f}(R_1), \hat{f}(R_2)\}, \{\hat{f}(R_2), \hat{f}(R_4)\}\}, \quad core(\hat{f}(\mathcal{R})) = \{\hat{f}(R_2)\}.$$

By Proposition 2.4, Theorem 4.2(2) and Theorem 4.6,

$$red(\mathcal{R}) = \{\{R_1, R_2\}, \{R_2, R_4\}\}, \quad core(\mathcal{R}) = \{R_2\}.$$

5 Conclusions

In this paper, we have investigated the original relation information system and image relation information system, and obtained some invariant characterizations of relation information systems under homomorphism. These results will be significant for establishing a framework of granular computing in knowledge bases and may have potential applications to knowledge discovery, decision making and reasoning about data. In the future, we will consider concrete applications of our results.

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Global stability in a discrete Lotka-Volterra competition model

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Abstract

We consider the Euler difference scheme for two-dimensional Lotka-Volterra competition equations and show that the difference scheme has positive and bounded solutions. In addition, we present sufficient conditions that the solutions of the scheme converge to the equilibrium points of the scheme. The convergence is shown based on the two approaches: first, partition of the domain used for the boundedness of the solutions and second, calculation of the movement of the species started in each partitioned region. Numerical examples are presented to verify the results.

Key words: Euler difference scheme, positivity, global stability, competition model

1. Introduction

The competition model in the two-dimensional case represents two species which are competing for a common resource; an additional term is included within the logistic prey growth Lotka-Volterra model to incorporate this interspecific competition for some limiting resource. This limiting resource can be anything for which supply is smaller than demand. The classic two-dimensional competition model is given by

$$\frac{dx}{dt} = x(t)(r_1 - a_{11}x(t) - a_{12}y(t)), \ \frac{dy}{dt} = y(t)(r_2 - a_{21}x(t) - a_{22}y(t)), \tag{1}$$

where $r_i > 0$ and $a_{ij} > 0$. Here $x(t)$ and $y(t)$ denote the population sizes or population density in the species x and y at time t; the parameters r_i's are the intrinsic growth rates for the two species x and y; a_{ii}'s measure the inhibiting effect on the two species; a_{12} and a_{21} are the interspecific acting coefficients.

The species x in the model (1) acts on y with functional response of type $a_{12}x(t)y(t)$. However other types of functional responses including Holling types [1–5], Beddington-DeAngelis type [6–8], Crowley-Martin type [9–11], and Ivlev-type of functional responses [12–14] have been applied to many population models

The dynamics of the model (1) is well-known [15–17] ; the solutions of (1) are positive and bounded, and the stability of the system (1) has been studied. There are a number of works on investigating continuous time Lokta-Volterra models, but relatively few theoretical papers are published on their discretized models [18–21]. The author in [22] has

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introduced a method to present global stability in discrete Lokta-Volterra predator-prey models for the case that all species coexist at a unique equilibrium. In [23], the authors have shown the global stability of the Euler difference scheme for a three-dimensional predator-prey model using a new approach.

As far as we know, there is no theoretical research on the global stability of the discrete-time competition model of (1), so that we consider the Euler difference scheme

$$x_{n+1} = F_{y_n}(x_n),\ y_{n+1} = G_{x_n}(y_n),\ n \geq 0 \tag{2}$$

with

$$F_y(x) = x\left\{1 + (r_1 - a_{11}x - a_{12}y)\Delta t\right\}, \tag{3}$$
$$G_x(y) = y\left\{1 + (r_2 - a_{21}x - a_{22}y)\Delta t\right\}, \tag{4}$$

where Δt is a time step size, $x_n = x_0 + n\Delta t$ and $y_n = y_0 + n\Delta t$ with $(x_0, y_0) = (x(0), y(0))$.

The paper is organized as follows. Section 2 gives the positivity and boundedness of solutions of (2). In Section 3, we partition the domain used for the boundedness of the discrete solutions and find the geometric properties of the movement of the solutions starting in the partitioned regions. Using the properties, we present sufficient conditions that the solutions converge to equilibrium points of (2). In Section 4, some numerical examples are presented to verify our results.

2. Positivity of the discrete solutions

In this section, we consider the positivity and boundedness of the solutions of (2). Note that if τ_1 and τ_2 are positive constants satisfying

$$U_1(\tau_2) = \frac{1 + r_1\Delta t - a_{12}\tau_2\Delta t}{2a_{11}\Delta t} > 0,\quad U_2(\tau_1) = \frac{1 + r_2\Delta t - a_{21}\tau_1\Delta t}{2a_{22}\Delta t} > 0, \tag{5}$$

then

$$F_{\tau_2}(x), G_{\tau_1}(y) \text{ are increasing on } 0 \leq x \leq U_1(\tau_2),\ 0 \leq y \leq U_2(\tau_1). \tag{6}$$

For the positivity and boundedness of the solutions (x_n, y_n) we assume

$$\max\{r_1, r_2\} < 1/\Delta t \tag{7}$$

and consider constants x^* and y^* such that

$$r_1 a_{11}^{-1} \leq x^* \leq U_1(y^*),\ r_2 a_{22}^{-1} \leq y^* \leq U_2(x^*). \tag{8}$$

Remark 1. For every point (x^*, y^*) satisfying

$$\frac{r_1}{a_{11}} \leq x^* \leq \min\left\{\frac{1 + r_1\Delta t}{4a_{11}\Delta t}, \frac{1 + r_2\Delta t}{2a_{21}\Delta t}\right\},\quad \frac{r_2}{a_{22}} \leq y^* \leq \min\left\{\frac{1 + r_1\Delta t}{2a_{12}\Delta t}, \frac{1 + r_2\Delta t}{4a_{22}\Delta t}\right\}, \tag{9}$$

the two conditions in (8) hold since

$$\begin{aligned}
U_1(y^*) &= \frac{1+r_1\Delta t - a_{12}y^*\Delta t}{2a_{11}\Delta t} \geq \frac{1+r_1\Delta t - a_{12}\min\left\{\frac{1+r_1\Delta t}{2a_{12}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\}\Delta t}{2a_{11}\Delta t} \\
&= \frac{1+r_1\Delta t}{2a_{11}\Delta t} - \frac{a_{12}}{2a_{11}}\min\left\{\frac{1+r_1\Delta t}{2a_{12}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \\
&= \max\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_1\Delta t}{2a_{11}\Delta t} - \frac{a_{12}}{2a_{11}}\frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \geq \frac{1+r_1\Delta t}{4a_{11}\Delta t} \\
&\geq \min\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{2a_{21}\Delta t}\right\} \geq x^*
\end{aligned}$$

and

$$\begin{aligned}
U_2(x^*) &= \frac{1+r_2\Delta t - a_{21}x^*\Delta t}{2a_{22}\Delta t} \geq \frac{1+r_2\Delta t - a_{21}\min\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{2a_{21}\Delta t}\right\}\Delta t}{2a_{22}\Delta t} \\
&= \frac{1+r_2\Delta t}{2a_{22}\Delta t} - \frac{a_{21}}{2a_{22}}\min\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{2a_{21}\Delta t}\right\} \\
&= \max\left\{\frac{1+r_2\Delta t}{2a_{22}\Delta t} - \frac{a_{21}}{2a_{22}}\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \geq \frac{1+r_2\Delta t}{4a_{22}\Delta t} \\
&\geq \min\left\{\frac{1+r_1\Delta t}{2a_{12}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \geq y^*.
\end{aligned}$$

Using x^* and y^* in (8), we can obtain the positivity and boundedness of (x_n, y_n).

Theorem 1. *Let (x_n, y_n) be the solution of (2). Assume that (7) and (8) hold.*

If $(x_0, y_0) \in (0, x^*) \times (0, y^*)$, then $(x_n, y_n) \in (0, x^*) \times (0, y^*)$ for all n.

Proof. Using the condition in this theorem and (5), we have

$$0 < x_0 < x^* \leq U_1(y^*) < U_1(y_0),\ 0 < y_0 < y^* \leq U_2(x^*) < U_2(x_0), \tag{10}$$

and then the increasing property (6) gives the positivity of x_1 and y_1:

$$x_1 = F_{y_0}(x_0) > F_{y_0}(0) = 0, \quad y_1 = G_{x_0}(y_0) > G_{x_0}(0) = 0. \tag{11}$$

Now, we claim that $x_1 < x^*$ and $y_1 < y^*$. If $r_1 - a_{11}x_0 - a_{12}y_0 \leq 0$, then

$$x_1 = F_{y_0}(x_0) \leq x_0 < x^*.$$

Otherwise, we get

$$0 < x_0 < (r_1 - a_{12}y_0)a_{11}^{-1} < (1 + r_1\Delta t - a_{12}y_0\Delta t)(2a_{11}\Delta t)^{-1} = U_1(y_0),$$

where the last inequality is obtained from $r_1\Delta t < 1$ in (7). Hence (6) and (8) imply the boundedness of x_1:

$$x_1 = F_{y_0}(x_0) < F_{y_0}\left((r_1 - a_{12}y_0)a_{11}^{-1}\right) = (r_1 - a_{12}y_0)a_{11}^{-1} < r_1 a_{11}^{-1} \leq x^*. \tag{12}$$

Similarly if $r_2 - a_{21}x_0 - a_{22}y_0 \le 0$, then $y_1 = G_{x_0}(y_0) \le y_0 < y^*$. Otherwise, we have

$$0 < y_0 < (r_2 - a_{21}x_0)a_{22}^{-1} < (1 + r_2\Delta t - a_{21}x_0\Delta t)(2a_{22}\Delta t)^{-1} = U_2(x_0),$$

where the last inequality is obtained from $r_2\Delta t < 1$ in (7). Thus (6) and (8) imply the boundedness of y_1 that

$$y_1 = G_{x_0}(y_0) < G_{x_0}\left((r_2 - a_{21}x_0)a_{22}^{-1}\right) = (r_2 - a_{21}x_0)a_{22}^{-1} < r_2a_{22}^{-1} \le y^*. \tag{13}$$

Hence using (11), (12) and (13), we have that

$$\text{if } (x_0, y_0) \in (0, x^*) \times (0, y^*), \text{ then } (x_1, y_1) \in (0, x^*) \times (0, y^*).$$

Therefore, using the mathematical induction, we can obtain the desired result. □

Remark 2. Due to (9), we can choose sufficiently large values of x^* and y^* when letting Δt be sufficiently small, so that the area of $(0, x^*) \times (0, y^*)$ for the initial state (x_0, y_0) in Theorem 1 can be taken large.

3. Stability of the discrete solutions

Let $\mathcal{D} = (0, x^*) \times (0, y^*)$ for x^* and y^* defined in (8). In order to discuss the stability of the Euler scheme (2) for each initial position (x_0, y_0) contained in $\mathcal{D}$, we partition $\mathcal{D}$ into the four regions

$$\begin{aligned} \mathrm{I} &= \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) \ge 0,\ g(\mathbf{x}) > 0\}, \quad \mathrm{II} = \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) < 0,\ g(\mathbf{x}) \ge 0\}, \\ \mathrm{III} &= \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) \le 0,\ g(\mathbf{x}) < 0\}, \quad \mathrm{IV} = \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) > 0,\ g(\mathbf{x}) \le 0\}, \end{aligned} \tag{14}$$

where $\mathbf{x} = (x, y)$ and

$$f(x, y) = r_1 - a_{11}x - a_{12}y,\ g(x, y) = r_2 - a_{21}x - a_{22}y. \tag{15}$$

Since the location of the regions depends on the x and y-intercepts of the two lines $f(x, y) = 0$ and $g(x, y) = 0$, we partition $\mathcal{D}$ by using the four categories $\mathcal{C}_i (1 \le i \le 4)$ as in Figure 1; we use the symbol $\mathcal{C}_1$ for the two conditions $r_1a_{11}^{-1} < r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} < r_2a_{22}^{-1}$, the symbol $\mathcal{C}_2$ for $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$, the symbol $\mathcal{C}_3$ for $r_1a_{11}^{-1} < r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$, and finally the symbol $\mathcal{C}_4$ for $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} < r_2a_{22}^{-1}$. The magenta circles in Figure 1 denote the stable points of the difference model (2) in the categories, which will be proved.

Remark 3. In the case of $\mathcal{C}_1$

$$r_1a_{11}^{-1} < r_2a_{21}^{-1},\ r_1a_{12}^{-1} < r_2a_{22}^{-1}, \tag{16}$$

the region IV is empty. In order to prove this emptiness, suppose, on the contrary, that there exists $(x, y) \in \mathrm{IV}$, which means, from (14), that

$$r_1 - a_{11}x - a_{12}y > 0,\ r_2 - a_{21}x - a_{22}y \le 0. \tag{17}$$

Eliminating x and y from (17), we have the two inequalities, respectively:

$$-r_1a_{21} + r_2a_{11} < (a_{11}a_{22} - a_{12}a_{21})y, \tag{18}$$

$$-r_1a_{22} + r_2a_{12} < (a_{12}a_{21} - a_{11}a_{22})x. \tag{19}$$

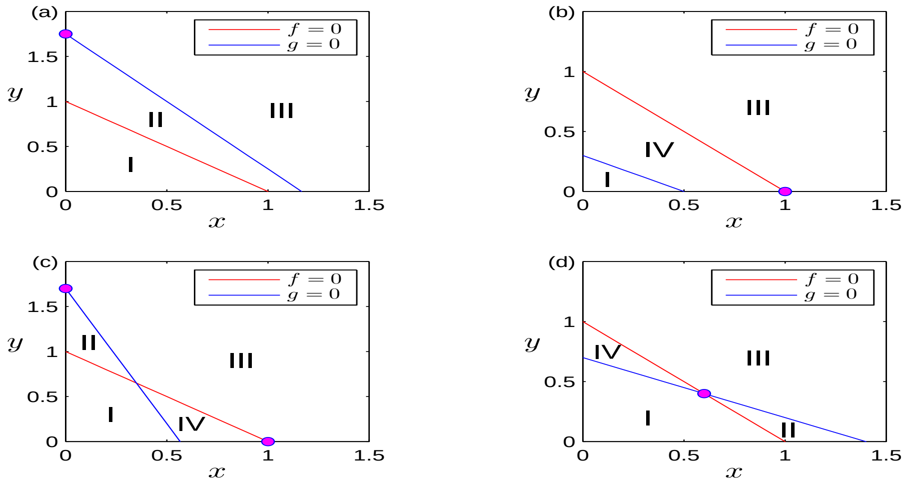

Figure 1: Two lines $f = 0$ and $g = 0$ and regions with stable points. (a) $r_2 = 3.5, a_{21} = 3.0, a_{22} = 2$ (b) $r_2 = 1.5, a_{21} = 3, a_{22} = 5$ (c) $r_2 = 1.7, a_{21} = 3, a_{22} = 1$ (d) $r_2 = 3.5, a_{21} = 2.5, a_{22} = 5$

We find a contradiction by using the following three cases:
Case 1. Let $a_{11}a_{22} - a_{12}a_{21} = 0$.
In this case, (18) becomes $-r_1a_{21} + r_2a_{11} < 0$, which contradicts (16).
Case 2. Let $a_{11}a_{22} - a_{12}a_{21} < 0$.
Using the positivity of y, (18) becomes $-r_1a_{21} + r_2a_{11} < 0$, which contradicts (16).
Case 3. Let $a_{11}a_{22} - a_{12}a_{21} > 0$.
Using the positivity of x, (19) becomes $-r_1a_{22} + r_2a_{12} < 0$, which contradicts (16).
Therefore it follows from Cases 1, 2 and 3 that the region IV is empty and then

$$\mathcal{D} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \text{ for } \mathcal{C}_1 \tag{20}$$

as in Figure 1-(a). Similarly we can obtain

$$\mathcal{D} = \mathrm{I} \cup \mathrm{III} \cup \mathrm{IV} \text{ for } \mathcal{C}_2 \tag{21}$$

as in Figure 1-(b).

For convenience, we use the difference equations

$$x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\}, \tag{22}$$
$$y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} \tag{23}$$

as well as (2), where $f(x, y)$ and $g(x, y)$ are defined in (15).
For the stability we need to assume

$$1 > \Delta t\,(a_{11}x^* + a_{22}y^* + x^*y^*|a_{12}a_{21} - a_{11}a_{22}|\Delta t)\,. \tag{24}$$

Lemma 1. *Let (x_n, y_n) be the solution of (2). Assume that (7), (8) and (24) hold.*

If $(x_k, y_k) \in \mathrm{I}$ for some k, then (x_{k+1}, y_{k+1}) is not contained in III.

Proof. The condition $(x_k, y_k) \in \mathrm{I}$ gives

$$g(x_k, y_k) > 0. \tag{25}$$

Suppose, on the contrary, that (x_{k+1}, y_{k+1}) is contained in III, which means

$$f(x_{k+1}, y_{k+1}) \leq 0 \text{ and } g(x_{k+1}, y_{k+1}) < 0.$$

Then (22) and (23) give

$$\begin{aligned} 0 \geq f(x_{k+1}, y_{k+1}) &= f\,(x_k + x_k f(x_k, y_k)\Delta t,\ y_k + y_k g(x_k, y_k)\Delta t) \\ &= f(x_k, y_k) + (-a_{11}) x_k f(x_k, y_k)\Delta t + (-a_{12}) y_k g(x_k, y_k)\Delta t \end{aligned} \tag{26}$$

and

$$\begin{aligned} 0 > g(x_{k+1}, y_{k+1}) &= g(x_k + x_k f(x_k, y_k)\Delta t,\ y_k + y_k g(x_k, y_k)\Delta t) \\ &= g(x_k, y_k) + (-a_{21}) x_k f(x_k, y_k)\Delta t + (-a_{22}) y_k g(x_k, y_k)\Delta t. \end{aligned} \tag{27}$$

We write (26) and (27) as

$$\begin{aligned} f(x_k, y_k)(1 - a_{11} x_k \Delta t) &\leq a_{12} y_k g(x_k, y_k)\Delta t, \\ g(x_k, y_k)(1 - a_{22} y_k \Delta t) &< a_{21} x_k f(x_k, y_k)\Delta t. \end{aligned} \tag{28}$$

Combining (24) and Theorem 1 gives

$$0 < 1 - a_{11} x^* \Delta t < 1 - a_{11} x_k \Delta t$$

and so (28) implies

$$g(x_k, y_k)(1 - a_{22} y_k \Delta t) < a_{21} x_k \Delta t \frac{a_{12} y_k g(x_k, y_k)\Delta t}{(1 - a_{11} x_k \Delta t)}. \tag{29}$$

Using (24) and (25), we can simplify (29) as follows.

$$\begin{aligned} 1 &< \Delta t\{a_{11} x_k (1 - a_{22} y_k \Delta t) + a_{22} y_k + a_{12} y_k a_{21} x_k \Delta t\} \\ &\leq \Delta t\{a_{11} x_k + a_{22} y_k + x_k y_k |a_{12} a_{21} - a_{11} a_{22}|\Delta t\}, \end{aligned} \tag{30}$$

where the last inequality contradicts (24). Hence (x_{k+1}, y_{k+1}) is not contained in III. □

Remark 4. Similarly to Lemma 1 under the same assumption, we can obtain that

$$\text{if } (x_k, y_k) \in \text{ III for some } k, \text{ then } (x_{k+1}, y_{k+1}) \text{ is not contained in I} \tag{31}$$

as follows. The condition $(x_k, y_k) \in \mathrm{III}$ gives

$$g(x_k, y_k) < 0. \tag{32}$$

Suppose, on the contrary, that

$$f(x_{k+1}, y_{k+1}) \geq 0 \text{ and } g(x_{k+1}, y_{k+1}) > 0. \tag{33}$$

Using (33) instead of $f(x_{k+1}, y_{k+1}) \leq 0$ and $g(x_{k+1}, y_{k+1}) < 0$ in the proof of Lemma 1 and following the proof of Lemma 1 with (32), we have

$$g(x_k, y_k)(1 - a_{22} y_k \Delta t) > a_{21} x_k \Delta t \frac{a_{12} y_k g(x_k, y_k)\Delta t}{(1 - a_{11} x_k \Delta t)}$$

and then obtain the contradiction (30) due to (32). Therefore we obtain (31).

Lemma 2. *Let (x_n, y_n) be the solution of (2). Assume that (7), (8) and (24) hold.*

$$\textit{If } (x_k, y_k) \in \mathrm{II} \textit{ for some } k, \textit{ then } (x_n, y_n) \in \mathrm{II} \textit{ for all } n \geq k.$$

Proof. Let $(x_k, y_k) \in \mathrm{II}$, which implies $f(x_k, y_k) < 0 \leq g(x_k, y_k)$ and then

$$x_{k+1} < x_k, \quad y_{k+1} \geq y_k. \tag{34}$$

It follows from Theorem 1, (34) and (10) that

$$0 < x_{k+1} < x_k < U_1(y_k), \; 0 < y_k \leq y_{k+1} < y^* < U_2(x_k). \tag{35}$$

Using the decreasing function $F_y(x)$ of y and combining (6) with (35), we have

$$x_{k+2} = F_{y_{k+1}}(x_{k+1}) \leq F_{y_k}(x_{k+1}) < F_{y_k}(x_k) = x_{k+1} \tag{36}$$

and then (22) gives

$$f(x_{k+1}, y_{k+1}) < 0. \tag{37}$$

Similarly, the strictly decreasing function $G_x(y)$ of x with (6) and (35) gives

$$y_{k+2} = G_{x_{k+1}}(y_{k+1}) > G_{x_k}(y_{k+1}) \geq G_{x_k}(y_k) = y_{k+1}. \tag{38}$$

Substituting (23) into (38) yields

$$g(x_{k+1}, y_{k+1}) > 0,$$

with which (37) gives

$$f(x_{k+1}, y_{k+1}) < 0 < g(x_{k+1}, y_{k+1}).$$

This implies

$$(x_{k+1}, y_{k+1}) \in \mathrm{II}.$$

Hence

$$\text{if } (x_k, y_k) \in \mathrm{II}, \text{ then } (x_{k+1}, y_{k+1}) \in \mathrm{II}.$$

Therefore using mathematical induction, we can obtain the desired result. □

Remark 5. Similarly to Lemma 2 under the same assumption, we can obtain that

$$\text{if } (x_k, y_k) \in \mathrm{IV} \text{ for some } k, \text{ then } (x_n, y_n) \in \mathrm{IV} \text{ for all } n \geq k \tag{39}$$

as follows. Let $(x_k, y_k) \in \mathrm{IV}$, which implies

$$f(x_k, y_k) > 0 \geq g(x_k, y_k). \tag{40}$$

Then replacing $f(x_k, y_k) < 0 \leq g(x_k, y_k)$ in the proof of Lemma 2 with (40) and following the proof of Lemma 2, we have

$$f(x_{k+1}, y_{k+1}) > 0 > g(x_{k+1}, y_{k+1}),$$

which implies

$$(x_{k+1}, y_{k+1}) \in \mathrm{IV}.$$

Hence mathematical induction gives (39).

In the following theorem, we show the global stability of the solutions of (2) for the category $\mathcal{C}_1$ as in Figure 1-(a); we present the condition that the species y always outcompetes the species x.

Theorem 2. *Assume that (7), (8) and (24) hold.*

$$\text{If } r_1 a_{11}^{-1} < r_2 a_{21}^{-1} \text{ and } r_1 a_{12}^{-1} < r_2 a_{22}^{-1}, \text{ then } \left(0, r_2 a_{22}^{-1}\right) \text{ is globally stable.}$$

Proof. The condition in this theorem is corresponding to $\mathcal{C}_1$, so that $\mathcal{D}$ is partitioned into the three regions I, II and III due to (20). We claim the global stability for $(x_0, y_0) \in \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$ by using mathematical induction as follows.

Case 2-1. Let $(x_0, y_0) \in \mathrm{II}$.

Using Lemma 2 and Theorem 1, we have that

$$0 < x_{n+1} < x_n, \ 0 < y_n \leq y_{n+1} < y^*, \tag{41}$$

which give the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively.

Note that the increasing property of $\{y_n\}$ gives $\omega_2 > 0$.

In addition, the limit ω_1 is zero, which can be obtained by indirect proof. Suppose, on the contrary, that ω_1 is nonzero. Taking the limit of (2) and using $\omega_i > 0$ $(i = 1, 2)$, we have

$$\left(a_{11}a_{22} - a_{12}a_{21}\right)(\omega_1, \omega_2) = \left(r_1 a_{22} - r_2 a_{12}, -r_1 a_{21} + r_2 a_{11}\right). \tag{42}$$

Since $r_1 a_{22} - r_2 a_{12} < 0$ and $-r_1 a_{21} + r_2 a_{11} > 0$ from the conditions in this theorem, the equality (42) with $\omega_i > 0$ gives

$$0 > a_{11}a_{22} - a_{12}a_{21} > 0, \tag{43}$$

which is a contradiction. Consequently, ω_1 is zero.

Taking the limit of the second equation in (2) with $\omega_1 = 0$ and $\omega_2 > 0$, we have $\omega_2 = r_2 a_{22}^{-1}$, which completes the proof for Case 2-1.

Case 2-2. Let $(x_0, y_0) \in \mathrm{I}$.

This case implies that $f(x_0, y_0) \geq 0$ and $g(x_0, y_0) > 0$. We use the following three steps to prove this theorem in this case.

Step 1. There exists a positive integer m_1 such that $(x_{m_1}, y_{m_1}) \notin \mathrm{I}$.

Suppose, on the contrary, that $(x_n, y_n) \in \mathrm{I}$ for all n, which means $f(x_n, y_n) \geq 0$ and $g(x_n, y_n) > 0$ for all n. Then

$$x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\} \geq x_n > 0, \quad y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} > y_n > 0$$

and hence the boundedness of (x_n, y_n) in Theorem 1 gives the convergence of the increasing sequences $\{x_n\}$ and $\{y_n\}$, which have positive limits ω_1 and ω_2, respectively. Therefore we have a contradiction by using (42)–(43).

Step 2. There exists a positive integer m such that $(x_m, y_m) \in \mathrm{II}$.

Using $(x_0, y_0) \in \mathrm{I}$ and Step 1, there exists a positive integer m_1 such that $(x_{m_1-1}, y_{m_1-1}) \in \mathrm{I}$ and $(x_{m_1}, y_{m_1}) \in \mathcal{D}-\mathrm{I}$. Since $\mathcal{D}-\mathrm{I} = \mathrm{II} \cup \mathrm{III}$, we have

$$(x_{m_1}, y_{m_1}) \in \mathrm{II} \text{ or } (x_{m_1}, y_{m_1}) \in \mathrm{III}. \tag{44}$$

Applying Lemma 1 with $(x_{m_1-1}, y_{m_1-1}) \in \mathrm{I}$, it is not true that $(x_{m_1}, y_{m_1}) \in \mathrm{III}$ and then $(x_{m_1}, y_{m_1}) \in \mathrm{II}$. Taking $m = m_1$ gives the desired result.

Step 3. If $(x_0, y_0) \in \mathrm{I}$, then $(x_m, y_m) \in \mathrm{II}$ for some positive integer m due to Step 2. Therefore the proof for Case 2-1 completes the proof for Case 2-2.

Case 2-3. Let $(x_0, y_0) \in \mathrm{III}$.

This case implies that $f(x_0, y_0) \leq 0$ and $g(x_0, y_0) < 0$. We use the following two steps to prove this theorem in this case.

Step 1. If $(x_n, y_n) \in \mathrm{III}$ for all n, then $\lim_{n\to\infty}(x_n, y_n) = (0, r_2 a_{22}^{-1})$.

Assume that $(x_n, y_n) \in \mathrm{III}$ for all n, which implies

$$f(x_n, y_n) \leq 0, \;\; g(x_n, y_n) < 0 \tag{45}$$

for all n. The assumption gives the decreasing property

$$0 < x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\} \leq x_n, \;\; 0 < y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} < y_n$$

and then Theorem 1 gives the convergence of $\{x_n\}$ and $\{y_n\}$ with the nonnegative limits ω_1 and ω_2, respectively. It is only possible that $\omega_1 = 0$ and $\omega_2 > 0$ as follows.

If $\omega_1 > 0$ and $\omega_2 > 0$, then (42)–(43) give a contradiction.

If $\omega_1 > 0$ and $\omega_2 = 0$, then $\omega_1 = r_1 a_{11}^{-1}$. This is impossible due to the unstability of $(r_1 a_{11}^{-1}, 0)$ since the linearized system of (2) at $(r_1 a_{11}^{-1}, 0)$ has the eigenvalue

$$1 + \Delta t a_{11}^{-1}(r_2 a_{11} - r_1 a_{21}) > 1$$

under the condition $a_{21} a_{11}^{-1} < r_2 {r_1}^{-1}$. Therefore $\{(x_n, y_n)\}$ cannot have the limit $(r_1 a_{11}^{-1}, 0)$.

If $\omega_1 = 0$ and $\omega_2 = 0$, then

$$\lim_{n\to\infty} f(x_n, y_n) = r_1 > 0, \quad \lim_{n\to\infty} g(x_n, y_n) = r_2 > 0,$$

which are contradictory to (45).

Therefore it remains that $\omega_1 = 0$ and $\omega_2 > 0$, which gives $(\omega_1, \omega_2) = (0, r_2 a_{22}^{-1})$.

Step 2. If $(x_m, y_m) \notin \mathrm{III}$ for some m, then $\lim_{n\to\infty}(x_n, y_n) = (0, r_2 a_{22}^{-1})$.

Since $(x_m, y_m) \in \mathcal{D}-\mathrm{III}$ and $\mathcal{D}-\mathrm{III} = \mathrm{I} \cup \mathrm{II}$, we have

$$(x_m, y_m) \in \mathrm{I} \text{ or } (x_m, y_m) \in \mathrm{II}.$$

However it is not true that $(x_m, y_m) \in \mathrm{I}$ due to Remark 4 and so we have $(x_m, y_m) \in \mathrm{II}$. Therefore, following the proof for Case 2-1, we obtain $\lim_{n\to\infty}(x_n, y_n) = (0, r_2 a_{22}^{-1})$.

Finally, we obtain the desired result from the proofs for Cases 2-1, 2-2 and 2-3. □

In the following theorem, we show the global stability of (2) for $\mathcal{C}_2$ as in Figure 1-(b) and present the condition that the species x always outcompetes the species y.

Theorem 3. *Assume that (7), (8) and (24) hold.*

$$\textit{If } r_1 a_{11}^{-1} > r_2 a_{21}^{-1} \textit{ and } r_1 a_{12}^{-1} > r_2 a_{22}^{-1}, \textit{ then } \left(r_1 a_{11}^{-1}, 0\right) \textit{ is globally stable.}$$

Proof. The condition in this theorem is corresponding to $\mathcal{C}_1$ and so $\mathcal{D}$ is partitioned into the three regions I, III and IV due to (21). We claim the global stability for $(x_0, y_0) \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{IV}$ by using mathematical induction as follows.

Case 3-1. Let $(x_0, y_0) \in \mathrm{IV}$.

In this case, (39) gives $(x_n, y_n) \in \mathrm{IV}$ for all n, with which (22) and (23) give $x_n < x_{n+1}$ and $y_{n+1} \leq y_n$. Then Theorem 1 gives

$$0 < x_n < x_{n+1} < x^*, \;\; 0 < y_{n+1} \leq y_n, \tag{46}$$

which imply the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively. The increasing property of $\{x_n\}$ gives $\omega_1 > 0$.
In addition, the limit ω_2 is zero, which can be obtained by indirect proof as in Case 2-1. Suppose, on the contrary, that ω_2 is nonzero. Taking the limit of (2) and using the positivity of ω_1 and ω_2, we have (42). Applying the conditions $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$ to (42) yields the contradiction (43) Consequently, ω_2 is zero.
Taking the limit of the first equation in (2) with $\omega_1 > 0$ and $\omega_2 = 0$, we have $\omega_1 = r_1a_{11}^{-1}$, which completes the proof for Case 3-1.
Case 3-2. Let $(x_0, y_0) \in \mathrm{I}$.
In this case we have $f(x_0, y_0) \geq 0$ and $g(x_0, y_0) > 0$, and use the following three steps.

Step 1. There exists a positive integer m_1 such that $(x_{m_1}, y_{m_1}) \notin \mathrm{I}$.
Suppose, on the contrary, that $(x_n, y_n) \in \mathrm{I}$ for all n, which means $f(x_n, y_n) \geq 0$ and $g(x_n, y_n) > 0$ for all n. Then

$$x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\} \geq x_n > 0, \quad y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} > y_n > 0,$$

and hence the boundedness of (x_n, y_n) in Theorem 1 gives the convergence of the increasing sequences $\{x_n\}$ and $\{y_n\}$, which have positive limits ω_1 and ω_2, respectively. Therefore we have the contradiction (43) as in Case 3-1.

Step 2. There exists a positive integer m such that $(x_m, y_m) \in \mathrm{IV}$.
Using $(x_0, y_0) \in \mathrm{I}$ and Step 1, there exists a positive integer m_1 such that $(x_{m_1-1}, y_{m_1-1}) \in \mathrm{I}$ and $(x_{m_1}, y_{m_1}) \in \mathcal{D}-\mathrm{I}$ for some m_1. Since $\mathcal{D}-\mathrm{I} = \mathrm{III} \cup \mathrm{IV}$, we have

$$(x_{m_1}, y_{m_1}) \in \mathrm{III} \text{ or } (x_{m_1}, y_{m_1}) \in \mathrm{IV}.$$

Applying Lemma 1 with $(x_{m_1-1}, y_{m_1-1}) \in \mathrm{I}$, it is not true that $(x_{m_1}, y_{m_1}) \in \mathrm{III}$ and then $(x_{m_1}, y_{m_1}) \in \mathrm{IV}$. Taking $m = m_1$ gives $(x_m, y_m) \in \mathrm{IV}$.

Step 3. If $(x_0, y_0) \in \mathrm{I}$, then $(x_m, y_m) \in \mathrm{IV}$ for some positive integer m due to Step 2. Therefore the proof used in Case 3-1 completes the proof for Case 3-2.
Case 3-3. Let $(x_0, y_0) \in \mathrm{III}$.
In this case we have $f(x_0, y_0) \leq 0$ and $g(x_0, y_0) < 0$,and use the following two steps.

Step 1. If $(x_n, y_n) \in \mathrm{III}$ for all n, then $\lim_{n\to\infty}(x_n, y_n) = (r_1a_{11}^{-1}, 0)$.
As in Step 1 of Case 2-3 in Theorem 2, $\{(x_n, y_n)\}$ has the limit (ω_1, ω_2). It is only possible that $\omega_1 > 0$ and $\omega_2 = 0$ as follows.
If $\omega_1 > 0$ and $\omega_2 > 0$, then (46)–(**??**) give a contradiction.
If $\omega_1 = 0$ and $\omega_2 > 0$, then $\omega_2 = r_2a_{22}^{-1}$. This is impossible due to the unstability of $(0, r_2a_{22}^{-1})$ since the linearized system of (2) at $(0, r_2a_{22}^{-1})$ has the eigenvalue

$$1 + \Delta t a_{22}^{-1}(r_1a_{22} - r_2a_{12}) > 1$$

under the condition $a_{22}a_{12}^{-1} > r_2{r_1}^{-1}$. Therefore $\{(x_n, y_n)\}$ cannot have the limit $(r_1a_{11}^{-1}, 0)$.
If $\omega_1 = 0$ and $\omega_2 = 0$, then

$$\lim_{n\to\infty} f(x_n, y_n) = r_1 > 0, \quad \lim_{n\to\infty} g(x_n, y_n) = r_2 > 0,$$

which are contradictory to (45).
It remains that $\omega_1 > 0$ and $\omega_2 = 0$, which yields the desired result $(\omega_1, \omega_2) = (r_1a_{11}^{-1}, 0)$.

Step 2. If $(x_m, y_m) \notin \mathrm{III}$ for some m, then $\lim_{n\to\infty}(x_n, y_n) = (r_1a_{11}^{-1}, 0)$.
Since $(x_m, y_m) \in \mathcal{D}-\mathrm{III} = \mathrm{I} \cup \mathrm{IV}$, we have

$$(x_m, y_m) \in \mathrm{I} \text{ or } (x_m, y_m) \in \mathrm{IV}.$$

However it is not true that $(x_m, y_m) \in \mathrm{I}$ due to Remark 4. Therefore, we have $(x_m, y_m) \in \mathrm{IV}$ and then $\lim_{n\to\infty}(x_n, y_n) = (r_1 a_{11}^{-1}, 0)$ by following the proof for Case 3-1.

Finally, we obtain the desired result from the proofs for Cases 3-1 and 3-2. □

In the following theorem, we show the convergence of the solutions of (2) for the category $\mathcal{C}_3$ as in Figure 1-(c) and the dependence of the limit on the region in which the initial state is located.

From now on, in the case that $a_{11}a_{22} - a_{12}a_{21} \neq 0$, we use the symbol (θ_1, θ_2) to mean

$$(\theta_1, \theta_2) = (a_{11}a_{22} - a_{12}a_{21})^{-1} (r_1 a_{22} - r_2 a_{12}, -r_1 a_{21} + r_2 a_{11}), \tag{47}$$

where (θ_1, θ_2) satisfies

$$f(\theta_1, \theta_2) = g(\theta_1, \theta_2) = 0. \tag{48}$$

Theorem 4. *Let the conditions (7), (8) and (24) hold. Assume that*

$$r_1 a_{11}^{-1} > r_2 a_{21}^{-1} \text{ and } r_1 a_{12}^{-1} < r_2 a_{22}^{-1}.$$

(a) *If* $(x_0, y_0) \in \mathrm{II}$, *then* $\lim_{n\to\infty}(x_n, y_n) = (0, r_2 a_{22}^{-1})$.

(b) *If* $(x_0, y_0) \in \mathrm{IV}$, *then* $\lim_{n\to\infty}(x_n, y_n) = (r_1 a_{11}^{-1}, 0)$.

(c) *If* $(x_0, y_0) \in \mathrm{I} \cup \mathrm{III}$, *then* $\{(x_n, y_n)\}$ *converges with the limit* $(r_1 a_{11}^{-1}, 0)$ *or* $(0, r_2 a_{22}^{-1})$.

Proof. For the proof of (a), let $(x_0, y_0) \in \mathrm{II}$. We have from Lemma 2 and Theorem 1 that

$$0 < x_{n+1} < x_n, \ 0 < y_n \leq y_{n+1} < y^*, \tag{49}$$

which gives the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively. The increasing property of $\{y_n\}$ gives $\omega_2 > 0$.

In addition, the limit ω_1 is zero, which can be obtained by indirect proof. Suppose, on the contrary, that ω_1 is nonzero. Taking the limit of (2) and using the positivity of ω_1 and ω_2, we have

$$(a_{11}a_{22} - a_{12}a_{21})\omega_1 = r_1 a_{22} - r_2 a_{12}. \tag{50}$$

Since $(x_0, y_0) \in \mathrm{II}$, the definition of the region II gives

$$f(x_0, y_0) < 0 \leq g(x_0, y_0). \tag{51}$$

Solving (51) for x_0, we obtain

$$(r_1 a_{22} - r_2 a_{12}) - (a_{11}a_{22} - a_{12}a_{21})x_0 < 0. \tag{52}$$

The conditions $a_{21}a_{11}^{-1} > r_2 r_1{}^{-1} > a_{22}a_{12}^{-1}$ in this theorem give

$$a_{11}a_{22} - a_{12}a_{21} < 0. \tag{53}$$

Applying (53) into both (52) and (50) yields

$$\omega_1 > x_0. \tag{54}$$

Combining (54) with (49), we have that for all n

$$\omega_1 > x_0 > x_n,$$

which is contradictory to $\lim_{n\to\infty} x_n = \omega_1$. Consequently, ω_1 is zero.
Taking the limit of the second equation in (2) with $\omega_1 = 0$ and $\omega_2 > 0$, we have $\omega_2 = r_2 a_{22}^{-1}$, which completes the proof of (a).

For the proof of (b), let $(x_0, y_0) \in \mathrm{IV}$. Using (46), we have the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively. The increasing property of $\{x_n\}$ gives $\omega_1 > 0$. In addition, the limit ω_2 is zero, which can be obtained by indirect proof. Suppose, on the contrary, that ω_2 is nonzero. Taking the limit of (2) and using the positivity of ω_1 and ω_2, we have

$$(a_{11}a_{22} - a_{12}a_{21})\omega_2 = -r_1 a_{21} + r_2 a_{11}. \tag{55}$$

Since $(x_0, y_0) \in \mathrm{IV}$, the definition of the region IV gives

$$f(x_0, y_0) > 0 \geq g(x_0, y_0). \tag{56}$$

Solving (56) for y_0, we obtain

$$(r_1 a_{21} - r_2 a_{11}) + (a_{11}a_{22} - a_{12}a_{21})y_0 > 0. \tag{57}$$

Applying (53) into (57) yields

$$\omega_2 > y_0. \tag{58}$$

Combining (58) with (46), we have that for all n

$$\omega_2 > y_0 > y_n,$$

which is contradictory to $\lim_{n\to\infty} y_n = \omega_2$. Consequently, ω_2 is zero.
Taking the limit of the first equation in (2) with $\omega_1 > 0$ and $\omega_2 = 0$, we have $\omega_1 = r_1 a_{11}^{-1}$, which completes the proof of (b).

For the proof of (c), we consider the following two cases.
Case 4-1. Let $(x_0, y_0) \in \mathrm{I}$.
We use the following three steps to obtain the desired result in this case.

Step 1. There exists a positive constant m such that $(x_m, y_m) \notin \mathrm{I}$.
Suppose, on the contrary, that $(x_n, y_n) \in \mathrm{I}$ for all n. Then $\{x_n\}$ and $\{y_n\}$ have the positive limits (θ_1, θ_2) defined in (47) by applying (53) and the approach used in Step1 of Case 2-2 in Theorem 2. However the system (2) under the condition $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ is unstable at the point (θ_1, θ_2) since the linearized system at (θ_1, θ_2) has the eigenvalue $1 + \Delta t a_{11}^{-1}(r_2 a_{11} - r_1 a_{21})$ greater than 1. Therefore $\{x_n\}$ and $\{y_n\}$ cannot have the positive limits θ_1 and θ_2, respectively, which is contradictory.

Step 2. There exists a positive constant m such that $(x_m, y_m) \in \mathrm{II} \cup \mathrm{IV}$.
Since $(x_0, y_0) \in \mathrm{I}$, Step 1 gives the existence of a positive integer m such that

$$(x_{m-1}, y_{m-1}) \in \mathrm{I} \text{ and } (x_m, y_m) \notin \mathrm{I},$$

which implies $(x_m, y_m) \in \mathrm{II} \cup \mathrm{IV}$ due to Lemma 1 and $\mathcal{D} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$.

Step 3. It follows from (a), (b) and Step 2 in this theorem that (x_n, y_n) converges and has the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$.
Case 4-2. Let $(x_0, y_0) \in \mathrm{III}$.
We use the following two steps to obtain the desired result in this case.

Step 1. If $(x_n, y_n) \in \mathrm{III}$ for all n, then $\{(x_n, y_n)\}$ converges with the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$. To prove this, note that we have the convergence of $\{(x_n, y_n)\}$ with the limit (ω_1, ω_2) by following the proof of Step 1 of Case 2-3 in Theorem 2.
If $\omega_1 > 0$ and $\omega_2 > 0$, then $(\omega_1, \omega_2) = (\theta_1, \theta_2)$. This is impossible due to the unstability of (θ_1, θ_2) since the linearized system of (2) at (θ_1, θ_2) has the eigenvalue greater than 1:

$$1 + 0.5\Delta t \left\{ -(a_{11}\theta_1 + a_{22}\theta_2) + \sqrt{(a_{11}\theta_1 + a_{22}\theta_2)^2 + \alpha} \right\} > 1$$

since $\alpha = 4(a_{12}a_{21} - a_{11}a_{22})\theta_1\theta_2 > 0$ under the condition $a_{21}a_{11}^{-1} > r_2 {r_1}^{-1} > a_{22}a_{12}^{-1}$. Therefore it is not possible that $\omega_1 > 0$ and $\omega_2 > 0$.
If $\omega_1 = 0$ and $\omega_2 = 0$, then we have the contradictions to to (45):

$$\lim_{n\to\infty} f(x_n, y_n) = r_1 > 0, \quad \lim_{n\to\infty} g(x_n, y_n) = r_2 > 0.$$

Therefore the remaining signs of ω_1 and ω_2 are

$$(+, 0) \text{ and } (0, +),$$

which give the desired result

$$(\omega_1, \omega_2) = (r_1 a_{11}^{-1}, 0) \text{ and } (0, r_2 a_{22}^{-1}),$$

respectively, by taking the limit of (2) and using the signs of ω_1 and ω_2.

Step 2. If $(x_m, y_m) \notin \mathrm{III}$ for some m, then $\{(x_n, y_n)\}$ converges with the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$. To prove this, we follow the proof used in Step 2 of Case 4-1.
Since $(x_0, y_0) \in \mathrm{III}$, using the condition $(x_m, y_m) \notin \mathrm{III}$ for some m, we can assume that there exists a positive constant m_1 such that

$$(x_{m_1-1}, y_{m_1-1}) \in \mathrm{III} \text{ and } (x_{m_1}, y_{m_1}) \notin \mathrm{III},$$

which implies

$$(x_{m_1}, y_{m_1}) \in \mathrm{II} \cup \mathrm{IV} \tag{59}$$

due to $\mathcal{D} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$ and Lemma 1. Therefore, using (59) and (a) and (b) in this theorem, we have that (x_n, y_n) converges and has the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$.

Finally, we obtain the desired result from the proofs for Cases 4-1 and 4-2. □

In the following theorem, we show the global stability of the solutions of (2) for the category $\mathcal{C}_4$ as in Figure 1-(d) where each component of the equilibrium point is positive.

Theorem 5. *Let the conditions (7), (8) and (24) hold. Assume that*

$$r_1 a_{11}^{-1} < r_2 a_{21}^{-1} \text{ and } r_1 a_{12}^{-1} > r_2 a_{22}^{-1}.$$

Then for (θ_1, θ_2) defined in (47)

$$(\theta_1, \theta_2) \text{ is globally stable.}$$

Proof. Note that the conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ in this theorem give

$$a_{11}a_{22} - a_{12}a_{21} > 0. \tag{60}$$

We prove this theorem by using the four cases and mathematical induction.
Case 5-1. Let $(x_0, y_0) \in \mathrm{II}$.
Lemma 2 and Theorem 1 give (49). Then we have

$$\lim_{n\to\infty}(x_n, y_n) = (\omega_1, \omega_2),\ \omega_2 > 0$$

and

$$f(x_n, y_n) < 0 \le g(x_n, y_n). \tag{61}$$

Solving (61) for x_n as in (51) and (52) and using (60), we have that for all n

$$0 < \theta_1 < x_n$$

and then $\omega_1 \ge \theta_1 > 0$. Since ω_1 and ω_2 are positive, we have

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Case 5-2. Let $(x_0, y_0) \in \mathrm{IV}$.
Using Remark 5 and Theorem 1, we have

$$0 < x_n < x_{n+1} < x^*,\ 0 < y_{n+1} \le y_n \tag{62}$$

and

$$\lim_{n\to\infty}(x_n, y_n) = (\omega_1, \omega_2),\ \omega_1 > 0.$$

The inequalities (62) implies

$$f(x_n, y_n) > 0 \ge g(x_n, y_n). \tag{63}$$

Solving (63) for y_n as in (56) and (57), we have that for all n

$$0 < \theta_2 < y_n$$

and then $\omega_2 \ge \theta_2 > 0$. Since ω_1 and ω_2 are positive, we have

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Case 5-3. Let $(x_0, y_0) \in \mathrm{I}$.
If $(x_m, y_m) \notin \mathrm{I}$ for some m, then

$$(x_m, y_m) \in \mathcal{D} - \mathrm{I} = \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$$

and further

$$(x_m, y_m) \in \mathrm{II} \cup \mathrm{IV}$$

due to Lemma 1. By Case 5-1 and 5-2, we have

$$\lim_{n\to\infty}(x_n, y_n) = (\theta_1, \theta_2).$$

On the other hand, if $(x_n, y_n) \in \mathrm{I}$ for all n, then we have the positive limits ω_1 and ω_2 of $\{x_n\}$ and $\{y_n\}$, respectively, due to the definition of I and Theorem 1. Taking the limit of (2) and using ω_i $(i = 1, 2)$, we have

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Case 5-4. Let $(x_0, y_0) \in \mathrm{III}$.
Replacing I in the proof of Case 5-3 with III, we can obtain

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Finally, we obtain the desired result from the proofs for Cases 5-1 to 5-4. □

4. Numerical examples

In this section, we provide simulations that illustrate our results in Theorem 2 to Theorem 5 for the difference scheme (2) with $\Delta t = 0.001$ and $(x^*, y^*) = (r_1 a_{11}^{-1}+50, r_2 a_{22}^{-1}+50)$. The values of parameters used in the following four examples satisfy the three conditions (7), (8) and (24).

Example 1. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 2, 3.5, 3, 2)$, which satisfies the two conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ in Theorem 2. Then the solutions (x_n, y_n) of (2) converge to $(0, r_2 a_{22}^{-1} = 1.75)$ as displayed in Figure 2-(a).

Example 2. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 1.5, 3, 5)$, which satisfies the two conditions $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ in Theorem 3. Then the solutions (x_n, y_n) of (2) converge to $(r_1 a_{11}^{-1} = 1, 0)$ as displayed in Figure 2-(b).

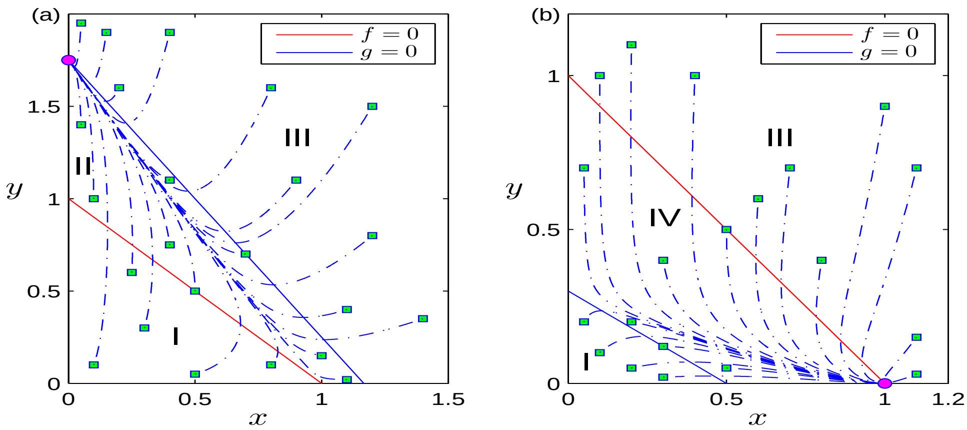

Figure 2: (a) Trajectories for different initial points in the regions I, II, III with $r_1 = 1, a_{11} = 1, a_{12} = 2, r_2 = 3.5, a_{21} = 3, a_{22} = 2$ in the category $\mathcal{C}_1$. (b) Trajectories for different initial points in the regions I, III, IV with $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 1.5, a_{21} = 3, a_{22} = 5$ in the category $\mathcal{C}_2$. The box and circle symbols denote initial and equilibrium points, respectively.

Example 3. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 1.7, 3, 1)$, which satisfies the two conditions $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ in Theorem 4. Then as displayed in Figure 3-(a), we obtain the results in Theorem 4. If $(x_0, y_0) \in \text{II}$, then the solutions (x_n, y_n) of (2) converge to $(0, r_2 a_{22}^{-1}) = (0, 1.7)$. If $(x_0, y_0) \in \text{IV}$, then $\lim_{n\to\infty}(x_n, y_n) = (r_1 a_{11}^{-1}, 0) = (1, 0)$. If $(x_0, y_0) \in \text{I} \cup \text{III}$, then $\{(x_n, y_n)\}$ converges with the limit $(r_1 a_{11}^{-1}, 0) = (1, 0)$ or $(0, r_2 a_{22}^{-1}) = (0, 1.7)$. Especially, Figure 3-(a) shows that there exist at least two initial points contained in I converging to $(r_1 a_{11}^{-1}, 0) = (1, 0)$ and $(0, r_2 a_{22}^{-1}) = (0, 1.7)$, respectively. In the region III, the same phenomenon happens. The outcome depends on the initial abundances of the two species.

Example 4. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 3.5, 2.5, 5)$, which satisfies the two conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ in Theorem 5. Then the solutions x_n and y_n of (2) converge to

$$(r_1 a_{22} - r_2 a_{12})(a_{11} a_{22} - a_{12} a_{21})^{-1} = 0.6$$

and

$$(-r_1 a_{21} + r_2 a_{11})(a_{11}a_{22} - a_{12}a_{21})^{-1} = 0.4,$$

respectively, as displayed in Figure 3-(b). Although the outcome in Example 3 depends on the initial abundances of the two species, the outcome in Example 4 is independent of the initial abundances.

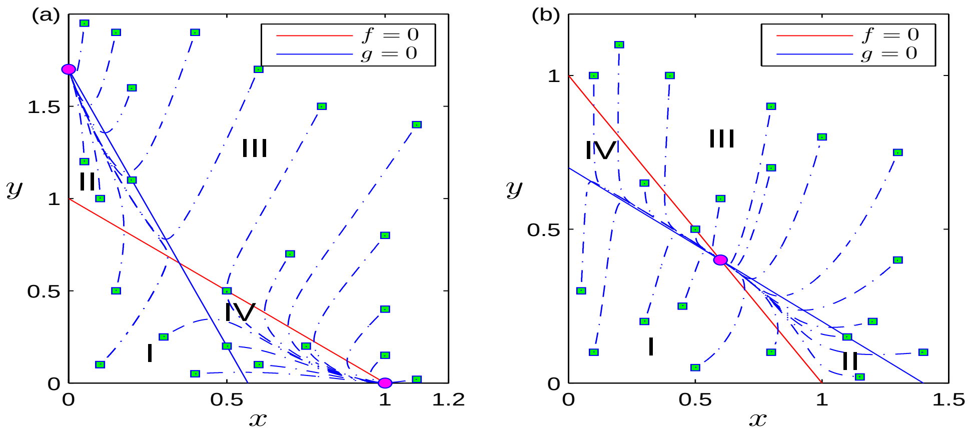

Figure 3: Trajectories for different initial points in the regions I, II, III and IV. The values of the parameters are (a) $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 1.7, a_{21} = 3, a_{22} = 1$ in the category $\mathcal{C}_3$. (b) $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 3.5, a_{21} = 2.5, a_{22} = 5$ in the category $\mathcal{C}_4$. The box and circle symbols denote initial and equilibrium points, respectively.

5. Conclusions and future work

In this paper, we have studied the Euler difference scheme for a two-dimensional Lotka-Volterra competition model and presented sufficient conditions for the global stability of the fixed points of a discrete competition model with two species. The main idea of our approach is to divide the domain used for the boundedness of solutions of the discrete model and to describe how to trace the trajectories with respect to each partition. Although we have applied our method for the two-dimensional discrete model, this method can be utilized to two-dimensional and other higher dimensional discrete models.

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Weighted Composition Operators from Bloch spaces into Zygmund spaces*

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Abstract

In this paper we characterize the boundedness and compactness of the weighted composition operator from the classical Bloch space β to the Zygmund space $\mathcal{Z}$, and from the little Bloch space β_0 to the little Zygmund space $\mathcal{Z}_0$, respectively.

Keywords Bloch space, Zygmund space; Weighted composition operator; Boundedness; Compactness

2010 MR Subject Classification 47B38, 30D99, 30H05

1 Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane and $H(D)$ denote the set of all analytic functions on D. Let $u, \varphi \in H(D)$, where φ is an analytic self-map of D. Then the well-known *weighted composition operator* uC_φ on $H(D)$ is defined by $uC_\varphi(f)(z) = u(z)\cdot(f\circ\varphi(z))$ for $f \in H(D)$ and $z \in D$. Weighted composition operators can be regarded as a generalization of multiplication operators and composition operators. In 2001, Ohno and Zhao studied the weighted composition operators on the classical Bloch space β in [14], which has led many researchers to study this operator on other Banach spaces of analytic functions. The boundedness and compactness of it have been studied on various Banach spaces of analytic functions, such as Hardy, Bergman, BMOA, Bloch-type spaces, see, e.g. [2, 4, 8, 18, 27].

In 2006, the boundedness of composition operators on the Zygmund space $\mathcal{Z}$ was first studied by Choe, Koo, and Smith in [1]. Later, many researchers have studied composition operators and weighted composition operators acting on the Zygmund space $\mathcal{Z}$. Li and Stević in [9] studied the boundedness and compactness of the generalized composition operators on Zygmund spaces and Bloch type spaces. They in [11] considered the boundedness and compactness of the weighted composition operators from Zygmund spaces to Bloch spaces. Ye and Hu in [22] characterized boundedness and compactness of weighted composition operators on the Zygmund space $\mathcal{Z}$. Esmaeili and Lindström in [7] studied weighted composition operators from Zygmund type spaces to Bloch type spaces and their essential norms. Sanatpour and Hassanlou in [17] gave the essential norms of this operators between Zygmund-type spaces and Bloch-type spaces. See also

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[5, 15, 16, 19, 20, 21, 23, 24, 25, 26] for corresponding results for weighted composition operators from one Banach space of analytic functions to another. It is well-known that $\mathcal{Z} \subset \beta$. It is more interesting to characterize u, φ such that this operator uC_φ has the pull-back properly, that is, $uC_\varphi f \in \mathcal{Z}$ whenever $f \in \beta$. In this paper we consider this question.

Now we give a detailed definition of these spaces. A function f analytic on the unit disk is said to belong to the *Bloch* space β if

$$b(f) = \sup_{z\in\mathbb{D}}\{(1-|z|^2)|f'(z)|\} < \infty,$$

and to the little *Bloch* space β_0 if $f \in \beta$ and

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

It is well known that β is a Banach space under the norm

$$\|f\|_\beta = |f(0)| + b(f),$$

and β_0 is a closed subspace of β.

The Zygmund space $\mathcal{Z}$ consists of all analytic functions f defined on D such that

$$z(f) = \sup\{(1-|z|^2)|f''(z)| : z \in D\}, \quad 0 < \alpha < +\infty.$$

From a theorem of Zygmund (see [29, vol. I, p. 263] or [6, Theorem 5.3]), we see that $f \in \mathcal{Z}$ if and only if f is continuous in the close unit disk $\overline{D} = \{z : |z| \leq 1\}$ and the boundary function $f(e^{i\theta})$ such that

$$\sup_{h>0,\theta} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$

An analytic function $f \in H(D)$ is said to belong to the little Zymund space $\mathcal{Z}_0$ consists of all $f \in \mathcal{Z}$ satisfying $\lim_{|z|\to 1^-}(1-|z|^2)|f''(z)| = 0$. It can easily proved that $\mathcal{Z}$ is a Banach space under the norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + z(f)$$

and the polynomials are norm-dense in closed subspace $\mathcal{Z}_0$ of $\mathcal{Z}$. For some other information on this space and some operators on it, see, for example, [9, 10, 11].

Throughout this paper, constants are denoted by C, they are positive and only depending on p, and may differ from one occurrence to the other.

2 Auxiliary results

In order to prove the main results of this paper. we need some auxiliary results. The first part of the following lemma is a well known.

Lemma 2.1 *Suppose that* $f \in \beta$, *then*

(i) $|f(z)| \leq \log \dfrac{e}{(1-|z|^2)}\|f\|_\beta$ *for every* $z \in D$;

(ii) $|f''(z)| \leq \dfrac{8}{(1-|z|^2)^2} b(f)$ *for every* $z \in D$.

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Proof For any $f \in \beta$. Fix $z \in D$ and let $\rho = \dfrac{1+|z|}{2}$, by the Cauchy integral formula, we obtain that

$$|f''(z)| = |\frac{1}{2\pi i}\int_{|\xi|=\rho}\frac{f'(\xi)}{(\xi - z)^2}\,d\xi| \leq \frac{b(f)}{1-\rho^2}\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho\,d\theta}{|\rho e^{i\theta} - z|^2} = \frac{\|f\|_\infty}{1-\rho^2}\frac{\rho}{\rho^2 - |z|^2} \leq \frac{8}{(1-|z|^2)^2}.$$

Hence (ii) holds.

Lemma 2.2 *[28] Suppose that* $f \in \beta_0$, *then*

(i) $\lim\limits_{|z|\to 1^-} \dfrac{|f(z)|}{\log(e/(1-|z|^2))} = 0$;
(ii) $\lim\limits_{|z|\to 1^-} (1-|z|^2)^2|f''(z)| = 0$.

Lemma 2.3 *Suppose* $uC_\varphi : \beta_0 \to \mathcal{Z}_0$ *is a bounded operator, then* $uC_\varphi : \beta \to \mathcal{Z}$ *is a bounded operator.*

The proof is similar to that of Lemma 2.3 in [21]. The details are omitted.

Lemma 2.4 *Suppose that* uC_φ *be a bounded operator from* β *to* $\mathcal{Z}$, *then* uC_φ *is compact if and only if for any bounded sequence* $\{f_n\}$ *in* β *which converges to* 0 *uniformly on compact subsets of* D. *We have* $\|uC_\varphi(f_n)\|_{\mathcal{Z}} \to 0$, *as* $n \to \infty$.

The proof is similar to that of Proposition 3.11 in [3] . The details are omitted.

Lemma 2.5 *Let* $U \subset \mathcal{Z}_0$. *Then* U *is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in U}(1-|z|^2)|f''(z)| = 0.$$

The proof is similar to that of Lemma 1 in [12], we omit it.

3 Main results

Theorem 3.1 *Let* u *be an analytic function on the unit disc* D, *and* φ *an analytic self-map of* D. *Then* uC_φ *is a bounded operator from the classical space* β *to the Zygmund space* $\mathcal{Z}$ *if and only if the following are satisfied:*

$$\sup_{z\in D}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2} < \infty; \tag{3.1}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{1-|\varphi(z)|^2} < \infty; \tag{3.2}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} < \infty. \tag{3.3}$$

Proof Suppose uC_φ is bounded from the Bloch space β to the Zygmund space $\mathcal{Z}$. Then we can easily obtain the following results by taking $f(z) = 1$ and $f(z) = z$ in β respectively:

$$u \in \mathcal{Z}; \tag{3.4}$$

$$\sup_{z\in D}(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z) + \varphi(z)u''(z)| < +\infty. \tag{3.5}$$

By (3.4), (3.5) and the boundedness of the function $\varphi(z)$, we get

$$K_1 = \sup_{z\in D}(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)| < +\infty. \tag{3.6}$$

Let $f(z) = z^2$ in β again, in the same way we have

$$\sup_{z\in D}(1-|z|^2)|4\varphi(z)\varphi'(z)u'(z) + \varphi^2(z)u''(z) + 2u(z)(\varphi(z)\varphi''(z) + (\varphi'(z))^2)| < \infty.$$

Using these facts and the boundedness of the function $\varphi(z)$ again, we get

$$K_2 = \sup_{z\in D}(1-|z|^2)|(\varphi'(z))^2u(z)| < +\infty. \tag{3.7}$$

Fix $a \in D$, we take the test functions

$$f_a(z) = 3\log\frac{e}{1-\bar{a}z} + \frac{3}{\log\frac{e}{1-|a|^2}}(\log\frac{e}{1-\bar{a}z})^2 - \frac{1}{\log^2\frac{e}{1-|a|^2}}(\log\frac{e}{1-\bar{a}z})^3 \tag{3.8}$$

for $z \in D$. By a directly calculation we obtain that $f_a \in \beta$ and $\sup_a \|f_a\|_\beta \le C < \infty$, where C is not depended on a. Since $f_a(a) = 5\log\frac{e}{1-|a|^2}$, $f_a'(a) = 0$, $f_a''(a) = 0$, we have

$$\begin{aligned} C\|f_a\|_\beta &\ge \|uC_\varphi f_a\|_{\mathcal{Z}} \ge \sup_{z\in D}(1-|z|^2)|(uC_\varphi f_a)''(z)| \\ &= \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)f_a'(\varphi(z)) \\ &+ f_a''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)f_a(\varphi(z))|. \end{aligned}$$

Let $a = \varphi(\lambda)$, it follows that

$$\begin{aligned} C\|f_a\|_\beta &\ge (1-|\lambda|^2)^\alpha|\big(2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda)\big)f_{\varphi(\lambda)}'(\varphi(\lambda)) \\ &+ f_{\varphi(\lambda)}''(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda) + u''(\lambda)f_{\varphi(\lambda)}(\varphi(\lambda))| \\ &= 5(1-|\lambda|^2)^\alpha|u''(\lambda)\log\frac{e}{1-|\varphi(\lambda)|^2}|. \end{aligned}$$

Hence (3.1) holds.

Next, we will show that (3.2) holds. Fix $a \in D$ with $|a| > \frac{1}{2}$, we take another test functions:

$$g_a(z) = \frac{8(1-|a|^2)^2}{(1-\bar{a}z)^2} - \frac{14(1-|a|^2)^3}{(1-\bar{a}z)^3} + \frac{6(1-|a|^2)^4}{(1-\bar{a}z)^4} \tag{3.9}$$

for $z \in D$. By a directly calculation we obtain that $g_a \in \beta$ and $\sup_a \|g_a\|_\beta \le C < \infty$, where C is not depended on a. Since $g_a(a) = 0$, $g_a'(a) = \dfrac{-2\bar{a}}{1-|a|^2}$, $g_a''(a) = 0$, it follows that for all $\lambda \in D$ with $|\varphi(\lambda)| > \frac{1}{2}$, we have

$$\begin{aligned} C\|g_a\|_\beta &\ge \|uC_\varphi g_a\|_{\mathcal{Z}} \ge \sup_{z\in D}(1-|z|^2)|(uC_\varphi g_a)''(z)| \\ &= \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)g_a'(\varphi(z)) \\ &+ g_a''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)g_a(\varphi(z))|. \end{aligned}$$

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Let $a = \varphi(\lambda)$, it follows that

$$\begin{aligned} C\|g_a\|_\beta &\geq (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)g'_{\varphi(\lambda)}(\varphi(\lambda)) \\ &+ g''_{\varphi(\lambda)}(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda)+u''(\lambda)g_{\varphi(\lambda)}(\varphi(\lambda))| \\ &= (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)\frac{-2\overline{\varphi(\lambda)^2}}{1-|\varphi(\lambda)|^2}| \\ &\geq \frac{1}{2}\frac{(1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)|}{1-|\varphi(\lambda)|^2}. \end{aligned}$$

For $\forall\lambda \in D$ with $|\varphi(\lambda)| \leq \frac{1}{2}$, by (3.6), we have

$$\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)|}{1-|\varphi(\lambda)|^2} \leq \frac{4}{3}\sup_{\lambda\in D}(1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)| < +\infty.$$

Hence (3.2) holds.

Finally we will show (3.3) holds. Fix $a \in D$ with $|a| > \frac{1}{2}$, we take the test functions:

$$h_a(z) = -\frac{3(1-|a|^2)^2}{(1-\bar{a}z)^2} + \frac{6(1-|a|^2)^3}{(1-\bar{a}z)^3} - \frac{3(1-|a|^2)^4}{(1-\bar{a}z)^4} \tag{3.10}$$

for $z \in D$. It is easily proved that $\sup_{\frac{1}{2}<|a|<1}\|h_a\|_\beta \leq C < \infty$, where C is not depended on a. For $w \in D$, let $a = \varphi(w)$, since

$$h_{\varphi(w)}(\varphi(w)) = 0, \quad h'_{\varphi(w)}(\varphi(w)) = 0, \quad h''_{\varphi(w)}(\varphi(w)) = \frac{-6(\overline{\varphi(w)})^2}{(1-|\varphi(w)|^2)^2},$$

then, for all $w \in D$ with $|\varphi(w)| > \frac{1}{2}$, we obtain that

$$C\|h_a\|_\beta \geq \|uC_\varphi g_a\|_{\mathcal{Z}} \geq (1-|w|^2)\frac{|6u(w)(\varphi'(w))^2(\overline{\varphi(w)})^2|}{(1-|\varphi(w)|^2)^2}.$$

Then, by (3.7), we have

$$\begin{aligned} &\sup_{w\in D}\frac{(1-|w|^2)|u(w)(\varphi'(w))^2|}{(1-|\varphi(w)|^2)^2} \leq \sup_{|\varphi(w)|>\frac{1}{2}}\frac{(1-|w|^2)|u(w)(\varphi'(w))^2|}{(1-|\varphi(w)|^2)^2} \\ &+ \sup_{|\varphi(w)|\leq\frac{1}{2}}\frac{(1-|w|^2)|u(w)(\varphi'(w))^2|}{(1-|\varphi(w)|^2)^2} \\ &\leq 4\sup_{|\varphi(w)|>\frac{1}{2}}(1-|w|^2)|\frac{|u(w)(\varphi'(w))^2(\overline{\varphi(w)})^2|}{(1-|\varphi(w)|^2)^2} + \frac{16}{9}\sup_{|\varphi(w)|\leq\frac{1}{2}}(1-|w|^2)|u(w)(\varphi'(w))^2| \\ &< \infty. \end{aligned}$$

Hence (3.3) holds.

Conversely, suppose that (3.1), (3.2), and (3.2) hold. For $f \in \beta$, by Lemma 2.1, we have the

following inequality:

$$\begin{aligned}(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z)) \\ &+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))| \\ &\leq (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))| \\ &+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)|+(1-|z|^2)|u''(z)f(\varphi(z))| \\ &\leq \frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2}b(f) \\ &+ 8\frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^2}b(f)+(1-|z|^2)|u''(z)|\log(\frac{e}{1-|\varphi(z)|^2})\|f\|_\beta \\ &\leq C\|f\|_\beta,\end{aligned}$$

and

$$\begin{aligned}&|u(0)f(\varphi(0))|+|u'(0)f(\varphi(0))|+|u(0)f'(\varphi(0))\varphi'(0)| \\ &\leq \big((|u(0)|+|u'(0)|)\log(\frac{e}{1-|\varphi(0)|^2})+\frac{|u(0)\varphi'(0)|}{1-|\varphi(0)|^2}\big)\|f\|_\beta.\end{aligned}$$

This shows that uC_φ is bounded. This completes the proof of Theorem 3.1.

Corollary 3.1 *Let φ be an analytic self-map of D. Then C_φ is a bounded operator from the Bloch space β to the Zygmund space $\mathcal{Z}$ if and only if*

$$\sup_{z\in D}\frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2}<\infty \quad\text{and}\quad \sup_{z\in D}\frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2}<\infty.$$

In the formulation of lemma, we use the notation M_u on $H(D)$ defined by $M_u f = uf$ for $f\in H(D)$.

Corollary 3.2 *The pointwise multiplier $M_u:\beta\to\mathcal{Z}$ is a bounded operator if and only if $u=0$.*

Theorem 3.2 *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Then uC_φ is a compact operator from β to $\mathcal{Z}$ if and only if uC_φ is a bounded operator and the following are satisfied:*

$$\lim_{|\varphi(z)|\to1^-}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2}=0; \tag{3.11}$$

$$\lim_{|\varphi(z)|\to1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2}=0; \tag{3.12}$$

$$\lim_{|\varphi(z)|\to1^-}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2}=0. \tag{3.13}$$

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Proof Suppose that uC_φ is compact from β to the Zygmund space $\mathcal{Z}$. Let $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. If such a sequence does not exist then (3.11), (3.12) and (3.13) are automatically satisfied. Without loss of generality we may suppose that $|\varphi(z_n)| > \frac{1}{2}$ for all n. We take the test functions

$$f_n(z) = \frac{6}{a_n} \log^2 \frac{e}{1-\overline{\varphi(z_n)}z} - \frac{8}{a_n^2} \log^3 \frac{e}{1-\overline{\varphi(z_n)}z} + \frac{3}{a_n^3} \log^4 \frac{e}{1-|\varphi(z_n)|^2}. \tag{3.14}$$

where $a_n = \log \dfrac{e}{1-|\varphi(z_n)|^2}$. By a directly calculation, we may easily prove that $\{f_n\}$ converges to 0 uniformly on compact subsets of D and $\sup_n \|f_n\|_\beta \le C < \infty$. Then $\{f_n\}$ is a bounded sequence in β which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty} \|uC_\varphi(f_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. Note that

$$f_n(\varphi(z_n)) = a_n, \quad f_n'(\varphi(z_n)) = 0, \quad f_n''(\varphi(z_n)) = 0.$$

It follows that

$$\|uC_\varphi f_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))f_n'(\varphi(z_n))$$

$$+u(z_n)f_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)f_n(\varphi(z_n))|$$

$$= (1-|z_n|^2)|u''(z_n)| \log \frac{e}{1-|\varphi(z_n)|^2}.$$

Then

$$\lim_{n\to\infty} (1-|z_n|^2)|u''(z_n)| \log \frac{e}{1-|\varphi(z_n)|^2} = 0.$$

Next, let

$$g_n(z) = \frac{8(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^2} - \frac{14(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^3} + \frac{6(1-|\varphi(z_n)|^2)^4}{(1-\overline{\varphi(z_n)}z)^4}.$$

By a directly calculation we obtain that $g_n \rightrightarrows 0$ $(n \to \infty)$ on compact subsets of D and $\sup_n \|g_n\|_\beta \le C < \infty$. Consequently, $\{g_n\}$ is a bounded sequence in β which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty} \|uC_\varphi(g_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. Note that $g_n(\varphi(z_n)) \equiv 0$, $g_n''(\varphi(z_n)) \equiv 0$ and $g_n'(\varphi(z_n)) = \dfrac{-2\overline{\varphi(z_n)}}{1-|\varphi(z_n)|^2}$.

It follows that

$$\|uC_\varphi g_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))g_n'(\varphi(z_n))$$

$$+u(z_n)g_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)g_n(\varphi(z_n))|$$

$$= 2(1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))\frac{\overline{\varphi(z_n)}}{1-|\varphi(z_n)|^2}|.$$

Then $\displaystyle\lim_{n\to\infty} \frac{(1-|z_n|^2)|2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)|}{1-|\varphi(z_n)|^2} = 0.$

Finally, let

$$h_n(z) = -\frac{3(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^2} + \frac{6(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^3} - \frac{3(1-|\varphi(z_n)|^2)^4}{(1-\overline{\varphi(z_n)}z)^4}.$$

By a directly calculation we obtain that $h_n \rightrightarrows 0$ $(n \to \infty)$ on compact subsets of D and $\sup_n \|h_n\|_{\mathcal{Z}} \le C < \infty$. Consequently, $\{h_n\}$ is a bounded sequence in $\mathcal{Z}$ which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty} \|uC_\varphi(h_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. Note that $h_n(\varphi(z_n)) \equiv 0$, $h_n'(\varphi(z_n)) \equiv 0$ and $h_n''(\varphi(z_n)) = \dfrac{-6(\overline{\varphi(z_n)})^2}{(1-|\varphi(z_n)|^2)^2}$. It follows that

$$\|uC_\varphi h_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))h_n'(\varphi(z_n))$$

$$+u(z_n)h_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)h_n(\varphi(z_n))|$$

$$= 6(1-|z_n|^2)|u(z_n)(\varphi'(z_n))^2|\frac{|\overline{\varphi(z_n)}|^2}{(1-|\varphi(z_n)|^2)^2}.$$

Then $\lim\limits_{n\to\infty}(1-|z_n|^2)\dfrac{|u(z_n)(\varphi'(z_n))^2|}{(1-|\varphi(z_n)|^2)^2} = 0$. The proof of the necessary is completed.

Conversely, suppose that (3.11), (3.12), and (3.13) hold. Since uC_φ is a bounded operator, by Theorem 3.1, we have

$$M_1 \triangleq \sup_{z\in D}(1-|z|^2)|u''(z)|\log\frac{1}{1-|\varphi(z)|^2} < \infty, \qquad M_3 \triangleq \sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} < \infty,$$

and

$$M_2 \triangleq \sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2} < \infty.$$

Let $\{f_n\}$ be a bounded sequence in β with $\|f_n\|_\beta \le 1$ and $f_n \to 0$ uniformly on compact subsets of D. We only prove $\lim\limits_{n\to\infty}\|uC_\varphi(f_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. By the assumption, for any $\epsilon > 0$, there is a constant δ, $0 < \delta < 1$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} < \epsilon, \qquad (1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2} < \epsilon,$$

and

$$\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2} < \epsilon.$$

Let $K = \{w \in D : |w| \le \delta\}$. Noting that K is a compact subset of D, we get that

$$\begin{aligned}
z(uC_\varphi f_n) &= \sup_{z\in D}(1-|z|^2)|(uC_\varphi f_n)''(z)| \\
&\le \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f_n'(\varphi(z))| \\
&+ \sup_{z\in D}(1-|z|^2)|f_n''(\varphi(z))(\varphi'(z))^2u(z)| + \sup_{z\in D}(1-|z|^2)|u''(z)f_n(\varphi(z))| \\
&\le 10\epsilon + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f_n'(\varphi(z))| \\
&+ \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|f_n''(\varphi(z))(\varphi'(z))^2u(z)| + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|u''(z)f_n(\varphi(z))| \\
&\le 10\epsilon + M_2\sup_{w\in K}|f_n'(w)| + M_3\sup_{w\in K}|f_n''(w)| + M_1\sup_{w\in K}|f_n(w)|.
\end{aligned}$$

As $n\to\infty$, $\|uC_\varphi f_n\|_{\mathcal{Z}} \to 0$. Hence uC_φ is compact. This completes the proof of Theorem 3.2.

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Corollary 3.3 *Let φ be an analytic self-map of D. Then C_φ is a compact operator from the Bloch space β to the Zygmund space $\mathcal{Z}$ if and only if C_φ is bounded,*

$$\lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} = 0 \quad \text{and} \quad \lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2} = 0.$$

Theorem 3.3 *Let u be an analytic function on the unit disc D, and φ an analytic self-map of D. Then $uC_\varphi : \beta_0 \to \mathcal{Z}_0$ is a bounded operator if and only if $u \in \mathcal{Z}_0$, (3.1), (3.2), and (3.3) hold, and the following are satisfied:*

$$\lim_{|z|\to 1^-} (1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)| = 0. \tag{3.15}$$

$$\lim_{|z|\to 1^-} (1-|z|^2)|u(z)(\varphi'(z))^2| = 0; \tag{3.16}$$

Proof Suppose that uC_φ is bounded from the little Bloch space β_0 to the little Zygmund type spaces $\mathcal{Z}_0$. Then $u = uC_\varphi 1 \in \mathcal{Z}_0$. Also $u\varphi = uC_\varphi z \in \mathcal{Z}_0$, thus

$$(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z) + \varphi(z)u''(z)| \longrightarrow 0 \quad (|z| \to 1^-).$$

Since $|\varphi| \le 1$ and $u \in \mathcal{Z}_0$, we have $\lim_{|z|\to 1^-}(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)| = 0$. Hence (3.15) holds.

Similarly, $uC_\varphi z^2 \in \mathcal{Z}_0$, then

$$(1-|z|^2)|4\varphi(z)\varphi'(z)u'(z) + \varphi^2(z)u''(z) + 2u(z)(\varphi(z)\varphi''(z) + (\varphi'(z))^2)| \longrightarrow 0 \quad (|z| \to 1^-).$$

By (3.15), $|\varphi| \le 1$ and $u \in \mathcal{Z}_0$, we get that $\lim_{|z|\to 1^-}(1-|z|^2)|u(z)(\varphi'(z))^2| = 0$, i. e. that (3.16) holds. On the other hand, from Lemma 2.3 and Theorem 3.1, we obtain that (3.1), (3.2), and (3.3) hold.

Conversely, for $\forall f \in \beta_0$, we have both $(1-|z|^2)^2|f''(z)| \to 0$ and $\frac{|f(z)|}{\ln \frac{e}{1-|z|^2}} \to 0$ as $|z| \to 1^-$ by Lemma 2.2. Given $\epsilon > 0$ there is a $0 < \delta < 1$ such that $(1-|z|^2)|f'(z)| < \frac{\epsilon}{3M_2}$, $(1-|z|^2)^2|f''(z)| < \frac{\epsilon}{3M_3}$ and $\frac{|f(z)|}{\log \frac{e}{1-|z|^2}} < \frac{\epsilon}{3M_1}$ for all z with $\delta < |z| < 1$, where $M_1,\ M_2,\ M_3$ are defined in above.

If $|\varphi(z)| > \delta$, it follows that

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|[2\varphi'(z)u'(z) + \varphi''(z)u(z)]f'(\varphi(z)) \\
&+ f''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)f(\varphi(z))| \\
&\le (1-|z|^2)|[2\varphi'(z)u'(z) + \varphi''(z)u(z)]f'(\varphi(z))| \\
&+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)| + (1-|z|^2)|u''(z)f(\varphi(z))| \\
&\le M_2(1-|\varphi(z)|^2)|f'(\varphi(z))| + M_3(1-|\varphi(z)|^2)^2|f''(\varphi(z))| + M_1\frac{|f(\varphi(z))|}{\log \frac{e}{1-|\varphi(z)|^2}} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

We know that there exists a constant M_4 such that $|f(z)| \leq M_3$, $|f'(z)| \leq M_4$ and $|f''(z)| \leq M_4$ for all $|z| \leq \delta$.

If $|\varphi(z)| \leq \delta$, it follows that

$$\begin{aligned}(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|[2\varphi'(z)u'(z)+\varphi''(z)u(z)]f'(\varphi(z)) \\ &+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))| \\ &\leq M_4(1-|z|^2)||2\varphi'(z)u'(z)+\varphi''(z)u(z)| \\ &+ M_4(1-|z|^2)|(\varphi'(z))^2u(z)|+M_4(1-|z|^2)|u''(z)|.\end{aligned}$$

Thus we conclude that $(1-|z|^2)|(uC_\varphi(f))''(z)| \to 0$ as $|z| \to 1^-$. Hence $uC_\varphi f \in \mathcal{Z}_0$ for all $f \in \beta_0$. On the other hand, uC_φ is a bounded operator from β to $\mathcal{Z}$ by Theorem 3.1. Hence uC_φ is a bounded operator from the little Bloch space β_0 to the little Zygmund space $\mathcal{Z}_0$.

Corollary 3.4 *Let φ be an analytic self-map of D. Then C_φ is a bounded operator from β_0 to $\mathcal{Z}_0$ if and only if C_φ is a bounded operator from β to $\mathcal{Z}$ and $\varphi \in \mathcal{Z}_0$.*

Proof By Theorem 3.3 we have that C_φ is a bounded operator from β_0 to $\mathcal{Z}_0$ if and only $C_\varphi : \beta \to \mathcal{Z}$ is bounded, $\varphi \in \mathcal{Z}_0$, and

$$\lim_{|z|\to 1^-}(1-|z|^2)|(\varphi'(z))^2| = 0.$$

However, That $\varphi \in \mathcal{Z}_0$ means $\varphi' \in \beta_0$. Then we have that $|\varphi'(z)| \leq \log\frac{e}{1-|z|^2}\|\varphi'\|_\beta$ by Lemma 2.1. It follows that

$$(1-|z|^2)|(\varphi'(z))^2| \leq (1-|z|^2)\log^2\frac{e}{1-|z|^2}\|\varphi'\|_\beta^2 \to 0,$$

as $|z| \to 1^-$.

Theorem 3.4 *Let u be an analytic function on the unit disc D, and φ an analytic self-map of D. Then uC_φ is compact from β_0 to $\mathcal{Z}_0$ if and only if the following are satisfied:*

$$\lim_{|z|\to 1^-}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2} = 0; \tag{3.17}$$

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2} = 0; \tag{3.18}$$

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} = 0. \tag{3.19}$$

Proof Assume (3.17), (3.18), and (3.19) hold. From Theorem 3.3, we know that uC_φ is bounded from β_0 to $\mathcal{Z}_0$. Suppose that $f \in \beta_0$ with $\|f\|_\beta \leq 1$. We obtain that

$$\begin{aligned}&(1-|z|^2)|(uC_\varphi f)''(z)| \leq (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))| \\ &+(1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)|+(1-|z|^2)|u''(z)f(\varphi(z))| \\ &\leq (1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|\frac{1}{1-|\varphi(z)|^2}b(f) \\ &+8\frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^2}b(f)+(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2}\|f\|_\beta,\end{aligned}$$

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thus

$$\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \beta_0, \|f\|_\beta \le 1\}$$

$$\le (1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|\frac{1}{1-|\varphi(z)|^2}$$

$$+\frac{8(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^2}+(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2},$$

and it follows that

$$\lim_{|z|\to 1^-}\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \beta_0, \|f\|_\beta \le 1\} = 0,$$

hence $uC_\varphi : \beta_0 \to \mathcal{Z}_0$ is compact by Lemma 2.5.

Conversely, suppose that $uC_\varphi : \beta_0 \to \mathcal{Z}_0$ is compact.

First, it is obvious $uC_\varphi : \beta_0 \to \mathcal{Z}_0$ is bounded, then by Theorem 3.3, we have $u \in \mathcal{Z}_0$ and that (3.15) and (3.16) hold. On the other hand, by Lemma 2.5 we have

$$\lim_{|z|\to 1^-}\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \beta_0, \|f\|_\beta \le M\} = 0,$$

for some $M > 0$.

Next, noting that the proof of Theorem 3.1 and the fact that the functions given in (3.8) are in β_0 and have norms bounded independently of a, we obtain that

$$\lim_{|z|\to 1^-}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2} = 0.$$

Similarly, noting that the functions given in (3.9) are in β_0 and have norms bounded independently of a, we obtain that

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2} = 0 \tag{3.20}$$

for $|\varphi(z)| > \frac{1}{2}$. However, if $|\varphi(z)| \le \frac{1}{2}$, by (3.15), we easily have

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2}$$

$$\le \frac{4}{3}\lim_{|z|\to 1^-}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)| = 0.$$

Thus (3.18) holds.

Also, the third statement, that (3.19), is proved similarly. We omitted it here. This completes the proof of Theorem 4.2.

Corollary 3.5 *Let φ be an analytic self-map of D. Then C_φ is a compact operator from β_0 to $\mathcal{Z}_0$ if and only if*

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} = 0$$

and

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2} = 0.$$

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Approximate homomorphisms and derivations on non-Archimedean Lie JC^*-algebras

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Abstract. In this paper, by using the fixed point method, we prove the Hyers-Ulam stability of homomorphisms in non-Archimedean Lie JC^*-algebras and derivations on non-Archimedean Lie JC^*-algebras associated with the following additive mapping:

$$\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\cdots,i_{n-k+1}}^{n}x_i-\sum_{r=1}^{n-k+1}x_{i_r}\Big)$$
$$+f\Big(\sum_{i=1}^{n}x_i\Big)=2^{n-1}f(x_1)$$

for a fixed positive integer n with $n\geqslant 2$.

1. Introduction

In 1896, Hensel [4] introduced a field with a valuation in which does not have the Archimedean property. Let $\mathcal{K}$ be a field. A non-Archimedean absolute value on$\mathcal{K}$ is a function $|\cdot|:\mathcal{K}\to[0,+\infty)$ such that, for any $a,b\in\mathcal{K}$, the following conditions are satisfying

(i) $|a|\geqslant 0$ and equality holds if and only if $a=0$,

(ii) $|ab|=|a||b|$,

(iii) $|a+b|\leqslant\max\{|a|,|b|\}$ (the strict triangle inequality).

Note that $|1|=|-1|=1$ and $|n|\leqslant 1$ for each integer n. We always assume, in addition, that $|\cdot|$ is non-trival, i.e., there exists an $a_0\neq 0,1$. A function $\|.\|:X\to[0,\infty)$ is called a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii) for any $r\in K, x\in X$, $\|rx\|=|r|\|x\|$;

(iii) the strong triangle inequality holds, namely,

$$\|x+y\|\leqslant\max\{\|x\|,\|y\|\}\quad(x,y\in X).$$

Then $(X,\|\cdot\|)$ is called a non-Archimedean normed space. From the fact that

$$\|x_n-x_m\|\leqslant\max\{\|n_n-x_m\|:m\leqslant j\leqslant n-1\}\quad(n>m),$$

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holds, a sequence $\{x_n\}$ is Cauchy if and only if $\{x_n - x_m\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

A non-Archimedean Banach algebra is a complete non-Archimedean algebra $\mathcal{A}$ which satisfies $\|ab\| \leqslant \|a\| \cdot \|b\|$ for all $a, b \in \mathcal{A}$. For more detailed definitions of non-Archimedean Banach algebras, we refer the reader to [15].

If $\mathcal{U}$ is a non-Archimedean Banach algebra, then an involution on $\mathcal{U}$ is mapping $t \to t^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

(i) $t^{**} = t$ for $t \in \mathcal{U}$;
(ii) $(\alpha s + \beta t)^* = \bar{\alpha} s^* + \bar{\beta} t^*$;
(iii) $(st)^* = t^* s^*$ for all $s, t \in \mathcal{U}$.

If, in addition, $\|t^* t\| = \|t\|^2$ for $t \in \mathcal{U}$, then $\mathcal{U}$ is a non-Archimedean C^*-algebra.

The stability problem of functional equations originated from a question of Ulam [16] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond)$ be a metric group (a metric is defined on a set with group property) with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $h(x * y) = h(x) * h(y)$ is stable (see also [3, 5, 9, 10, 12, 13, 14]).

For explicitly later use, we recall a fundamental result in fixed point theory.

Theorem 1.1. *(The fixed point alternative theorem [2]) Let (Ω, d) be a complete generalized metric space and $J : \Omega \to \Omega$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$, that is,*

$$d(Jx, Jy) \leqslant L d(x, y), \quad x, y \in \Omega.$$

Then, for each given $x \in \Omega$, either

$$d(J^n x, J^{n+1} x) = \infty, \quad \forall n \geqslant 0,$$

or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geqslant n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $\Delta = \{y \in \Omega : d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ for all $y \in \Delta$.*

A non-Archimedean C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy - yx}{2}$ and endowed with *anticommutator product* (Jordan product) $x \circ y := \frac{xy + yx}{2}$ on $\mathcal{C}$, is called a non-Archimedean Lie JC^*-algebra (see [6, 7, 8]).

Jordan algebras as coordinates for Lie algebras were created to illuminate a particular aspect of physics, quantum-mechanical observables, but turned out to have illuminating connections with many areas of mathematics.

In this paper, we prove the Hyers-Ulam stability of homomorphisms and derivations in non-Archimedean Lie JC^*-algebras associated with the following additive functional equation:

$$\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k+1}}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\cdots,i_{n-k+1}}^{n}x_i-\sum_{r=1}^{n-k+1}x_{i_r}\Big)$$
$$+f\Big(\sum_{i=1}^{n}x_i\Big)=2^{n-1}f(x_1) \tag{1.1}$$
for a fixed positive integer n with $n\geqslant 2$.

2. Stability of homomorphisms in non-Archimedean Lie JC^*-algebras

Definition 2.1. [7] *Let $\mathcal{A}$ and $\mathcal{B}$ be non-Archimedean Lie JC^*-algebras. A $\mathbb{C}$-linear mapping $H:\mathcal{A}\to\mathcal{B}$ is called a (non-Archimedean Lie JC^*-algebra) homomorphism if H satisfies*
$$H([x,y])=[H(x),H(y)],$$
$$H(x\circ y)=H(x)\circ h(y),$$
$$H(x^*)=H(x)^*$$
for all $x,y\in\mathcal{A}$.

Throughout this section, assume that A and B are two non-Archimedean Lie JC^*-algebras, respectively, with norm $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$.

For a given mapping $f:\mathcal{A}\to\mathcal{B}$, we define
$$D_\mu f(x_1,\cdots,x_n):=\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k+1}}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\cdots,i_{n-k+1}}^{n}\mu x_i-\sum_{r=1}^{n-k+1}\mu x_{i_r}\Big)$$
$$+f\Big(\sum_{i=1}^{n}\mu x_i\Big)-2^{n-1}f(\mu x_1)$$
for all $\mu\in\mathbb{T}^1:=\{\lambda\in\mathbb{C}:|\lambda|=1\}$ and all $x_1,\cdots,x_n\in\mathcal{A}$.

We recall the following needed lemmas in this paper.

Lemma 2.2. [11] *Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces and $f:\mathcal{V}\to\mathcal{W}$ be an additive mapping such that $f(\mu x)=\mu f(x)$ for all $x\in\mathcal{V}$ and $\mu\in\mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.3. [7] *A mapping $f:\to\mathcal{A}\to\mathcal{B}$ with $f(0)=0$ satisfies the functional equation (1.1) if and only if $f:\mathcal{A}\to\mathcal{B}$ is additive.*

We prove the Hyers-Ulam stability of homomorphisms in non-Archimedean Lie JC^*-algebras for the functional equation $D_\mu f(x_1,\cdots,x_n)=0$.

Theorem 2.4. *Let $f:\mathcal{A}\to\mathcal{B}$ be a mapping for which there are functions $\varphi:\mathcal{A}^n\to[0,\infty),\psi:\mathcal{A}^2\to[0,\infty)$, and $\eta:\mathcal{A}\to[0,\infty)$ such that $|2|<1$ is far from zero and*
$$\lim_{m\to\infty}\frac{1}{|2|^m}\varphi(2^m x_1,2^m x_2,\cdots,2^m x_n)=0, \tag{2.1}$$
$$\lim_{m\to\infty}\frac{1}{|2|^{2m}}\psi(2^m x,2^m y)=0, \tag{2.2}$$

$$\lim_{m\to\infty}\frac{1}{|2|^m}\eta(2^m x)=0, \tag{2.3}$$

$$\|D_\mu f(x_1,\cdots,x_n)\|_{\mathcal{B}}\leqslant\varphi(x_1,\cdots,x_n), \tag{2.4}$$

$$\|f([x,y])-[f(x),f(y)]\|_{\mathcal{B}}\leqslant\psi(x,y), \tag{2.5}$$

$$\|f(x\circ y)-f(x)\circ f(y)\|_{\mathcal{B}}\leqslant\psi(x,y), \tag{2.6}$$

$$\|f(x^*)-f(x)^*\|_{\mathcal{B}}\leqslant\eta(x), \tag{2.7}$$

for all $x,y,x_1,\cdots,x_n\in\mathcal{A}$ and $\mu\in\mathbb{T}^1$. If there exists a constant $0<L<1$ such that $\varphi(x_1,x_2,\cdots,x_n)\leqslant\alpha L\varphi(\frac{x_1}{2},\frac{x_2}{2},\cdots,\frac{x_n}{2})$ for all $x_1,x_2,\cdots,x_n\in\mathcal{A}$, where $\alpha=|2|^{n-1}$, then there exists a unique homomorphism $H:\mathcal{A}\to\mathcal{B}$ such that

$$\|f(x)-H(x)\|\leqslant\frac{L}{1-L}\varphi(\frac{x}{2},\frac{x}{2},0,\cdots,0) \tag{2.8}$$

for all $x\in\mathcal{A}$.

Proof. Let $\mu=1$. Using the following relation

$$1+\sum_{k=1}^{n-k}\binom{n-k}{k}=\sum_{k=0}^{n-k}\binom{n-k}{k}=2^{n-k} \tag{2.9}$$

for all $n>k$ and putting $x_1=x_2=x$ and $x_3=x_4=\cdots=x_n=0$ in (2.4), we obtain

$$\|\frac{\alpha}{2}f(2x)-\alpha f(x)\|_{\mathcal{B}}\leqslant\varphi(x,x,0,\cdots,0)$$

for all $x\in\mathcal{A}$. So

$$\|\frac{1}{2}f(2x)-f(x)\|_{\mathcal{B}}\leqslant\frac{1}{\alpha}\varphi(x,x,0,\cdots,0)\leqslant L\varphi\Big(\frac{x}{2},\frac{x}{2},0,\cdots,0\Big) \tag{2.10}$$

for all $x\in\mathcal{A}$. Let define $\Omega:=\{g:\mathcal{A}\to\mathcal{B}\}$ and introduce a generalized metric on Ω as follows

$$d(g,h)=\inf\{k\in(0,\infty):\|g(x)-h(x)\|_{\mathcal{B}}<k\varphi\Big(\frac{x}{2},\frac{x}{2},0,\cdots,0\Big),\forall x\in\mathcal{A}\}.$$

It is easy to show that (Ω,d) is a generalized complete metric space (see [1]).

Now we consider the function $J:\Omega\to\Omega$ define by $Jg(x)=\frac{1}{|2|}g(2x)$ for all $x\in\mathcal{A}$ and $g\in\Omega$. Let for all $g,h\in\Omega$ and an arbitrary constant $k\in[0,\infty)$ with $d(x,y)\leqslant k$, we have

$$\|g(x)-h(x)\|_{\mathcal{B}}\leqslant k\varphi\Big(\frac{x}{2},\frac{x}{2},0,\cdots,0\Big)$$

for all $x\in\mathcal{A}$. Then we can write

$$\|Jg(x)-Jh(x)\|_{\mathcal{B}}=\frac{1}{|2|}\|g(2x)-h(2x)\|_{\mathcal{B}}\leqslant\frac{k}{|2|}\varphi(x,x,0,\cdots,0)\leqslant\frac{\alpha kL}{|2|}\varphi\Big(\frac{x}{2},\frac{x}{2},0,\cdots,0\Big)$$

for all $x\in\mathcal{A}$. So we conclude that $d(Jg,Jh)\leqslant\frac{\alpha}{|2|}Ld(g,h)$ for all $g,h\in\Omega$. It follows from (2.9) that $d(Jf,f)\leqslant L$, that is, J is a self-function of Ω with the Lipchitz constant L. Therefore, from Theorem 1.1, there eists a fixed point H of J set $\Omega_1=\{h\in X:d(f,h)<\infty\}$ such that

$$H(x)=\lim_{m\to\infty}\frac{1}{|2|^m}f(2^m x) \tag{2.11}$$

for all $x \in \mathcal{A}$, since $\lim_{m\to\infty} d(J^n f, H) = 0$. Also $2H(\frac{x}{2}) = H(x)$ for all $x \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is the unique fixed point of J in Ω_1 such that

$$d(H, f) \leqslant \frac{1}{1-L} d(Jf, f) \leqslant \frac{L}{1-L},$$

i.e., H satisfies (2.8) for all $x \in \mathcal{A}$. It follows from the definition of H, (2.1) and (2.4) that

$$\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k+1}}^{n}\Big)H\Big(\sum_{i=1,i\neq i_1,\cdots,i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}\Big) + H\Big(\sum_{i=1}^{n} x_i\Big) = 2^{n-1}H(x_1)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$. Since $H(0) = 0$, by Lemma 2.3, the mapping H is additive.

Put $x_1 = x$ and $x_2 = x_3 = \cdots = 0$ in (2.4). It follows from (2.9) that

$$\|f(\mu x) - \mu f(x)\| \leqslant \frac{1}{\alpha}\varphi(x, 0, \cdots, 0) \tag{2.12}$$

for all $x \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Also we conclude

$$\|\frac{1}{2^m}(f(\mu 2^m x) - \mu f(2^m x))\|_{\mathcal{B}} \leqslant \frac{1}{\alpha|2|^m}\varphi(2^m x, 0, \cdots, 0)$$

for all $x \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. The right hand side of the above inequality tends to zero as $m \to \infty$, and so we obtain

$$H(\mu x) = \lim_{m\to\infty}\frac{1}{|2|^m} f(\mu 2^m x) = \lim_{m\to\infty}\frac{1}{|2|^m}\mu f(2^m x) = \mu H(x)$$

for all $x \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Hence by Lemma 2.2, the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

It follows from (2.2), (2.5), (2.6) and (2.11) that

$$\begin{aligned}\|H([x, y]) - [H(x), H(y)]\|_{\mathcal{B}} &= \lim_{m\to\infty}\frac{1}{|2|^{2m}}\|f([2^m x, 2^m y]) - [f(2^m x), f(2^m y)]\|_{\mathcal{B}} \\ &\leqslant \lim_{m\to\infty}\frac{1}{|2|^{2m}}\psi(2^m x, 2^m y) = 0\end{aligned}$$

and

$$\begin{aligned}\|H(x \circ y) - H(x) \circ H(y)\|_{\mathcal{B}} &= \lim_{m\to\infty}\frac{1}{|2|^{2m}}\|f(2^m x \circ 2^m y) - f(2^m x) \circ f(2^m y)\|_{\mathcal{B}} \\ &\leqslant \lim_{m\to\infty}\frac{1}{|2|^{2m}}\psi(2^m x, 2^m y) = 0\end{aligned}$$

for all $x, y \in \mathcal{A}$. So

$$H([x, y]) = [H(x), H(y) \quad \text{and} \quad H(x \circ y) = H(x) \circ H(y)$$

for all $x, y \in \mathcal{A}$.

Similarly, by (2.3), (2.7) and (2.11), we have

$$\|H(x^*) - H(x)^*\|_{\mathcal{B}} = \lim_{m\to\infty}\frac{1}{|2|^m}\|f(2^m x^*) - f(2^m x)^*\|_{\mathcal{B}} \leqslant \lim_{m\to\infty}\frac{1}{|2|^m}\eta(2^m x) = 0$$

and so $H(x^*) = H(x)^*$ for all $x, y \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is the desired homomorphism satisfying (2.8). □

Corollary 2.5. *Let $r > 1$ and θ be nonnegative real number, and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\begin{aligned} \|D_\mu f(x_1, x_2, \cdots, x_n)\|_{\mathcal{B}} &\leqslant \theta(\||x_1\|_{\mathcal{A}}^r + \||x_2\|_{\mathcal{A}}^r + \cdots + \||x_n\|_{\mathcal{A}}^r), \\ \|f([x, y]) - [f(x), f(y)]\|_{\mathcal{B}} &\leqslant \theta \cdot \||x\|_{\mathcal{A}}^r \cdot \||y\|_{\mathcal{A}}^r, \\ \|f(x \circ y) - f(x) \circ f(y)\|_{\mathcal{B}} &\leqslant \theta \cdot \||x\|_{\mathcal{A}}^r \cdot \||y\|_{\mathcal{A}}^r, \\ \|f(x^*) - f(x)^*\|_{\mathcal{B}} &\leqslant \theta \cdot \||x\|_{\mathcal{A}}^r, \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and $x, y, x_1, \cdots, x_n \in \mathcal{A}$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - H(x)\|_{\mathcal{B}} \leqslant \frac{|2|\theta}{|2| - |2|^r} \|x\|_{\mathcal{A}}^r$$

for all $x \in \mathcal{A}$.

Proof. The proof follows from Theorem 2.4 by taking

$$\begin{aligned} \varphi(x_1, x_2, \cdots, x_n) &:= \theta(\||x_1\|_{\mathcal{A}}^r + \||x_2\|_{\mathcal{A}}^r + \cdots + \||x_n\|_{\mathcal{A}}^r), \\ \psi(x, y) &:= \theta \cdot \||x\|_{\mathcal{A}}^r \cdot \||y\|_{\mathcal{A}}^r, \\ \eta(x) &:= \theta \cdot \||x\|_{\mathcal{A}}^r \end{aligned}$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{A}$ and $L = |2|^{r-1}$. □

3. Stability of derivations on non-Archimedean Lie JC^*-algebras

Definition 3.1. [7] *Let $\mathcal{A}$ be a non-Archimedean Lie JC^*-algebra. A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a (non-Archimedean Lie JC^*-algebra) derivation if δ satisfies*

$$\begin{aligned} \delta([x, y]) &= [\delta(x), y] + [x, \delta(y)], \\ \delta(x \circ y) &= \delta(x) \circ y + x \circ \delta(y), \\ \delta(x^*) &= \delta(x)^* \end{aligned}$$

for all $x \in \mathcal{A}$.

Throughout this section, assume that $\mathcal{A}$ is a non-Archimedean Lie JC^*-algebra with norm $\| \cdot \|_{\mathcal{A}}$.

We prove the Hyers-Ulam stability of derivation on non-Archimedean Lie JC^*-algebras for the functional equation $D_\mu f(x_1, \cdots, x_n) = 0$.

Theorem 3.2. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are function $\varphi : \mathcal{A}^n \to [0, \infty)$, $\psi : \mathcal{A}^2 \to [0, \infty)$ and $\eta : \mathcal{A} \to [0, \infty)$ such that (2.1), (2.2), (2.3). (2.4) and (2.7) hold and*

$$\|f([x, y]) - [f(x), y]0[x, f(y)]\|_{\mathcal{A}} \leqslant \psi(x, y), \tag{3.1}$$

$$\|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_{\mathcal{A}} \leqslant \psi(x, y) \tag{3.2}$$

for all $x, y \in \mathcal{A}$. *If there exists a constant* $0 < L < 1$ *such that* $\varphi(x_1, x_2, \cdots, x_n) \leqslant \alpha L \varphi(\frac{x_1}{2}, \frac{x_2}{2}, \cdots, \frac{x_n}{2})$ *for all* $x_1, x_2, \cdots, x_n \in \mathcal{A}$, *where* $\alpha = |2|^{n-1}$, *then there exists a unique derivation* $\delta : \mathcal{A} \to \mathcal{A}$ *such that*

$$\|f(x) - \delta(x)\| \leqslant \frac{L}{1-L} \varphi\Big(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\Big) \tag{3.3}$$

for all $x \in \mathcal{A}$.

Proof. By the same reasoning as in the proof of Theorem 2.4, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying in the desired inequality (3.3) and the mapping $\delta : \mathcal{A} \to \mathcal{A}$ is defined by

$$\delta(x) = \lim_{m\to\infty} \frac{1}{|2|^m} f(2^m x) \tag{3.4}$$

for all $x \in \mathcal{A}$.

It follows from (2.2), (3.1), (3.3) and (3.4) that

$$\begin{aligned} &\|\delta([x, y]) - [\delta(x), y] - [x, \delta(y)]\|_{\mathcal{A}} \\ &\qquad = \lim_{m\to\infty} \frac{1}{|2|^{2m}} \|f([2^m x, 2^m y]) - [f(2^m x), 2^m y] - [2^m x, f(2^m y)]\|_{\mathcal{A}} \\ &\qquad \leqslant \lim_{m\to\infty} \frac{1}{|2|^{2m}} \psi(2^m x, 2^m y) = 0 \end{aligned}$$

and

$$\begin{aligned} &\|\delta(x \circ y) - \delta(x) \circ y - x \circ \delta(y)\|_{\mathcal{A}} \\ &\qquad = \lim_{m\to\infty} \frac{1}{|2|^{2m}} \|f(2^m x \circ 2^m y) - f(2^m x) \circ 2^m y - 2^m x \circ f(2^m y)\|_{\mathcal{A}} \\ &\qquad \leqslant \lim_{m\to\infty} \frac{1}{|2|^{2m}} \psi(2^m x, 2^m y) = 0 \end{aligned}$$

for all $x, y \in \mathcal{A}$. So

$$\begin{aligned} \delta([x, y]) &= [\delta(x), y] + [x, \delta(y)], \\ \delta(x \circ y) &= \delta(x) \circ y + x \circ \delta(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$.

Similarly, as in the proof of Theorem 2.4, one can show $\delta(x^*) = \delta(x)^*$ for all $x \in \mathcal{A}$. Therefore, $\delta : \mathcal{A} \to \mathcal{A}$ is a non-Archimedean Lie JC^*-algebra derivation satisfying (3.4). □

Corollary 3.3. *Let* $r > 1$ *and* θ *be nonnegative and real number, and let* $f : \mathcal{A} \to \mathcal{A}$ *be a mapping such that*

$$\begin{aligned} \|D_\mu f(x_1, x_2, \cdots, x_n)\|_{\mathcal{B}} &\leqslant \theta(\|x_1\|_{\mathcal{A}}^r + \|x_2\|_{\mathcal{A}}^r + \cdots, \|x_n\|_{\mathcal{A}}^r), \\ \|f([x, y]) - [f(x), y] - [x, f(y)]\|_{\mathcal{B}} &\leqslant \theta \cdot \|x\|_{\mathcal{A}}^r \cdot \|y\|_{\mathcal{A}}^r, \\ \|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_{\mathcal{B}} &\leqslant \theta \cdot \|x\|_{\mathcal{A}}^r \cdot \|y\|_{\mathcal{A}}^r, \\ \|f(x^*) - f(x)^*\|_{\mathcal{B}} &\leqslant \theta \cdot \|x\|_{\mathcal{A}}^r \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ *and* $x, y, x_1, \cdots, x_n \in \mathcal{A}$*. Then there exists a unique homomorphism* $H : \mathcal{A} \to \mathcal{A}$ *such that*

$$\|f(x) - \delta(x)\|_{\mathcal{B}} \leqslant \frac{|2|\theta}{|2| - |2|^r}\|x\|_{\mathcal{A}}^r$$

for all $x \in \mathcal{A}$*.*

Proof. The proof follows from Theorem 3.2 by taking

$$\begin{aligned}\varphi(x_1, x_2, \cdots, x_n) &:= \theta.(\|x_1\|_{\mathcal{A}}^r + \|x_2\|_{\mathcal{A}}^r + \cdots + \|x_n\|_{\mathcal{A}}^r),\\ \psi(x, y) &:= \theta.(\|x\|_{\mathcal{A}}^r.\|y\|_{\mathcal{A}}^r),\\ \eta(x) &:= \theta.\|x\|_{\mathcal{A}}^r\end{aligned}$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{A}$ and $L = |2|^{r-1}$. □

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ON DISTRIBUTION AND PROBABILITY DENSITY FUNCTIONS OF ORDER STATISTICS ARISING FROM INDEPENDENT BUT NOT NECESSARILY IDENTICALLY DISTRIBUTED RANDOM VECTORS

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ABSTRACT

In this study, joint probability density and distribution functions of any ***d*** order statistics of *innid* continuous random vectors are expressed. Then, some results connecting distributions of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors are given.

Keywords: Order Statistics, Distribution Function, Probability Density Function, Continuous Random Variable.

MSC 2010: 62G30, 62E15.

1. Introduction

Several identities and recurrence relations for probability density function (*pdf*) and distribution function (*df*) of order statistics of independent and identically distributed (*iid*) random variables were established by numerous authors including (Arnold et al., 1992; Balasubramanian, Beg, 2003; David, 1981; Reiss, 1989). Furthermore, (Arnold et al., 1992; David, 1981; Gan, Bain, 1995; Khatri, 1962) obtained the probability function (*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. (Corley, 1984) defined a multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution. (Goldie, Maller, 1999) derived expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df*. (Guilbaud, 1982) expressed the probability of the functions of independent but not necessarily identically distributed (*innid*) random vectors as a linear combination of probabilities of the functions of *iid* random vectors and thus also for order statistics of random variables.

(Cao, West, 1997) obtained recurrence relationships among the distribution functions of order statistics arising from *innid* random variables. (Vaughan, Venables, 1972) derived the joint *pdf* and marginal *pdf* of order statistics of *innid* random variables by means of permanents. (Balakrishnan, 2007; Bapat, Beg, 1989) obtained the joint *pdf* and *df* of order statistics of *innid* random variables by means of permanents. (Childs, Balakrishnan, 2006) obtained, using multinomial arguments, the *pdf* of $X_{r:n+1}$ $(1 \le r \le n+1)$ by adding another independent random variable to the original n variables $X_1, X_2, ..., X_n$. Also,

(Balasubramanian et al.,1994) established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators.

In this paper, joint *df* and *pdf* of order statistics from *innid* continuous random vectors are obtained.

As far as we know, these approaches have not been considered in the framework of order statistics from *innid* continuous random vectors.

From now on, subscripts and superscripts are defined in first place in which they are used and these definitions will be valid unless they are redefined.

Consider $\mathrm{x} = (x^{(1)}, x^{(2)}, ..., x^{(b)})$ and $\mathrm{y} = (y^{(1)}, y^{(2)}, ..., y^{(b)})$, then it can be written as;

$\mathrm{x} \le \mathrm{y}$ if $x^{(v)} \le y^{(v)}$ (v=1, 2, …, b) and $\mathrm{x} + \mathrm{y} = (x^{(1)} + y^{(1)}, x^{(2)} + y^{(2)}, ..., x^{(b)} + y^{(b)})$.

Let $\xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, ..., \xi_i^{(b)})$ (i=1,2,…,n) be n *innid* continuous random vectors which components of ξ_i are independent.

$$X_{r:n}^{(v)} = Z_{r:n}(\xi_1^{(v)}, \xi_2^{(v)}, ..., \xi_n^{(v)}) \tag{1.1}$$

is stated as rth order statistic of vth components of $\xi_1, \xi_2, \ldots, \xi_n$.

From (1.1), ordered values of vth components of $\xi_1, \xi_2, \ldots, \xi_n$ are expressed as

$$X_{1:n}^{(v)} \le X_{2:n}^{(v)} \le ... \le X_{n:n}^{(v)}. \tag{1.2}$$

From (1.2), we can write $\mathrm{X}_{r:n} = (X_{r:n}^{(1)}, X_{r:n}^{(2)}, ..., X_{r:n}^{(b)})$ $(1 \le r \le n)$.

Also, $\mathrm{x}_w = (x_w^{(1)}, x_w^{(2)}, \ldots, x_w^{(b)})$, $x_w^{(v)} \in R$ ($w = 1,2,...,$d; $d = 1,2,...,$n).

Let F_i and f_i be *df* and *pdf* of $\xi_i^{(v)}$, respectively.

Moreover, $X_{1:n}^{(v),s}, X_{2:n}^{(v),s}, ..., X_{n:n}^{(v),s}$ are order statistics of *iid* continuous random variables with *df* F^s and *pdf* f^s, respectively, defined by

$$F^s = \frac{1}{n_s} \sum_{i \in s} F_i \tag{1.3}$$

and

$$f^s = \frac{1}{n_s} \sum_{i \in s} f_i \,. \tag{1.4}$$

Here, s is a subset of integers {1, 2,…, n} with $n_s \ge 1$ elements.

In follows, *df* and *pdf* of $\mathrm{X}_{r_1:n},\mathrm{X}_{r_2:n},...,\mathrm{X}_{r_d:n}$ ($1\le r_1<r_2<...<r_d\le n$) are given. Let $\mathrm{X}^{(v)}=(X^{(v)}_{r_1:n},X^{(v)}_{r_2:n},...,X^{(v)}_{r_d:n})$ and $\mathrm{x}^{(v)}=(x^{(v)}_1,x^{(v)}_2,...,x^{(v)}_d)$. For notational convenience we write $\sum\sum$ and $\sum\limits_{m_d,...,m_2,m_1}^{n,...,m_3,m_2}$ instead of $\sum\limits_{\kappa=1}^{n}(-1)^{n-\kappa}\dfrac{\kappa^n}{n!}\sum\limits_{n_s=\kappa}$ and $\sum\limits_{m_d=r_d}^{n}...\sum\limits_{m_2=r_2}^{m_3}\sum\limits_{m_1=r_1}^{m_2}$ in the expressions below, respectively.

2. Distribution function of order statistics from *innid* random vectors

In this section, *df* of $\mathrm{X}_{r_1:n},\mathrm{X}_{r_2:n},...,\mathrm{X}_{r_d:n}$ and its results are given. The results connect *df* of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors using (1.3).

Now, we give the following theorem for establish joint *df* of *d* order statistics of *innid* continuous random vectors.

Theorem 2.1.

$$F_{r_1,r_2,...,r_d:n}(\mathrm{x}_1,\mathrm{x}_2,...,\mathrm{x}_d)=\prod_{v=1}^{b}\{\sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2}C\sum_P\prod_{w=1}^{d+1}\prod_{l=m_{w-1}+1}^{m_w}[F_{j_l}(x^{(v)}_w)-F_{j_l}(x^{(v)}_{w-1})]\},\tag{2.1}$$

$\mathrm{x}_1<\mathrm{x}_2<...<\mathrm{x}_d$, where $C=[\prod\limits_{w=1}^{d+1}(m_w-m_{w-1})!]^{-1}$, $m_0=0$, $m_{d+1}=n$, $\sum\limits_P$ denotes sum over all $n!$ permutations $(j_1,j_2,...,j_n)$ of (1,2,…,n), $F_{j_l}(x^{(v)}_0)=0$ and $F_{j_l}(x^{(v)}_{d+1})=1$.

Proof. It can be written

$$\begin{aligned}F_{r_1,r_2,...,r_d:n}(\mathrm{x}_1,\mathrm{x}_2,...,\mathrm{x}_d)&=P\{\mathrm{X}_{r_1:n}\le\mathrm{x}_1,\mathrm{X}_{r_2:n}\le\mathrm{x}_2,...,\mathrm{X}_{r_d:n}\le\mathrm{x}_d\}\\&=P\{\mathrm{X}^{(1)}\le\mathrm{x}^{(1)},\mathrm{X}^{(2)}\le\mathrm{x}^{(2)},...,\mathrm{X}^{(b)}\le\mathrm{x}^{(b)}\}\\&=\prod_{v=1}^{b}P\{\mathrm{X}^{(v)}\le\mathrm{x}^{(v)}\}\\&=\prod_{v=1}^{b}P\{X^{(v)}_{r_1:n}\le x^{(v)}_1,X^{(v)}_{r_2:n}\le x^{(v)}_2,...,X^{(v)}_{r_d:n}\le x^{(v)}_d\}\ .\end{aligned}\tag{2.2}$$

(2.2) can be expressed as

$$F_{r_1,r_2,...,r_d:n}(\mathrm{x}_1,\mathrm{x}_2,...,\mathrm{x}_d)=$$
$$\prod_{v=1}^{b}\{\sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2}C\sum_P\left(\prod_{l=1}^{m_1}F_{j_l}(x^{(v)}_1)\right)\left(\prod_{l=m_1+1}^{m_2}[F_{j_l}(x^{(v)}_2)-F_{j_l}(x^{(v)}_1)]\right)...\prod_{l=m_d+1}^{n}[1-F_{j_l}(x^{(v)}_d)]\}.$$

Thus, (2.1) is obtained.

The approach in Theorem 2.1 can also be adapted to Theorem 2.2 for *iid* case.

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Theorem 2.2.

$$F_{r_1,r_2,\ldots,r_d:n}(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_d)=\prod_{v=1}^{b}\{\sum\sum\sum_{m_d,\ldots,m_2,m_1}^{n,\ldots,m_3,m_2} n!C\prod_{w=1}^{d+1}[F^s(x_w^{(v)})-F^s(x_{w-1}^{(v)})]^{m_w-m_{w-1}}\}. \quad (2.3)$$

Proof. (2.2) can be expressed as

$$F_{r_1,r_2,\ldots,r_d:n}(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_d)=\prod_{v=1}^{b}[\sum\sum P\{X_{r_1:n}^{(v),s}\le x_1^{(v)},X_{r_2:n}^{(v),s}\le x_2^{(v)},\ldots,X_{r_d:n}^{(v),s}\le x_d^{(v)}\}]. \quad (2.4)$$

(2.3) is obtained from (2.1) and (2.4).

We now obtain the following three results for *df* of order statistics of *innid* continuous random vectors from the above theorems.

Result 2.1.

$$F_{r_1:n}(x_1^{(1)})=\sum_{m_1=r_1}^{n}\frac{1}{m_1!(n-m_1)!}\sum_P\left(\prod_{l=1}^{m_1}\left(F_{j_l}(x_1^{(1)})\right)\right)\prod_{l=m_1+1}^{n}[1-F_{j_l}(x_1^{(1)})]$$

$$=\sum\sum\sum_{m_1=r_1}^{n}\binom{n}{m_1}[F^s(x_1^{(1)})]^{m_1}[1-F^s(x_1^{(1)})]^{n-m_1}. \quad (2.5)$$

Proof. In (2.1) and (2.3), if $b=1$, $d=1$, (2.5) is obtained.

In addition,

$$F_{r_1:n}(x_1^{(1)})=\sum_{m_1=r_1}^{n}\frac{1}{m_1!(n-m_1)!}\sum_P\left(\prod_{l=1}^{m_1}F_{j_l}(x_1^{(1)})\right)\prod_{l=m_1+1}^{n}[1-F_{j_l}(x_1^{(1)})]$$

$$=\sum_{m_1=r_1}^{n}\frac{1}{m_1!(n-m_1)!}\sum_P\left(\prod_{l=1}^{m_1}F_{j_l}(x_1^{(1)})\right)\sum_{t=m_1}^{n}(-1)^{n-t}\sum_{n_\tau=n-t}\prod_{l=1}^{n-t}F_{\tau_l}(x_1^{(1)}),$$

where $\sum_{n_\tau=n-t}$ denotes sum over all $\binom{n-m_1}{n-t}$ subsets $\tau=\{\tau_1,\tau_2,\ldots,\tau_{n-t}\}$ of $\{j_{m_1+1},j_{m_1+2},\ldots,j_n\}$.

Result 2.2.

$$F_{1:n}(x_1^{(1)})=1-\frac{1}{n!}\sum_P\prod_{l=1}^{n}[1-F_{j_l}(x_1^{(1)})]$$

$$=\sum\sum[1-(1-F^s(x_1^{(1)}))^n]. \quad (2.6)$$

Proof. In (2.5), if $r_1=1$, (2.6) is obtained.

Result 2.3.

$$F_{n:n}(x_1^{(1)}) = \frac{1}{n!}\sum_P \prod_{l=1}^{n} F_{j_l}(x_1^{(1)})$$

$$= \sum\sum [F^s(x_1^{(1)})]^n . \tag{2.7}$$

Proof. In (2.5), if $r_1 = n$, (2.7) is obtained.

3. Probability density function of order statistics from *innid* random vectors

In this section, *pdf* of $\mathrm{X}_{r_1:n}, \mathrm{X}_{r_2:n}, ..., \mathrm{X}_{r_d:n}$ and its results are given. The results connect *pdf* of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors using (1.3) and (1.4).

Joint *pdf* of *d* order statistics of *innid* continuous random vectors is expressed in the following theorem.

Theorem 3.1.

$$f_{r_1,r_2,...,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) = \prod_{v=1}^{b}\{D\sum_P \left(\prod_{w=1}^{d+1}\prod_{l=r_{w-1}+1}^{r_w-1}[F_{j_l}(x_w^{(v)}) - F_{j_l}(x_{w-1}^{(v)})]\right)\prod_{w=1}^{d} f_{j_{r_w}}(x_w^{(v)})\}, \tag{3.1}$$

$\mathrm{x}_1 < \mathrm{x}_2 < ... < \mathrm{x}_d$, where $D = [\prod_{w=1}^{d+1}(r_w - r_{w-1} - 1)!]^{-1}$, $r_0 = 0$ and $r_{d+1} = n+1$.

Proof. Let $\delta\mathrm{x}_w = (\delta x_w^{(1)}, \delta x_w^{(2)}, ..., \delta x_w^{(b)})$ and $\delta\,\mathrm{x}^{(v)} = (\delta x_1^{(v)}, \delta x_2^{(v)}, ..., \delta x_d^{(v)})$.

Consider

$\{\mathrm{x}_1 < \mathrm{X}_{r_1:n} \le \mathrm{x}_1 + \delta\mathrm{x}_1, \mathrm{x}_2 < \mathrm{X}_{r_2:n} \le \mathrm{x}_2 + \delta\mathrm{x}_2, ..., \mathrm{x}_d < \mathrm{X}_{r_d:n} \le \mathrm{x}_d + \delta\mathrm{x}_d\}$.

It can be written

$$P\{\mathrm{x}_1 < \mathrm{X}_{r_1:n} \le \mathrm{x}_1 + \delta\,\mathrm{x}_1, \mathrm{x}_2 < \mathrm{X}_{r_2:n} \le \mathrm{x}_2 + \delta\,\mathrm{x}_2, ..., \mathrm{x}_d < \mathrm{X}_{r_d:n} \le \mathrm{x}_d + \delta\,\mathrm{x}_d\}$$

$$= P\{\mathrm{x}^{(1)} < \mathrm{X}^{(1)} \le \mathrm{x}^{(1)} + \delta\,\mathrm{x}^{(1)}, \mathrm{x}^{(2)} < \mathrm{X}^{(2)} \le \mathrm{x}^{(2)} + \delta\,\mathrm{x}^{(2)}, ..., \mathrm{x}^{(b)} < \mathrm{X}^{(b)} \le \mathrm{x}^{(b)} + \delta\,\mathrm{x}^{(b)}\}$$

$$= \prod_{v=1}^{b} P\{\mathrm{x}^{(v)} < \mathrm{X}^{(v)} \le \mathrm{x}^{(v)} + \delta\,\mathrm{x}^{(v)}\}$$

$$= \prod_{v=1}^{b} P\{x_1^{(v)} < X_{r_1:n}^{(v)} \le x_1^{(v)} + \delta\, x_1^{(v)}, x_2^{(v)} < X_{r_2:n}^{(v)} \le x_2^{(v)} + \delta x_2^{(v)}, ..., x_d^{(v)} < X_{r_d:n}^{(v)} \le x_d^{(v)} + \delta x_d^{(v)}\}. \tag{3.2}$$

Dividing (3.2) by $\prod_{v=1}^{b}\prod_{w=1}^{d}\delta x_w^{(v)}$ and then letting $\delta x_1^{(v)}, \delta x_2^{(v)}, ..., \delta x_d^{(v)}$ tend to zero, we obtain

$$f_{r_1,r_2,...,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) = \prod_{v=1}^{b}\{D\sum_P F_{j_1}(x_1^{(v)})...F_{j_{r_1-1}}(x_1^{(v)}) f_{j_{r_1}}(x_1^{(v)})[F_{j_{r_1+1}}(x_2^{(v)}) - F_{j_{r_1+1}}(x_1^{(v)})]$$

$$...[F_{j_{r_2}-1}(x_2^{(v)}) - F_{j_{r_2}-1}(x_1^{(v)})] f_{j_{r_2}}(x_2^{(v)})... f_{j_{r_d}}(x_d^{(v)})[1 - F_{j_{r_d+1}}(x_d^{(v)}]...[1 - F_{j_n}(x_d^{(v)})]\}. \quad (3.3)$$

From (3.3), we can write

$$f_{r_1,r_2,...,r_d:n}(\mathrm{x}_1, \mathrm{x}_2,..., \mathrm{x}_d) = \prod_{v=1}^{b}\{D\sum_{p}\left(\prod_{l=1}^{r_1-1}[F_{j_l}(x_1^{(v)})]\right) f_{j_{r_1}}(x_1^{(v)})$$

$$\cdot\left(\prod_{l=r_1+1}^{r_2-1}[F_{j_l}(x_2^{(v)}) - F_{j_l}(x_1^{(v)})]\right) f_{j_{r_2}}(x_2^{(v)})... f_{j_{r_d}}(x_d^{(v)}) \prod_{l=r_d+1}^{n}[1 - F_{j_l}(x_d^{(v)})]\}. \quad (3.4)$$

Thus, (3.1) is obtained.

Next theorem shows that *pdf* of d order statistics of *innid* continuous random vectors can be expressed in terms of *pdf* of d order statistics of *iid* continuous random vectors.

Theorem 3.2.

$$f_{r_1,r_2,...,r_d:n}(\mathrm{x}_1, \mathrm{x}_2,..., \mathrm{x}_d) = \prod_{v=1}^{b}\{\sum\sum n!D\left(\prod_{w=1}^{d+1}[F^s(x_w^{(v)}) - F^s(x_{w-1}^{(v)})]^{r_w - r_{w-1}-1}\right)\prod_{w=1}^{d} f^s(x_w^{(v)})\}. \quad (3.5)$$

Proof. (3.2) can be expressed as

$$\prod_{v=1}^{b}[\sum\sum P\{x_1^{(v)} < X_{r_1:n}^{(v),s} \le x_1^{(v)} + \delta x_1^{(v)}, x_2^{(v)} < X_{r_2:n}^{(v),s} \le x_2^{(v)} + \delta x_2^{(v)},..., x_d^{(v)} < X_{r_d:n}^{(v),s} \le x_d^{(v)} + \delta x_d^{(v)}\}]. \quad (3.6)$$

Dividing (3.6) by $\prod_{v=1}^{b}\prod_{w=1}^{d}\delta x_w^{(v)}$ and then letting $\delta x_1^{(v)}, \delta x_2^{(v)},..., \delta x_d^{(v)}$ tend to zero, (3.5) is obtained.

The following five results of which first three are belong to *pdf* of single order statistic and last two are belong to joint *pdf* of d order statistics of *innid* continuous random vectors can be written from last two theorems.

Result 3.1.

$$f_{r_1:n}(x_1^{(1)}) = \frac{1}{(r_1-1)!(n-r_1)!}\sum_{P}\left(\prod_{l=1}^{r_1-1}F_{j_l}(x_1^{(1)})\right)\left(\prod_{l=r_1+1}^{n}[1 - F_{j_l}(x_1^{(1)})]\right) f_{j_{r_1}}(x_1^{(1)})$$

$$= \sum\sum r_1\binom{n}{r_1}[F^s(x_1^{(1)})]^{r_1-1}[1 - F^s(x_1^{(1)})]^{n-r_1} f^s(x_1^{(1)}). \quad (3.7)$$

Proof. In (3.1) and (3.5), if $b = 1$, $d = 1$, (3.7) is obtained.

Result 3.2.

$$f_{1:n}(x_1^{(1)}) = \frac{1}{(n-1)!}\sum_{P}\left(\prod_{l=2}^{n}[1 - F_{j_l}(x_1^{(1)})]\right) f_{j_1}(x_1^{(1)})$$

$$= \sum\sum n[1 - F^s(x_1^{(1)})]^{n-1} f^s(x_1^{(1)}). \quad (3.8)$$

Proof. In (3.7), if $r_1 = 1$, (3.8) is obtained.

Result 3.3.

$$f_{n:n}(x_1^{(1)}) = \frac{1}{(n-1)!}\sum_P \left(\prod_{l=1}^{n-1} F_{j_l}(x_1^{(1)})\right) f_{j_n}(x_1^{(1)})$$

$$= \sum\sum n[F^s(x_1^{(1)})]^{n-1} f^s(x_1^{(1)}). \quad (3.9)$$

Proof. In (3.7), if $r_1 = n$, (3.9) is obtained.

Result 3.4.

$$f_{1,n:n}(x_1^{(1)}, x_2^{(1)}) = \frac{1}{(n-2)!}\sum_P \left(\prod_{l=2}^{n-1} [F_{j_l}(x_2^{(1)}) - F_{j_l}(x_1^{(1)})]\right) f_{j_1}(x_1^{(1)}) f_{j_n}(x_2^{(1)})$$

$$= \sum\sum n(n-1)[F^s(x_2^{(1)}) - F^s(x_1^{(1)})]^{n-2} f^s(x_1^{(1)}) f^s(x_2^{(1)}). \quad (3.10)$$

Proof. In (3.1) and (3.5), if $b=1$, $d=2$ and $r_1 = 1$, $r_2 = n$, (3.10) is obtained.

Result 3.5.

$$f_{1,2,\dots,k:n}(\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_\mathrm{k}) = \prod_{v=1}^{b}\{\frac{1}{(n-k)!}\sum_P \left(\prod_{l=k+1}^{n} [1 - F_{j_l}(x_k^{(v)})]\right) f_{j_1}(x_1^{(v)}) f_{j_2}(x_2^{(v)}) \dots f_{j_k}(x_k^{(v)})\}$$

$$= \prod_{v=1}^{b}\{\sum\sum \frac{n!}{(n-k)!}[1 - F^s(x_k^{(v)})]^{n-k} f^s({x_1}^{(v)}) f^s({x_2}^{(v)}) \dots f^s({x_k}^{(v)})\}. \quad (3.11)$$

Proof. In (3.1) and (3.5), if $d = k$ and $r_1 = 1$, $r_2 = 2, \dots, r_k = k$, (3.11) is obtained.

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Stability of homomorphisms and derivations in non-Archimedean random C^*-algebras via fixed point method

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Abstract. In this paper, using the fixed point method, we investigate the Hyers-Ulam stability of homomorphisms in non-Archimedean random C^*-algebras and non-Archimedean random Lie JC^*-algebras and of derivations on non-Archimedean random C^*-algebras and non-Archimedean random Lie JC^*-algebras related to the generalized Cauchy-Jensen additive functional equation.

1. **Introduction**

A non-Archimedean field is a field like $\mathcal{K}$ equipped is a function $|\cdot| : \mathcal{K} \to [0, +\infty)$ such that $|a| = 0$ if and only if $a = 0$, $|ab| = |a||b|$ and $|a + b| \leqslant \max\{|a|, |b|\}$ for all $a, b \in \mathcal{K}$. Note that $|1| = |-1| = 1$ and $|n| \leqslant 1$ for each integer n. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. We always assume, in addition, that $|\cdot|$ is non-trivial, i.e., there exists an $a_0 \neq 0, 1$. A function $\|.\| : X \to [0, \infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:
(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) for any $r \in K, x \in X, \|rx\| = |r|\|x\|$;
(iii) the strong triangle inequality holds; namely,

$$\|x + y\| \leqslant \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|.\|)$ is called a non-Archimedean normed space. From the fact that

$$\|x_n - x_m\| \leqslant \max\{\|n_n - x_m\| : m \leqslant j \leqslant n - 1\} \quad (n > m)$$

holds, a sequence $\{x_n\}$ is Cauchy if and only if $\{x_n - x_m\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.

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A non-Archimedean Banach algebra is a complete non-Archimedean algebra $\mathcal{A}$ which satisfies $\|ab\| \leqslant \|a\| \cdot \|b\|$ for all $a, b \in \mathcal{A}$. For more detailed definitions of non-Archimedean Banach algebras, we refer the reader to [25].

If $\mathcal{U}$ is a non-Archimedean Banach algebra, then an involution on $\mathcal{U}$ is mapping $t \to t^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

(i) $t^{**} = t$ for $t \in \mathcal{U}$;

(ii) $(\alpha s + \beta t)^* = \bar{\alpha} s^* + \bar{\beta} t^*$;

(iii) $(st)^* = t^* s^*$ for all $s, t \in \mathcal{U}$.

If, in addition, $\|t^* t\| = \|t\|^2$ for $t \in \mathcal{U}$, then $\mathcal{U}$ is a non-Archimedean C^*-algebra.

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond)$ be a metric group (a metric is defined on a set with group property) with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we we would say that the equation of homomorphism $h(x * y) = h(x) * h(y)$ is stable (see also [10, 11, 14, 18, 19, 20, 21, 22]).

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

For explicitly later use, we recall a fundamental result in fixed point theory.

Theorem 1.1. [9] *Let (Ω, d) be a complete generalized metric space and $J : \Omega \to \Omega$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given $x \in \Omega$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative n or there exists a positive integer n_0 such that*

(1) *$d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geqslant n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $\Delta = \{y \in \Omega : d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ for all $y \in \Delta$.*

A C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy - yx}{2}$ and endowed with *anticommutator product* (Jordan product) $x \circ y := \frac{xy + yx}{2}$ on $\mathcal{C}$, is called a Lie JC^*-algebra (see [15, 16, 17]).

Jordan algebras as coordinates for Lie algebras were created to illuminate a particular aspect of physics, quantum-mechanical observables, but turned out to have illuminating connections with many areas of mathematics.

In this paper, using the fixed point method, we prove the Hyers-Ulam stability of homomorphisms and derivations in non-Archimedean random C^*-algebras and non-Archimedean random Lie JC^*-algebras associated with $f : X \to Y$ satisfying the following functional equation (see [1])

$$\sum_{\substack{1 \leqslant i_1 < \cdots < i_m \leqslant n \\ 1 \leqslant k_l (\neq i_j, \forall j \in \{1, \cdots, m\}) \leqslant n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) = \frac{(n-m+1)}{n} \binom{n}{m} \sum_{i=1}^n f(x_i) \quad (1.1)$$

for all $x_1, \cdots, x_n \in X$, where $m, n \in \mathbb{N}$ are fixed integer with $n \geqslant 2$, $1 \leqslant m \leqslant n$. In particular, it is shown that in the case $m = 1$, (1.1) yields the Cauchy additive equation $f(\sum_{l=1}^{n} x_{k_l}) = \sum_{l=1}^{n} f(x_i)$ and also in the case $m = n$, (1.1) yields the Jensen additive equation $f(\frac{\sum_{j=1}^{n} x_j}{n}) = \frac{1}{n}\sum_{l=1}^{n} f(x_i)$. Then (1.1) is a generalized form of the Cauchy-Jensen additive equation, and thus every solution of the equation (1.1) may be analogously called general (m, n)-Cauchy-Jensen additive. For each m with $1 \leqslant m \leqslant n$, a mapping $f : X \to Y$ satisfies (1.1) for all $n \geqslant 2$ if and only if $f(x) - f(0) = A(x)$ is Cauchy additive, where $f(0) = 0$ if $m < n$. In particular, we have $f((n - m + 1)x) = (n - m + 1)f(x)$ and $f(mx) = mf(x)$ for all $x \in X$.

2. **Random spaces**

In this section, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces as in [2, 3, 6, 7, 8]. Throughout this paper, Δ^+ is the space of distribution functions, that is the space of all mapping $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. And D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leqslant G$ if and only if $F(t) \leqslant G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \leqslant 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Definition 2.1. [23] *A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm norm (briefly, a continuous t-norm) if T satisfies the following conditions:*
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$;
(d) $T(a, b) \leqslant T(c, d)$ whenever $a \leqslant c$ and $b \leqslant d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t-norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz t-norm).

Definition 2.2. [24] *A non-Archimedean random normed space (briefly, NA-RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:*
$(RN1)$ $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
$(RN2)$ $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X, \alpha \neq 0$.
$(RN3)$ $\mu_{x+y}(t) \geqslant T(\mu_x(t), \mu_y(t))$ for all $x, y \in X$ and all $t \geqslant 0$.

Every normed space $(X, \|\cdot\|)$ defines a non-Archimedean random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 2.3. [12] *A non-Archimedean random normed algebra* (X, μ, T, T') *is a non-Archimedean random normed space* (X, μ, T) *with an algebraic structure such that*

$(RN4)$ $\mu_{xy}(t) \geqslant T'(\mu_x(t), \mu_y(t))$ *for all* $x, y \in X$ *and all* $t > 0$*, in which* T' *is a continuous* t*-norm.*

Every non-Archimedean normed algebra $(X, \|\cdot\|)$ defines a non-Archimedean random normed algebra (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ if and only if

$$\|xy\| \leqslant \|x\| \, \|y\| + t\|x\| + t\|y\| \;\; (x, y \in X; t > 0).$$

This space is called an induced non-Archimedean random normed algebra.

Definition 2.4. *Let* (X, μ, T_M) *and* (Y, μ, T_M) *be non-Archimedean random normed algebras.*

(1) *An* $\mathbb{R}$*-linear mapping* $f : X \to Y$ *is called a* homomorphism *if* $f(xy) = f(x)f(y)$ *for all* $x, y \in X$*.*
(2) *An* $\mathbb{R}$*-linear mapping* $f : X \to Y$ *is called a* derivation *if* $f(xy) = f(x)y + xf(y)$ *for all* $x, y \in X$*.*

Definition 2.5. *Let* $(\mathcal{U}, \mu, T)$ *be a non-Archimedean random Banach algebra. Then an involution on* $\mathcal{U}$ *is mapping* $u \to u^*$ *from* $\mathcal{U}$ *into* $\mathcal{U}$ *which satisfies*

(i) $u^{**} = u$ *for* $u \in \mathcal{U}$*;*
(ii) $(\alpha u + \beta v)^* = \bar{\alpha} u^* + \bar{\beta} v^*$*;*
(iii) $(uv)^* = v^* u^*$ *for all* $u, v \in \mathcal{U}$*.*

If, in addition, $\mu_{u^*u}(t) = T'(\mu_u(t), \mu_u(t))$ *for* $u \in \mathcal{U}$*, then* $\mathcal{U}$ *is a non-Archimedean random* C^**-algebra.*

Definition 2.6. *Let* (X, μ, T) *be an* NA*-*RN*-space.*

(1) *A sequence* $\{x_n\}$ *in* X *is said be* convergent *to* x *in* X *if, for every* $\epsilon > 0$ *and* $\lambda > 0$*, there exists a positive integer* N *such that* $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ *whenever* $n \geqslant N$*.*
(2) *A sequence* $\{x_n\}$ *in* X *is called a* Cauchy sequence *if, for every* $\epsilon > 0$ *and* $\lambda > 0$*, there exists a positive integer* N *such that* $\mu_{x_n - x_{n+1}}(\epsilon) > 1 - \lambda$ *whenever* $n \geqslant m \geqslant N$*.*
(3) *An* RN*-space* (X, μ, T) *is said to be* complete *if and only if every Cauchy sequence in* X *is convergent to a point in* X*.*

3. Stability of homomorphisms and derivations in non-Archimedean random C^*-algebras

Throughout this section, we suppose that $\mathcal{A}$ and $\mathcal{B}$ are non-Archimedean random C^*-algebras, respectively, with norms $\mu^{\mathcal{A}}_{\cdot}$ and $\mu^{\mathcal{B}}_{\cdot}$.

We use the following abbreviation for a given mapping $f : \mathcal{A} \to \mathcal{B}$:

$$D_\lambda f(x_1, \cdots, x_n) := \sum_{\substack{1 \leqslant i_1 < \cdots < i_m \leqslant n \\ 1 \leqslant k_l (\neq i_j, \forall j \in \{1, \cdots, m\}) \leqslant n}} f\left(\frac{\sum_{j=1}^m \lambda x_{i_j}}{m} + \sum_{l=1}^{n-m} \lambda x_{k_l}\right) - \frac{(n-m+1)\binom{n}{m} \sum_{i=1}^n \lambda f(x_i)}{n}$$

for all $\lambda \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$ and all $x_1, \cdots, x_n \in \mathcal{A}$.

It is well-known that a $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a *random homomorphism* in non-Archimedean random C^*-algebras if H satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in \mathcal{A}$.

We prove the Hyers-Ulam stability oh homomorphisms in non-Archimedean random C^*-algebras for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 3.1. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+, \psi : \mathcal{A}^2 \to D^+$, and $\eta : \mathcal{A} \to D^+$ such that $|\mathcal{M}| = |n - m + 1| < 1$ and $|\mathcal{N}| = |(n - m + 1)\binom{n}{m}| < 1$ are far from zero and*

$$\mu^{\mathcal{B}}_{D_\lambda f(x_1, \cdots, x_n)}(t) \geqslant \varphi_{x_1, \cdots, x_n}(t), \tag{3.1}$$

$$\mu^{\mathcal{B}}_{f(xy) - f(x)f(y)}(t) \geqslant \psi_{x,y}(t), \tag{3.2}$$

$$\mu^{\mathcal{B}}_{f(x^*) - f(x)^*}(t) \geqslant \eta_x(t), \tag{3.3}$$

for all $\lambda \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$ and all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ such that

$$\varphi_{\mathcal{M}x_1, \cdots, \mathcal{M}x_n}(|\mathcal{M}|Lt) \geqslant \varphi_{x_1, \cdots, x_n}(t), \tag{3.4}$$

$$\psi_{\mathcal{M}x, \mathcal{M}y}(|\mathcal{M}|^2 Lt) \geqslant \psi_{x,y}(t), \tag{3.5}$$

$$\eta_{\mathcal{M}x}(|\mathcal{M}|Lt) \geqslant \eta_x(t), \tag{3.6}$$

for all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$, then there exists a unique random homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x) - H(x)}(t) \geqslant \varphi_{x, \cdots, x}((|\mathcal{N}| - |\mathcal{N}|L)t) \tag{3.7}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof. It follows from (3.4), (3.5), (3.6), and $L < 1$ that

$$\lim_{m \to \infty} \varphi_{\mathcal{M}^m x_1, \cdots, \mathcal{M}^m x_n}(|\mathcal{M}|^m t) = 1, \tag{3.8}$$

$$\lim_{m \to \infty} \psi_{\mathcal{M}^m x, \mathcal{M}^m y}(|\mathcal{M}|^{2m} t) = 1, \tag{3.9}$$

$$\lim_{m \to \infty} \eta_{\mathcal{M}^m x}(|\mathcal{M}|^m t) = 1, \tag{3.10}$$

for all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$.

Now we define $\Omega := \{g : \mathcal{A} \to \mathcal{B}; g(0) = 0\}$ and introduce a generalized metric on Ω as following:

$$d(g, h) = \inf\{k \in (0, \infty) : \mu^{\mathcal{B}}_{g(x) - h(x)}(kt) > \varphi_{x, x, \cdots, x}(t), \forall x \in \mathcal{A},\ t > 0\}$$

where $\inf \emptyset = +\infty$. By the same technique as in the proof of [13, Theorem 3.2], we can show that (Ω, d) is a complete generalized metric space. We define $J : \Omega \to \Omega$ by $Jg(x) = \frac{1}{\mathcal{M}} g(\mathcal{M}x)$ for all $x \in \mathcal{A}$ and $g \in \Omega$. Note that for all $g, h \in \Omega$, from (3.4), we have

$$\begin{aligned}
d(g,h) \leqslant k &\Rightarrow \mu^{\mathcal{B}}_{g(x)-h(x)}(kt) > \varphi_{x,\cdots,x}(t) \\
&\Rightarrow \mu^{\mathcal{B}}_{\frac{1}{\mathcal{M}}g(\mathcal{M}x)-\frac{1}{\mathcal{M}}h(\mathcal{M}x)}(kt) > \varphi_{\mathcal{M}x,\cdots,\mathcal{M}x}(|\mathcal{M}|t) \\
&\Rightarrow \mu^{\mathcal{B}}_{\frac{1}{\mathcal{M}}g(\mathcal{M}x)-\frac{1}{\mathcal{M}}h(\mathcal{M}x)}(kLt) > \varphi_{x,\cdots,x}(t) \\
&\Rightarrow d(Jg, Jh) < kL.
\end{aligned}$$

Then one can show that $d(Jg, Jh) \leqslant Ld(g,h)$ for all $g, h \in \Omega$ and so J is self-function of Ω with the the Lipschitz constant L.

Letting $\lambda = 1$ and putting $x_1 = x_2 = \cdots = x_n = x$ in (3.1), we obtain

$$\mu^{\mathcal{B}}_{\binom{n}{m} f((n-m+1)x) - \binom{n}{m}(n-m+1) f(x)}(t) \geqslant \varphi_{x,x,\cdots,x}(t)$$

for all $x \in \mathcal{A}$ and $t > 0$. Then

$$\mu^{\mathcal{B}}_{f(x)-\frac{1}{\mathcal{M}}f(\mathcal{M}x)}(t) \geqslant \varphi_{x,x,\cdots,x}(|\mathcal{N}|t)$$

for all $x \in \mathcal{A}$ and $t > 0$. This implies that $d(Jf, f) \leqslant \frac{1}{|\mathcal{N}|} < \infty$. By The fixed point alternative theorem, Theorem 1.1, J has a unique fixed point $H : \mathcal{A} \to \mathcal{B}$ in $\Omega_0 := \{h \in \Omega : d(h,f) < \infty\}$ such that

$$H(x) = \lim_{m\to\infty} \frac{1}{|\mathcal{M}|^m} f(\mathcal{M}^m x) \tag{3.11}$$

for all $x \in \mathcal{A}$, since $\lim_{m\to\infty} d(J^m f, H) = 0$.

On the other hand, it follows from (3.1), (3.8) and (3.11) that

$$\begin{aligned}
\mu^{\mathcal{B}}_{D_\lambda H(x_1,\cdots,x_n)}(t) &= \lim_{m\to\infty} \mu_{\frac{1}{\mathcal{M}^m} D_\lambda f(\mathcal{M}^m x_1,\cdots,\mathcal{M}^m x_n)}(t) \\
&\geqslant \lim_{m\to\infty} \varphi_{\mathcal{M}^m x_1,\cdots,\mathcal{M}^m x_n}(|\mathcal{M}|^m t) = 1.
\end{aligned}$$

By a similar method to the above, we can get $\lambda H(\mathcal{M}x) = H(\lambda \mathcal{M}x)$ for all $\lambda \in \mathbb{T}$ and all $x \in \mathcal{A}$. Then by using the same technique as in the proof of [10, Theorem 2.1], we can show that H is $\mathbb{C}$-linear.

It follows from (3.2), (3.9) and (3.11) that

$$\begin{aligned}
\mu^{\mathcal{B}}_{H(xy)-H(x)H(y)}(t) &= \lim_{m\to\infty} \mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}xy)-f(\mathcal{M}^m x)f(\mathcal{M}^m y)}\left(|\mathcal{M}|^{2m} t\right) \\
&\geqslant \lim_{m\to\infty} \psi_{\mathcal{M}^m x, \mathcal{M}^m y}\left(|\mathcal{M}|^{2m} t\right) = 1
\end{aligned}$$

for all $x, y \in \mathcal{A}$. Therefore, we conclude that $H(xy) = H(x)H(y)$ for all $x, y \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is a homomorphism satisfying (3.7).

By same method as above, from (3.3),(3.10) and (3.11), we can write

$$\begin{aligned}
\mu^{\mathcal{B}}_{H(x^*)-H(x)^*}(t) &= \lim_{m\to\infty} \mu^{\mathcal{B}}_{\frac{1}{\mathcal{M}^m}(f(\mathcal{M}^m x^*)-f(\mathcal{M}^m x)^*)}(t) \\
&\geqslant \lim_{m\to\infty} \eta_{\mathcal{M}^m x}\left(|\mathcal{M}|^m t\right) = 1
\end{aligned}$$

for all $x \in \mathcal{A}$ and all $t > 0$. Then we conclude that $H(x^*) = H(x)^*$ and the proof is complete, as desired. □

Corollary 3.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\mu^{\mathcal{B}}_{D_\lambda f(x_1,\cdots,x_n)}(t) \geqslant \frac{t}{t+\theta(\|x_1\|^r_{\mathcal{A}} + \|x_2\|^r_{\mathcal{A}} + \cdots + \|x_n\|^r_{\mathcal{A}})},$$
$$\mu^{\mathcal{B}}_{f(xy)-f(x)f(y)}(t) \geqslant \frac{t}{t+\theta(\|x\|^r_{\mathcal{A}}.\|y\|^r_{\mathcal{A}})},$$
$$\mu^{\mathcal{B}}_{f(x^*)-f(x)^*}(t) \geqslant \frac{t}{t+\theta\|x\|^r_{\mathcal{A}}}$$

for all $\lambda \in \mathbb{T}^1$, all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$. Then there exists a unique random homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x)-H(x)}(t) \geqslant \frac{(|\mathcal{N}| - |\mathcal{N}|^r)t}{(|\mathcal{N}| - |\mathcal{N}|^r)t + n\theta\|x\|^r_{\mathcal{A}}}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof. Letting

$$\varphi_{x_1,\cdots,x_n}(t)\frac{t}{t+\theta(\|x_1\|^r_{\mathcal{A}} + \|x_2\|^r_{\mathcal{A}} + \cdots + \|x_n\|^r_{\mathcal{A}})},$$
$$\psi_{x,y}(t) = \frac{t}{t+\theta(\|x\|^r_{\mathcal{A}}.\|y\|^r_{\mathcal{A}})},$$
$$\eta_x(t) = \frac{t}{t+\theta\|x\|^r_{\mathcal{A}}}$$

for all $x_1, \cdots, x_n, x, y \in \mathcal{A}$, $L = |\mathcal{N}|^{r-1}$ and $t > 0$ in Theorem 3.1, we get the desired result. □

In the following theorem, we investigate the Hyers-Ulam stability of derivations on non-Archimedean random C^*-algebras for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 3.3. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$, $\psi : \mathcal{A}^2 \to D^+$, satisfying (3.1), (3.3), and $\eta : \mathcal{A} \to D^+$ such that $|\mathcal{M}| < 1$ and $|\mathcal{N}| < 1$ are far from zero and*

$$\mu^{\mathcal{A}}_{f(xy)-f(x)y-xf(y)}(t) \geqslant \psi_{x,y}(t), \tag{3.12}$$

for all $\lambda \in \mathbb{T}^1$ and all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ such that (3.4), (3.5) and (3.6) hold, then there exists a unique random derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\mu^{\mathcal{A}}_{f(x)-\delta(x)}(t) \geqslant \varphi_{x,\cdots,x}\left((|\mathcal{N}| - |\mathcal{N}|L)\right) \tag{3.13}$$

for all $x \in^{\mathcal{A}}$ and $t > 0$.

Proof. By the same argument as in the proof of Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying (3.13). The mapping δ is given by

$$\delta(x) = \lim_{m\to\infty} \frac{1}{|\mathcal{M}|^m} f(\mathcal{M}^m x) \tag{3.14}$$

for all $x \in \mathcal{A}$.

It follows from (3.12), (3.9) and (3.14) that

$$\begin{aligned}\mu^{\mathcal{B}}_{\delta(xy)-\delta(x)y-x\delta(y)}(t) &= \lim_{m\to\infty} \mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}xy)-f(\mathcal{M}^m x)\mathcal{M}^m y-\mathcal{M}^m x f(\mathcal{M}^m y)}\left(|\mathcal{M}|^{2m}t\right)\\ &\geqslant \lim_{m\to\infty} \psi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right) = 1\end{aligned}$$

for all $x, y \in \mathcal{A}$. Therefore, we conclude that $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. The remainder of the proof is similar to the proof of Theorem 3.1. $\square$

4. Stability of homomorphisms and derivations in non-Archimedean random Lie JC^*-algebras

A non-Archimedean random C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy-yx}{2}$ and endowed with *anticommutator product* (Jordan product) $x \circ y := \frac{xy+yx}{2}$ on $\mathcal{C}$, is called a non-Archimedean random Lie JC^*-algebra.

Definition 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be non-Archimedean random Lie JC^*-algebras. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a random Lie JC^*-algebra homomorphism if H satisfies*

$$\begin{aligned}&H([x, y]) = [H(x), H(y)],\\ &H(x \circ y) = H(x) \circ H(y),\\ &H(x^*) = H(x)^*\end{aligned}$$

for all $x, y \in \mathcal{A}$.

Throughout this section, assume that $\mathcal{A}$ and $\mathcal{B}$ are two non-Archimedean random Lie JC^*-algebras respectively with norm $\mu^{\mathcal{A}}_{\cdot}$ and $\mu^{\mathcal{B}}_{\cdot}$.

In the following theorem, we prove the Hyers-Ulam stability of homomorphisms in non-Archimedean random Lie JC^*-algebra for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 4.2. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$ and $\psi : \mathcal{A}^2 \to D^+$ satisfying (3.1), (3.3) and*

$$\mu^{\mathcal{B}}_{f([x,y])-[f(x),f(y)]}(t) \geqslant \psi_{x,y}(t), \tag{4.1}$$

$$\mu^{\mathcal{B}}_{H(x\circ y)-H(x)\circ H(y)}(t) \geqslant \phi_{x,y}(t) \tag{4.2}$$

for all $\lambda \in \mathbb{T}^1$, all $x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ such that (3.4), (3.5) and (3.6) hold, and also

$$\phi_{\mathcal{M}x,\mathcal{M}y}(|\mathcal{M}|^2 Lt) \geqslant \phi_{x,y}(t), \tag{4.3}$$

for all $x, y \in \mathcal{A}$ and $t > 0$, then there exists a unique random Lie JC^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.7).

Proof. It follows from (4.3) and $L < 1$ that

$$\lim_{m\to\infty} \phi_{\mathcal{M}^m x,\mathcal{M}^m y}(|\mathcal{M}|^{2m}t) = 1, \tag{4.4}$$

for all $x, y \in \mathcal{A}$ and $t > 0$.

By the same argument as in the proof of Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $H:\mathcal{A}\to\mathcal{B}$ satisfying (3.7). The mapping H is given by

$$H(x)=\lim_{m\to\infty}\frac{f(\mathcal{M}^m x)}{|\mathcal{M}|^m} \tag{4.5}$$

for all $x\in\mathcal{A}$. It follows from (3.9), (4.4) and (4.5) that

$$\begin{aligned}\mu^{\mathcal{B}}_{H([x,y])-[H(x),H(y)]}(t)&=\lim_{m\to\infty}\mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}[x,y])-[f(\mathcal{M}^m x),f(\mathcal{M}^m y)]}\left(|\mathcal{M}|^{2m}t\right)\\&\geqslant\lim_{m\to\infty}\psi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right)=1\end{aligned}$$

and

$$\begin{aligned}\mu^{\mathcal{B}}_{H(x\circ y)-H(x)\circ H(y)}&=\lim_{m\to\infty}\mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}(x\circ y))-f(\mathcal{M}x)\circ f(\mathcal{M}y)}\left(|\mathcal{M}|^{2m}t\right)\\&\geqslant\lim_{m\to\infty}\phi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right)=1\end{aligned}$$

for all $x,y\in\mathcal{A}$ and $t>0$, then it is concluded that

$$H([x,y])=[H(x),H(y)]\quad;\quad H(x\circ y)=H(x)\circ H(y)$$

for all $x,y\in\mathcal{A}$. Therefore, $H:\mathcal{A}\to\mathcal{B}$ is the unique random Lie JC^*-algebra homomorphism satisfying (3.7). □

Corollary 4.3. *Let $r>1$ and θ be nonnegative real numbers, and $f:\mathcal{A}\to\mathcal{B}$ be a mapping such that*

$$\begin{aligned}&\mu^{\mathcal{B}}_{D_\lambda f(x_1,\cdots,x_n)}(t)\geqslant\frac{t}{t+\theta(\|x_1\|^r_{\mathcal{A}}+\cdots+\|x_n\|^r_{\mathcal{A}})},\\&\mu^{\mathcal{B}}_{f([x,y])-[f(x),f(y)]}\geqslant\frac{t}{t+\theta(\|x\|^r_{\mathcal{A}}.\|y\|^r_{\mathcal{A}})},\\&\mu^{\mathcal{B}}_{f(x^8)-f(x)^*}(t)\geqslant\frac{t}{t+\theta.\|x\|^r_{\mathcal{A}}}\end{aligned}$$

for all $\lambda\in\mathbb{T}^1$, all $x_1,\cdots,x_n,x,y\in\mathcal{A}$ and $t>0$. Then there exists a unique random Lie JC^*-algebra homomorphism $H:\mathcal{A}\to\mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x)-H(x)}\geqslant\frac{(|\mathcal{N}|-|\mathcal{N}|^r)t}{(|\mathcal{N}|-|\mathcal{N}|^r)t+n\theta\|x\|^r_{\mathcal{A}}}$$

for all $x\in\mathcal{A}$ and $t>0$.

Proof. By the same reasoning as in the proof of Theorem 4.2 and a technique similar to Corollary 3.2, by putting $L=|\mathcal{N}|^{r-1}$, the proof will be completed. □

Definition 4.4. *Let $\mathcal{A}$ be a non-Archimedean random Lie JC^*-algebra. A $\mathbb{C}$-linear mapping $\delta:\mathcal{A}\to\mathcal{A}$ is called a random Lie JC^*-algebra derivation if δ satisfies*

$$\begin{aligned}&\delta([x,y])=[\delta(x),y]+[x,\delta(y)],\\&\delta(x\circ y)=\delta(x)\circ y+x\circ\delta(y),\\&\delta(x^*)=\delta(x)^*\end{aligned}$$

for all $x,y\in\mathcal{A}$.

In the following theorem, we prove the Hyers-Ulam stability of derivation on non-Archimedean random Lie JC^*-algebras for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 4.5. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$ and $\psi : \mathcal{A}^2 \to D^+$ such that (3.1) and (3.3) hold and*

$$\mu^{\mathcal{A}}_{f([x,y])-[f(x),y]-[x,f(y)]}(t) \geqslant \psi_{x,y}(t), \tag{4.6}$$

$$\mu^{\mathcal{A}}_{f(x\circ y)-f(x)\circ y-x\circ f(y)}(t) \geqslant \phi_{x,y}(t) \tag{4.7}$$

for all $x, y \in \mathcal{A}$. If there exists an $L < 1$ and (3.4), (3.5), (3.6) and (4.3) hold, then there exists a unique random Lie JC^*-algebra derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that (3.13) holds.

Proof. By the same argument as in the proof of Theorem 4.2, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying (3.13), and is given by

$$\delta(x) = \lim_{m\to\infty} \frac{f(\mathcal{M}^m x)}{|\mathcal{M}|^m} \tag{4.8}$$

for all $x \in \mathcal{A}$.

It follows from (3.9), (4.4) and (4.8) that

$$\begin{aligned} \mu_{\delta([x,y])-[\delta(x),y]-[x,\delta(y)]}(t) &= \lim_{m\to\infty} \mu^{\mathcal{A}}_{f(\mathcal{M}^{2m}[x,y])-[f(\mathcal{M}^m x),\mathcal{M}^m y]-[\mathcal{M}^m x, f(\mathcal{M}^m y)]}\left(|\mathcal{M}|^{2m} t\right) \\ &\geqslant \lim_{m\to\infty} \psi_{\mathcal{M}^m x, \mathcal{M}^m y}\left(|\mathcal{M}|^{2m} t\right) = 1 \end{aligned}$$

and

$$\begin{aligned} \mu^{\mathcal{A}}_{\delta(x\circ y)-\delta(x)\circ y-x\circ\delta(y)}(t) &= \lim_{m\to\infty} \mu^{\mathcal{A}}_{f(\mathcal{M}^{2m}(x\circ y))-f(\mathcal{M}^m x)\circ y-x\circ f(\mathcal{M}^m y)}\left(|\mathcal{M}|^{2m} t\right) \\ &\geqslant \lim_{m\to\infty} \phi_{\mathcal{M}^m x, \mathcal{M}^m y}\left(|\mathcal{M}|^{2m} t\right) = 1 \end{aligned}$$

for all $x, y \in \mathcal{A}$ and $t > 0$, and so we conclude that

$$\delta([x,y]) = [\delta(x), y] + [x, \delta(y)], \quad \delta(x \circ y) = \delta(x) \circ y + x \circ \delta(y)$$

for all $x, y \in \mathcal{A}$. Therefore, $\delta : \mathcal{A} \to \mathcal{A}$ is the unique desired random Lie JC^*-algebra derivation satisfying (3.13). □

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ON THE FUZZY STABILITY PROBLEMS OF GENERALIZED SEXTIC MAPPINGS

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Abstract. We introduce a fuzzy anti-β-norm and generalized sextic mapping and then investigate the Hyers-Ulam-Rassias stability in quasi β-Banach space and the fuzzy stability by using a fixed point in fuzzy anti-β Banach space for the generalized sextic function.

1. Introduction

The concept of stability problem of a functional equation was first posed by Ulam [33] concerning the stability of group homomorphisms. In the next year, Hyers [14] gave a partial answer to the question of Ulam. Hyers' theorem was generalized in various directions. The very first author who generalized Hyers' theorem to the case of unbounded control functions was Aoki [1]. Rassias [28] succeeded in extending the result of Hyers' theorem by weakening the condition for the Cauchy difference operator $CDf(x, y) = f(x + y) - [f(x) + f(y)]$ to be controlled by $\varepsilon(||x||^p + ||y||^p)$. Rassias' paper [28] has provided a lot of influence in the development of Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. In 1996, Isac and Rassias [16] were first to provide applications of new fixed point theorems for the proof of stability theory of functional equations. By using fixed point methods the stability problems of several functional equations have been extensively investigated by a number of authors; see [6], [7], [25] and [26]. Recently, the stability problem of functional equations was investigated by using shadowing properties; see [20] and [31].

During the last three decades, several stability problems of a large variety of functional equations have been extensively studied and generalized by a number of authors [9], [12], [15], [28], and [2]. In particular, Xu and et al. [37] introduced the sextic functional equation

$$f(x+3y)+f(x-3y)-6[f(x+2y)+f(x-2y)]+15[f(x+y)+f(x-y)] = 20f(x)+720f(y)\,. \tag{1.1}$$

In fact, Xu and et al. [37] and Gordji and et al. [13] introduced a quintic mapping and sextic mapping.

In this paper, we deal with the following functional equation

$$\begin{aligned} &f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay) \\ &\quad = a^2(a^2+1)[f(x+y)+f(x-y)]+2(a^2-1)(a^4-1)[f(x)+f(y)] \end{aligned} \tag{1.2}$$

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holds for all $x,\, y \in X$ and all $a \in \mathbb{Z}\,(a \neq 0, \pm 1)$.

We will use the following definition to prove Hyers-Ulam-Rassias stability for the generalized sextic functional equation in the quasi β-normed space. Let β be a real number with $0 < \beta \leq 1$ and $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. *Let X be a linear space over a field $\mathbb{K}$. A quasi β-norm $||\cdot||$ is a real-valued function on X satisfying the following statements:*

(1) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$.

(2) $||\lambda x|| = |\lambda|^{\beta} \cdot ||x||$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $||x+y|| \leq K(||x||+||y||)$ for all $x,\, y \in X$.

The pair $(X, ||\cdot||)$ is called a *quasi β-normed space* if $||\cdot||$ is a quasi β-norm on X. The smallest possible K is called the *modulus of concavity* of $||\cdot||$. A *quasi β-Banach space* is a complete quasi-β-normed space.

A quasi β-norm $||\cdot||$ is called a $(\beta,\, p)$-norm $(0 < p \leq 1)$ if (3) takes the form $||x + y||^p \leq ||x||^p + ||y||^p$ for all $x,\, y \in X$. In this case, a quasi β-Banach space is called a (β, p)-Banach space; see [5], [29] and [27].

In 1984, Katsaras [18] and Wu and Fang [35] independently introduced a notion of a fuzzy norm. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view; see [3], [11], [19], [36] and [23]. In 2003, Bag and Samanta [3] modified the definition of Cheng and Mordeson [8]. Bag and Samanta [3] introduced the following definition of fuzzy normed spaces. The notion of fuzzy stability of functional equations was given in the paper [24]. Jebril and Samanta [17] introduced a fuzzy anti-norm linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [4] and investigated their important properties.

We will use the definition of fuzzy anti-normed spaces to investigate a fuzzy version of Hyers-Ulam-Rassias stability in the fuzzy anti-normed algebra setting.

Definition 1.2. [17] *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,\, 1]$ is called a fuzzy anti-norm on X if for all $x,\, y \in X$ and all $s,\, t \in \mathbb{R}$,*

(aN1) *$N(x,t) = 1$ for $t \leq 0$*

(aN2) *$N(x,t) = 0$ if and only if $x = 0$ for all $t > 0$*

(aN3) *$N(cx,t) = N(x, \frac{t}{|c|})$ for $c \neq 0$*

(aN4) *$N(x+y, s+t) \leq max\{N(x,s)\,, N(y,t)\}$*

(aN5) *$N(x,t)$ is a non-increasing function of $t \in \mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 0$,*

(aN6) *for $x \neq 0$, $N(x,\, \cdot)$ is continuous on $\mathbb{R}$.*

The pair (X, N) is called a fuzzy anti-normed space.

The property (aN3) implies that $N(-x,t) = N(x,t)$ for all $x \in X$ and $t > 0$. It is easy to show that (aN4) is equivalent the following condition:

$$N(x+y,t) \leq \max\{N(x,t)\,, N(y,t)\}\,, \text{ for all } x, y \in X \text{ and } t \in \mathbb{R}.$$

Definition 1.3. *Let X be a real vector space. A fuzzy anti-norm $N : X\times\mathbb{R} \to [0,\, 1]$ is called a fuzzy anti-β-norm on X if (aN_3) in Definition 1.2 takes the form*

$$(aN_3') \quad N(cx,\, t) = N(x,\, \frac{t}{|c|^{\beta}}) \quad (c \neq 0,\, 0 < \beta \leq 1)\,.$$

Example 1.4. *Let $(X,\, ||\cdot||)$ be a β-normed space. Define*

$$N(x,t) = \begin{cases} \frac{||x||}{t+||x||} & when\ t > 0,\, t \in \mathbb{R} \\ 1 & when\ t \leq 0\,, \end{cases}$$

where $x \in X$. *We note that*

$$N(cx,\, t) = \frac{||cx||}{t + ||cx||} = \frac{||x||}{\frac{t}{|c|^\beta} + ||x||} = N(x,\, \frac{t}{|c|^\beta})\,,$$

for all $x \in X$ *and* $c \in \mathbb{R}\,(c \neq 0\,, 0 < \beta \leq 1)$. *Then* (X, N) *is a fuzzy anti-*β*-normed space induced by the* β*-norm* $||\cdot||$.

Definition 1.5. *Let* $(X,\, N)$ *be a fuzzy anti-*β*-normed vector space. A sequence* $\{x_n\}$ *in* X *is said to be convergent or converge if there exists an* $x \in X$ *such that* $\lim_{n\to\infty} N(x_n - x,\, t) = 0$ *for all* $t > 0$. *In this case,* x *is called the limit of the sequence* $\{x_n\}$ *and we denote it by* $N\text{-}\lim_{n\to\infty} x_n = x$.

Definition 1.6. *Let* $(X,\, N)$ *be a fuzzy anti-*β*-normed vector space. A sequence* $\{x_n\}$ *in* X *is called Cauchy if for each* $\varepsilon > 0$ *and each* $t > 0$ *there exists an* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *and all integer* $d > 0$, *we have* $N(x_{n+d} - x_n,\, t) < \varepsilon$.

It is well-known that every convergent sequence in a fuzzy anti-β-normed vector space is Cauchy. If each Cauchy sequaence is convergent, then the fuzzy anti-β-normed space is said to be *fuzzy anti-β complete* and the fuzzy anti-β-normed vector space is called a *fuzzy anti-β Banach space.*

Now, we will state the theorem, the alternative of fixed point in a generalized metric space.

Definition 1.7. *Let* X *be a set. A function* $d : X \times X \to [0, \infty]$ *is called a generalized metric on* X *if* d *satisfies*

(1) $d(x,\, y) = 0$ *if and only if* $x = y$;
(2) $d(x,\, y) = d(y,\, x)$ *for all* $x,\, y \in X$;
(3) $d(x,\, z) \leq d(x,\, y) + d(y,\, z)$ *for all* $x,\, y,\, z \in X$.

Theorem 1.8 (The alternative of fixed point [21], [30])**.** *Suppose that we are given a complete generalized metric space* (X, d) *and a strictly contractive mapping* $J : X \to X$ *with Lipschitz constant* $0 < L < 1$. *Then for each given* $x \in X$, *either*

$$d(J^n x,\, J^{n+1}x) = \infty \text{ for all } n \geq 0\,,$$

or there exists a natural number n_0 *such that*

(1) $d(J^n x,\, J^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(2) *The sequence* $\{J^n x\}$ *is convergent to a fixed point* y^* *of* J ;
(3) y^* *is the unique fixed point of* J *in the set*

$$Y = \{y \in X | d(J^{n_0}x, y) < \infty\}\,;$$

(4) $d(y, y^*) \leq \frac{1}{1-L}\, d(y, Jy)$ *for all* $y \in Y$.

In this paper, we investigate the Hyers-Ulam-Rassias stability in quasi β-normed space and then the fuzzy stability by using a fixed point in fuzzy anti-β Banach space for the generalized sextic function $f : X \to Y$ satisfying the equation (1.2). Let us fix some notations which will be used throughout this paper. Let $a \in \mathbb{Z}\,(a \neq 0\,, \pm 1)$.

2. A sextic functional equation

In this section let X and Y be real vector spaces and we investigate the general solution of the functional equation (1.2). Before we proceed, we would like to introduce some basic definitions concerning n-additive symmetric mappings and key concepts which are found in [32] and [34]. A function $A : X \to Y$ is said to be

additive if $A(x+y) = A(x) + A(y)$ for all $x, y \in X$. Let n be a positive integer. A function $A_n : X^n \to Y$ is called *n-additive* if it is additive in each of its variables. A function A_n is said to be *symmetric* if $A_n(x_1, \cdots, x_n) = A_n(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ for every permutation $\{\sigma(1), \cdots, \sigma(n)\}$ of $\{1, 2, \cdots, n\}$. If $A_n(x_1, x_2, \cdots, x_n)$ is an n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, x, \cdots, x)$ and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and all $r \in \mathbb{Q}$. such a function $A^n(x)$ will be called a *monomial function* of degree n (assuming $A^n \not\equiv 0$). Furthermore the resulting function after substitution $x_1 = x_2 = \cdots = x_s = x$ and $x_{s+1} = x_{s+2} = \cdots = x_n = y$ in $A_n(x_1, x_2, \cdots, x_n)$ will be denoted by $A^{s,n-s}(x, y)$.

Theorem 2.1. *A function $f : X \to Y$ is a solution of the functional equation (1.2) if and only if f is of the form $f(x) = A^6(x)$ for all $x \in X$, where $A^6(x)$ is the diagonal of the 6-additive symmetric mapping $A_6 : X^6 \to Y$.*

Proof. Assume that f satisfies the functional equation (1.2). Letting $x = y = 0$ in the equation (1.2), we have

$$2a^2(2a^2+1)(a^2-1)f(0) = 0\,,$$

that is, $f(0) = 0$. Let $y = 0$ in the equation (1.2). Then we get

$$f(ax) = a^6 f(x) \tag{2.1}$$

for all $x \in X$. Putting $x = 0$ in the equation (1.2), we get

$$(a^4-1)(a^2-1)\Big(f(y) - f(-y)\Big) = 0 \tag{2.2}$$

for all $y \in X$. Hence we have $f(y) = f(-y)$, for all $y \in X$. That is, f is even. We can rewrite the functional equation (1.2) in the form

$$\begin{aligned} & f(x) - \frac{1}{2(a^2-1)(a^4-1)} f(ax+y) - \frac{1}{2(a^2-1)(a^4-1)} f(ax-y) \\ & - \frac{1}{2(a^2-1)(a^4-1)} f(x+ay) - \frac{1}{2(a^2-1)(a^4-1)} f(x-ay) \\ & + \frac{a^2(a^2+1)}{2(a^2-1)(a^4-1)} f(x+y) + \frac{a^2(a^2+1)}{2(a^2-1)(a^4-1)} f(x-y) + f(y) = 0 \end{aligned}$$

for all $x, y \in X$ and an integer $a (a \neq 0, \pm 1)$. By Theorem 3.5 and 3.6 in [34], f is a generalized polynomial function of degree at most 6, that is, f is of the form

$$f(x) = A^6(x) + A^5(x) + A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x) \tag{2.3}$$

for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric mapping $A_i : X^i \to Y$ for $i = 1, 2, 3, 4, 5, 6$. By $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0, A^5(x) = 0, A^3(x) = 0$ and $A^1(x) = 0$. It follows that

$$f(x) = A^6(x) + A^4(x) + A^2(x)$$

for all $x \in X$. By (2.1) and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and $r \in \mathbb{Q}$, we obtain that $A^2(x) = -\frac{a^2}{a^2+1} A^4(x)$ for all $x \in X$ and an integer $a\,(a \neq 0, \pm 1)$. Hence we get $A^4(x) = A^2(x) = 0$, for all $x \in X$. Thus we have $f(x) = A^6(x)$ for all $x \in X$.

Conversely, assume that $f(x) = A^6(x)$ for all $x \in X$, where $A^6(x)$ is the diagonal of a 6-additive symmetric mapping $A_6 : X^6 \to Y$. Note that

$$\begin{aligned} A^6(qx+ry) &= q^6A^6(x) + 6q^5rA^{5,1}(x,y) + 15q^4r^2A^{4,2}(x,y) + 20q^3r^3A^{3,3}(x,y) \\ &+ 15q^2r^4A^{2,4}(x,y) + 6qr^5A^{1,5}(x,y) + r^6A^6(y) \end{aligned}$$

$$c^s A^{s,t}(x,y) = A^{s,t}(cx,y)\,, \quad c^t A^{s,t}(x,y) = A^{s,t}(x,cy)$$

where $1 \le s,t \le 5$ and $c \in \mathbb{Q}$. Thus we may conclude that f satisfies the equation (1.2). □

We note that a mapping $f : X \to Y$ is called *generalized sextic* if f satisfies the functional equation (1.2).

3. Hyers-Ulam-Rassias stability over a quasi β-Banach space

Throughout this section, let X be a real linear space and let Y be a quasi β-Banach space with a quasi β-norm $||\cdot||_Y$. Let K be the modulus of concavity of $||\cdot||_Y$. We will investigate the Hyers-Ulam-Rassias stability for the functional equation (1.2); see also the paper [10].

For a given mapping $f : X \to Y$ and all fixed integer a ($a \neq 0, \pm 1$), let

$$D_a f(x,y) := f(ax+y) + f(ax-y) + f(x+ay) + f(x-ay) \tag{3.1}$$
$$-a^2(a^2+1)\Big(f(x+y)+f(x-y)\Big) - 2(a^2-1)(a^4-1)\Big(f(x)+f(y)\Big)$$

for all $x,\, y \in X$.

Theorem 3.1. *Suppose that there exists a mapping $\phi : X^2 \to [0,\infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$||D_a f(x,y)||_Y \le \phi(x,y) \tag{3.2}$$

and the series $\sum_{j=0}^{\infty} \left(\frac{K}{|a|^{6\beta}}\right)^j \phi(a^j x, a^j y)$ converges for all $x,\, y \in X$. Then there exists a unique generalized sextic mapping $S : X \to Y$ satisfying the equation (1.2) and the inequality

$$||f(x) - S(x)||_Y \le \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=0}^{\infty} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0)\,, \tag{3.3}$$

for all $x \in X$.

Proof. By letting $y = 0$ in inequality (3.2), since $f(0) = 0$ we have

$$\begin{aligned} ||D_a f(x,0)||_Y &= ||2f(ax) + 2f(x) - 2a^2(a^2+1)f(x) - 2(a^2-1)(a^4-1)f(x)||_Y \\ &= 2^\beta |a|^{6\beta} ||f(x) - \frac{1}{a^6} f(ax)||_Y \le \phi(x,0)\,, \end{aligned}$$

that is,

$$||f(x) - \frac{1}{a^6} f(ax)||_Y \le \frac{1}{2^\beta |a|^{6\beta}}\, \phi(x,0)\,, \tag{3.4}$$

for all $x \in X$.

We note that putting $x = ax$ and multiplying $\frac{1}{|a|^{6\beta}}$ in the inequality (3.4), we get

$$\frac{1}{|a|^{6\beta}} ||f(ax) - \frac{1}{a^6} f(a^2 x)||_Y \le \frac{1}{2^\beta |a|^{6\beta}}\, \frac{1}{|a|^{6\beta}} \phi(ax,0)\,, \tag{3.5}$$

for all $x \in X$.

Combining two inequalities (3.4) and (3.5), we have

$$||f(x) - \Big(\frac{1}{a^6}\Big)^2 f(a^2 x)||_Y \le \frac{K}{2^\beta |a|^{6\beta}} \Big(\phi(x,0) + \frac{1}{|a|^{6\beta}} \phi(ax,0)\Big)\,, \tag{3.6}$$

for all $x \in X$.

Since $K \geq 1$, inductively using the previous note we have the following inequalities

$$||f(x) - \Big(\frac{1}{a^6}\Big)^k f(a^k x)||_Y \leq \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=0}^{k-1} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0), \tag{3.7}$$

for all $x \in X$, $k \in \mathbb{N}$ and also

$$||\Big(\frac{1}{a^6}\Big)^k f(a^k x) - \Big(\frac{1}{a^6}\Big)^t f(a^t x)||_Y \leq \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=k}^{t-1} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0), \tag{3.8}$$

for all $x \in X$ and k, $t \in \mathbb{N}\,(k < t)$.

Since the right-hand side of the previous inequality (3.8) tends to 0 as $t \to \infty$, hence $\{\Big(\frac{1}{a^6}\Big)^n f(a^n x)\}$ is a Cauchy sequence in the quasi β-Banach space Y. Thus we may define

$$S(x) = \lim_{n\to\infty} \Big(\frac{1}{a^6}\Big)^n f(a^n x),$$

for all $x \in X$. Since $K \geq 1$, replacing x and y by $a^n x$ and $a^n y$ respectively and dividing by $|a|^{6\beta n}$ in the inequality (3.2), we have

$$\begin{aligned}
&\Big(\frac{1}{|a|^{6\beta}}\Big)^n ||D_a f(a^n x, a^n y)||_Y \\
&= \Big(\frac{1}{|a|^{6\beta}}\Big)^n ||f(a^n(ax+y)) + f(a^n(ax-y)) + f(a^n(x+ay)) + f(a^n(x-ay)) \\
&-a^2(a^2+1)\Big(f(a^n(x+y)) + f(a^n(x-y))\Big) \\
&-2(a^2-1)(a^4-1)\Big(f(a^n x) + f(a^n y)\Big)||_Y \\
&\leq \Big(\frac{K}{|a|^{6\beta}}\Big)^n \phi(a^n x, a^n y)
\end{aligned}$$

for all x, $y \in X$.

By taking $n \to \infty$, the definition of S implies that S satisfies (1.2) for all x, $y \in X$, that is, S is the generalized sextic mapping. Also, the inequality (3.7) implies the inequality (3.3).

Now, it remains to show the uniqueness. Assume that there exists $T : X \to Y$ satisfying (1.2) and (3.3). Then

$$\begin{aligned}
||T(x) - S(x)||_Y &= \Big(\frac{1}{|a|^{6\beta}}\Big)^n ||T(a^n x) - S(a^n x)||_Y \\
&\leq \Big(\frac{1}{|a|^{6\beta}}\Big)^n K\Big(||T(a^n x) - f(a^n x)||_Y + ||f(a^n x) - S(a^n x)||_Y\Big) \\
&\leq \frac{2K^2}{2^\beta |a|^{6\beta} K^n} \sum_{j=n}^{\infty} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0)
\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$, we immediately have the uniqueness of S. □

Corollary 3.2. *Let $\theta \geq 0$, $p < 6$ be a real number and X be a normed linear space with norm $||\cdot||$. Suppose $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and*

$$||D_a f(x,y)||_Y \leq \theta(||x||^p + ||y||^p) \tag{3.9}$$

for all $x,\, y \in X$ *and all* $t > 0$. *Then* $S(x) := N\text{-}\lim_{n\to\infty} \frac{1}{a^{6n}} f(a^n\, x)$ *exists for each* $x \in X$ *and defines a generalized sextic mapping* $S : X \to Y$ *such that*

$$||f(x) - S(x)||_Y \le \frac{\theta\, K\, ||x||^p}{2^\beta(|a|^{6\beta} - K|a|^{p\beta})}$$

for all $x \in X$ *and all* $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking $\phi(x,\, y) = \theta(||x||^p + ||y||^p)$ for all $x\,, y \in X$. □

4. Fuzzy fixed point stability over a Fuzzy Banach space

Let us fix some notations which will be used throughout this section. We assume X is a vector space and $(Y,\, N)$ is a fuzzy anti-β Banach space. Using fixed point method, we will prove the Hyers-Ulam stability of the functional equation satisfying equation (1.2) in fuzzy anti-β Banach space.

Theorem 4.1. *Let* $\phi : X^2 \to [0, \infty)$ *be a function such that there exists an* $0 < L < 1$ *with*

$$\phi(x, y) \le \frac{L}{|a|^{6\beta}}\, \phi(ax, ay) \tag{4.1}$$

for all $x,\, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and*

$$N(D_a f(x, y),\, t) \le \frac{\phi(x, y)}{t + \phi(x, y)} \tag{4.2}$$

for all $x,\, y \in X$ *and all* $t > 0$. *Then* $S(x) := N\text{-}\lim_{n\to\infty} a^{6n} f\Big(\frac{x}{a^n}\Big)$ *exists for each* $x \in X$ *and defines a generalized sextic mapping* $S : X \to Y$ *such that*

$$N(f(x) - S(x),\, t) \le \frac{L\, \phi(x, 0)}{2^\beta |a|^{6\beta}(1 - L)\, t + L\, \phi(x, 0)} \tag{4.3}$$

for all $x \in X$ *and all* $t > 0$.

Proof. By letting $y = 0$ in the inequality (4.2), we have

$$N\Big(2f(ax) - 2a^6 f(x),\, t\Big) \;\le\; \frac{\phi(x,\, 0)}{t + \phi(x,\, 0)} \tag{4.4}$$

for all $x \in X$ and all $t > 0$.

We note that by letting $x = \frac{x}{a}$ in the inequality (4.4) we have

$$N\Big(2f(x) - 2a^6 f\Big(\frac{x}{a}\Big),\, t\Big) \;\le\; \frac{\phi(\frac{x}{a},\, 0)}{t + \phi(\frac{x}{a},\, 0)}\,.$$

The inequality (4.1) implies that

$$N\Big(f(x) - a^6 f\Big(\frac{x}{a}\Big),\, \frac{t}{2^\beta}\Big) \;\le\; \frac{\frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}{t + \frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}\,.$$

By putting $t = \frac{L}{|a|^{6\beta}}\, t$, we have

$$N\Big(f(x) - a^6 f\Big(\frac{x}{a}\Big),\, \frac{L}{2^\beta |a|^{6\beta}}\, t\Big) \;\le\; \frac{\frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}{\frac{L}{|a|^{6\beta}}\, t + \frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}\,,$$

that is,

$$N\Big(f(x) - a^6 f\Big(\frac{x}{a}\Big),\ \frac{L}{2^\beta |a|^{6\beta}}\, t\Big) \ \le\ \frac{\phi(x,\,0)}{t+\phi(x,\,0)}\,, \tag{4.5}$$

for all $x \in X$ and all $t > 0$.

We consider the set

$$F := \{g : X \to X\}$$

and the mapping d defined on $F \times F$ by

$$d(g,\,h) = \inf\{\mu \in \mathbb{R}^+ \mid N\Big(g(x) - h(x),\, \mu t\Big) \le \frac{\phi(x,\,0)}{t+\phi(x,\,0)}\,, \forall x \in X \text{ and } t > 0\}$$

where $\inf \emptyset = +\infty$, as usual. Then $(F,\,d)$ is a complete generalized metric space; see [22, Lemma 2.1]. Now let's consider the linear mapping $J : F \to F$ such that

$$Jg(x) := a^6 g\Big(\frac{x}{a}\Big)$$

for all $x \in X$. Let $g\,, h \in F$ be given such that $d(g\,,h) = \varepsilon$. Then

$$N\Big(g(x) - h(x),\, \varepsilon t\Big) \le \frac{\phi(x,\,0)}{t+\phi(x,\,0)}$$

for all $x \in X$ and all $t > 0$.

$$\begin{aligned} N\Big(Jg(x) - Jh(x),\, L\varepsilon t\Big) &= N\Big(a^6 g\Big(\frac{x}{a}\Big) - a^6 h\Big(\frac{x}{a}\Big),\, L\varepsilon t\Big) \\ = \quad N\Big(g\Big(\frac{x}{a}\Big) - h\Big(\frac{x}{a}\Big),\, \frac{L}{|a|^{6\beta}}\varepsilon t\Big) &\le \frac{\phi(\frac{x}{a},\,0)}{\frac{L}{|a|^{6\beta}}t + \phi(\frac{x}{a},\,0)} \\ \le \quad \frac{\frac{L}{|a|^{6\beta}}\phi(x,\,0)}{\frac{L}{|a|^{6\beta}}t + \frac{L}{|a|^{6\beta}}\phi(x,\,0)} &= \frac{\phi(x,\,0)}{t+\phi(x,\,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. $d(g,\,h) = \varepsilon$ implies that $d(Jg,\,Jh) \le L\varepsilon$. Hence we get

$$d(Jg,\,Jh) \le L\,d(g,\,h)$$

for all $g,\,h \in F$. The inequality (4.5) implies that $d(f,\,Jf) \le \frac{L}{2^\beta |a|^{6\beta}}$. By Theorem 1.8, there exists a mapping $S : X \to Y$ such that

(1) S is a fixed point of J, that is,

$$S\Big(\frac{x}{a}\Big) = \frac{1}{a^6}\,S(x) \tag{4.6}$$

for all $x \in X$. The mapping S is a unique fixed point of J in the set $M = \{g \in F \mid d(f,\,g) < \infty\}$. This means that S is a unique mapping satisfying the equation (4.6) such that there exists a $\mu \in (0,\,\infty)$ satisfying

$$N\Big(f(x) - S(x),\, \mu t\Big) \le \frac{\phi(x,\,0)}{t+\phi(x,\,0)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f,\,S) \to 0$ as $n \to \infty$. This implies the following equality

$$\text{N-}\lim_{n\to\infty} a^{6n} f\Big(\frac{x}{a^n}\Big) = S(x)$$

for all $x \in X$;

(3) $d(f,\,S)\le \frac{1}{1-L}\,d(f,\,Jf)$, which implies the inequality

$$d(f,\,S)\le \frac{1}{1-L}\cdot\frac{L}{2^{\beta}|a|^{6\beta}}=\frac{L}{2^{\beta}|a|^{6\beta}(1-L)}\,.$$

It implies that

$$N\Big(f(x)-S(x),\,\frac{L}{2^{\beta}|a|^{6\beta}(1-L)}t\Big)\le\frac{\phi(x,\,0)}{t+\phi(x,\,0)}$$

for all $x\in X$ and all $t>0$. By replacing t by $\frac{2^{\beta}|a|^{6\beta}(1-L)}{L}\,t$, we have

$$N\Big(f(x)-S(x),\,t\Big)\le\frac{L\phi(x,\,0)}{2^{\beta}|a|^{6\beta}(1-L)\,t+L\phi(x,\,0)}$$

for all $x\in X$ and all $t>0$. That is, the inequality (4.3) holds. By letting $x=\frac{x}{a^n}$ and $y=\frac{y}{a^n}$ in the inequality (4.2), we have

$$N\Big(a^{6n}D_af\Big(\frac{x}{a^n},\,\frac{y}{a^n}\Big),\,|a|^{6\beta n}\,t\Big)\le\frac{\phi(\frac{x}{a^n},\,\frac{y}{a^n})}{t+\phi(\frac{x}{a^n},\,\frac{y}{a^n})}$$

for all $x,\,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. Replacing t by $\frac{t}{|a|^{6\beta n}}$,

$$N\Big(a^{6n}D_af\Big(\frac{x}{a^n},\,\frac{y}{a^n}\Big),\,t\Big)\le\frac{\phi(\frac{x}{a^n},\,\frac{y}{a^n})}{\frac{t}{|a|^{6\beta n}}+\phi(\frac{x}{a^n},\,\frac{y}{a^n})}\le\frac{L^n\,\phi(x,\,y)}{t+L^n\,\phi(x,\,y)}$$

for all $x,\,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\frac{L^n\,\phi(x,\,y)}{t+L^n\,\phi(x,\,y)}=0$ for all $x,\,y\in X$ and all $t>0$, we may conclude that

$$N\Big(D_aS(x,\,y),\,t\Big)=0$$

for all $x,\,y\in X$ and all $t>0$. Thus the mapping $S:X\to Y$ is the generalized sextic mapping. □

Corollary 4.2. *Let $\theta\ge 0$, $p>6$ be a real number and X be a normed linear space with norm $||\cdot||$. Suppose $f:X\to Y$ is a mapping satisfying $f(0)=0$ and*

$$N(D_af(x,y),\,t)\le\frac{\theta(||x||^p+||y||^p)}{t+\theta(||x||^p+||y||^p)}\tag{4.7}$$

for all $x,\,y\in X$ and all $t>0$. Then $S(x):=N\text{-}\lim_{n\to\infty}a^{6n}f\Big(\frac{x}{a^n}\Big)$ exists for each $x\in X$ and defines a generalized sextic mapping $S:X\to Y$ such that

$$N(f(x)-S(x),\,t)\le\frac{\theta\,||x||^p}{2^{\beta}(|a|^{p\beta}-|a|^{6\beta})\,t+\theta\,||x||^p}$$

for all $x\in X$ and all $t>0$.

Proof. The proof follows from Theorem 4.1 by taking $\phi(x,\,y)=\theta(||x||^p+||y||^p)$ for all $x\,,y\in X$ and $L=|a|^{(6-p)\beta}$. □

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Existence and uniqueness of solutions to SFDEs driven by G-Brownian motion with non-Lipschitz conditions

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Abstract

The main aim of this paper is to study the existence, uniqueness and stability of solution for stochastic functional differential equations driven by G-Brownian motion (in short G-SFDEs). The existence-and-uniqueness theorem is established for G-SFDEs under non-Lipschitz condition and weakened linear growth condition. We have used the Picard approximation scheme, Gronwall's inequality, Bihari's inequality and Burkholder-Davis-Gundy (in short BDG) inequalities to develop the existence theory for the above mentioned stochastic dynamical systems. In addition, the mean square stability of solutions for these systems has been obtained.

Key words: Existence, uniqueness, stability, G-Brownian motion, stochastic functional differential equations.

1 Introduction

Responding to the contemporary developments in the fields of physics, control engineering, economics, and social sciences, a growing concern has recently been witnessed in both stochastic differential and deterministic models. The applications of functional differential equations have been applied in a number of cases in physical phenomena, such as in the relocation of soil moisture, where the fluid flows through the crack of rocks, and the problem of conduction of heat as well as its share in order fluids is investigated. The idea of G-Brownian motion as well as the associated stochastic differential equations were introduced by Peng [8, 10]. These equations were extended to stochastic functional differential equations, which are driven by G-Brownian motion (in short G-SFDEs) by Ren, Bi and Sakthivel [12]. While Faizullah, developed the existence-and-uniqueness theorem for G-SFDEs with Cauchy-Maruyama approximation scheme [3], they used the strong Lipschitz and linear growth conditions to develop the mentioned theory. In this article, we have generalized the existence theory for functional stochastic dynamical systems, driven by G-Brownian motion. We have used non-Lipschitz condition and weak linear growth condition to study the existence, uniqueness and stability theory for G-SFDEs. We have considered the following stochastic dynamical system that

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is driven by G-Brownian motion. Let $0 \leq t \leq T < \infty$. Suppose $g : [0, T] \times BC([-\theta, 0]; \mathbb{R}^n) \to \mathbb{R}^n$, $h : [0, T] \times BC([-\theta, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ and $w : [0, T] \times BC([-\theta, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ are Borel measurable. Consider stochastic functional differential equation driven by G-Brownian motion of the type

$$dX(t) = g(t, X_t)dt + h(t, X_t)d\langle B, B\rangle(t) + w(t, X_t)dB(t), \tag{1.1}$$

where $X(t)$ is the value of stochastic process at time t and $X_t = \{X(t + \delta) : -\theta \leq \delta \leq 0, \theta > 0\}$ is a $BC([-\theta, 0]; \mathbb{R}^n)$-valued stochastic process, which presents the family of bounded continuous $\mathbb{R}^n$-valued functions φ defined on $[-\theta, 0]$ having norm $\|\varphi\| = \sup_{-\theta\leq\delta\leq 0} | \varphi(\delta) |$. $\{\langle B, B\rangle(t), t \geq 0\}$ is the quadratic variation process of G-Brownian motion $\{B(t), t \geq 0\}$ and $g, h, w \in M^2_G([-\tau, T]; \mathbb{R}^n)$. Denote the space of all $\mathcal{F}_t$-adapted process $X(t), 0 \leq t \leq T$, such that $\| X \|_{L^2} = \sup_{-\theta\leq t\leq T} |X(t)| < \infty$ by L^2. The initial data of equation (1.1) is given as follows

$$\begin{aligned} X_{t_0} =& \zeta = \{\zeta(\delta) : -\theta < \delta \leq 0\}\ is\ \mathcal{F}_0 - measurable,\ BC([-\theta, 0]; \mathbb{R}^n) -\ valued \\ &random\ variable\ such\ that\ \zeta \in M^2_G([-\theta, 0]; \mathbb{R}^n). \end{aligned} \tag{1.2}$$

The integral form of G-SFDE (1.1) with initial data (1.2) is given by

$$X(t) = \zeta(0) + \int_0^t g(s, X_s)ds + \int_0^t h(s, X_s)d\langle B, B\rangle(s) + \int_0^t w(s, X_s)dB(s).$$

The solution of G-SFDE (1.1) with initial data (1.2) is an $\mathbb{R}^n$ valued stochastic processes $X(t)$, $t \in [-\theta, T]$ such that

- **(i)** $X(t)$ is $\mathcal{F}_t$-adapted and continuous for all $t \in [0, T]$;
- **(ii)** $g(t, X_t) \in \mathcal{L}^1([o, T]; \mathbb{R}^n)$ and $h(t, X_t), w(t, X_t) \in \mathcal{L}^2([0, T]; \mathbb{R}^n)$;
- **(iii)** $X_0 = \zeta$ and for each $t \in [0, T]$, $dX(t) = g(t, X_t)dt + h(t, X_t)d\langle B, B\rangle(t) + w(t, X_t)dB(t)$ q.s.

$X(t)$ is called a unique solution if it is indistinguishable from any other solution $Y(t)$, that is,

$$E[\sup_{-\theta\leq q\leq t} |X(q) - Y(q)|^2] = 0.$$

Throughout this paper we assume the following two conditions, known as non-uniform Lipschitz condition and weakened linear growth condition respectively.

($\mathbf{A_i}$) For all $\varphi, \psi \in BC([-\theta, 0]; \mathbb{R}^d)$ and $t \in [0, T]$,

$$|g(t, \varphi) - g(t, \psi)|^2 + |h(t, \varphi) - |h(t, \psi)|^2 + |w(t, \varphi) - w(t, \psi)|^2 \leq \lambda(|\varphi - \psi|^2), \tag{1.3}$$

where $\lambda(.) : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing and concave function such that $\lambda(0) = 0$, $\lambda(v) > 0$ for $v > 0$ and

$$\int_{0+} \frac{dv}{\lambda(v)} = \infty. \tag{1.4}$$

As λ is concave and $\lambda(0) = 0$, there exists two positive constants c and d such that

$$\lambda(v) \leq c + dv, \tag{1.5}$$

for all $v \geq 0$.

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($\mathbf{A_{ii}}$) For all $t \in [0,T]$, $g(t,0), h(t,0), w(t,0) \in L^2$ and

$$|g(t,0)|^2 + |h(t,0)|^2 + |w(t,0)|^2 \leq K, \tag{1.6}$$

where K is a positive constant.

We have organized the rest of the paper as follows. In section 2, some well-known basic notions and results are included. In section 3, several important lemmas are developed. In section 4, the existence-and-uniqueness theorem is proved. In section 5, the mean square stability for the solution of G-SFDEs is given.

2 Preliminaries

The main purpose of this section is to give some basic concepts and results, which are used in the subsequent sections of this paper. For more detailed literature of G-expectation, we refer the readers to book [9] and papers [1, 2, 4, 5, 13].

Definition 2.1. Let $\mathcal{H}$ be a linear space of real valued functions defined on a nonempty basic space Ω. Then a sub-linear expectation E is a real valued functional on $\mathcal{H}$ with the following properties:

(**i**) For all $X, Y \in \mathcal{H}$, if $X \leq Y$ then $E[X] \leq E[Y]$.

(**ii**) For any real constant α, $E[\alpha] = \alpha$.

(**iii**) For all $X, Y \in \mathcal{H}$, $E[X+Y] \leq E[X] + E[Y]$.

(**iv**) For any $\theta > 0$ $E[\theta X] = \theta E[X]$.

Let $C_{b.Lip}(\mathbb{R}^{l\times d})$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^{l\times d}$ and

$$L_G^p(\Omega_T) = \{\phi(B_{t_1}, B_{t_2}, ..., B_{t_l}/l \geq 1, t_1, t_2, ..., t_l \in [0,T], \phi \in C_{b.Lip}(\mathbb{R}^{l\times d}))\}.$$

Let $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, ..., N-1$ then $M_G^0(0,T)$ denotes the collection of processes of the following type: For a given partition $\pi_T = \{t_0, t_1, ..., t_N\}$ of $[0,T]$,

$$\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1}]}(t).$$

Under the norm $\|\eta\| = \{\int_0^T E[|\eta_u|^p]du\}^{1/p}$, $M_G^p(0,T)$, $p \geq 1$, is the completion of $M_G^0(0,T)$. For every $\eta_t \in M_G^{2,0}(0,T)$, the G-Itô's integral $I(\eta)$ and G-quadratic variation process $\{\langle B\rangle_t\}_{t\geq 0}$ are respectively given by

$$I(\eta) = \int_0^T \eta_u dB_u = \sum_{i=0}^{N-1} \xi_i (B_{t_{i+1}} - B_{t_i}),$$

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_u dB_u.$$

The following definition and lemmas are borrowed from [7, 11].

Definition 2.2. A solution $X(t)$ of dynamical system (1.1) with initial data (1.2) is said to be stable in mean square if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $E|\zeta - \xi|^2 \leq \delta(\epsilon)$ follows that $E|X(t) - Y(t)|^2 < \epsilon$ for all $t \geq 0$, where $Y(t)$ is an other solution of system (1.1) having initial data $\xi \in M^2([-\theta, 0] : \mathbb{R}^l)$.

Lemma 2.3. *(Hölder's inequality) If $\frac{1}{q} + \frac{1}{r} = 1$ for any $q, r > 1$, $g \in L^2$ and $h \in L^2$ then $gh \in L^1$ and*

$$\int_c^d gh \leq (\int_c^d |g|^q)^{\frac{1}{q}} (\int_c^d |h|^r)^{\frac{1}{r}}.$$

Lemma 2.4. *(Gronwall's inequality) Let $C \geq 0$, $h(t) \geq 0$ and $w(t)$ be a real valued continuous function on $[c, d]$. If for all $c \leq t \leq d$, $w(t) \leq C + \int_c^d h(s)w(s)ds$, then*

$$w(t) \leq Ce^{\int_c^t h(s)ds},$$

for all $c \leq t \leq d$.

Lemma 2.5. *(Bihari's inequality) Suppose $T \geq 0$ and $h_0 \geq 0$. Assume $h(t)$ and $w(t)$ be continuous functions on $[0, T]$. Let $\lambda(.) : \mathbb{R}^+ \to \mathbb{R}^+$ be non-decreasing and concave continuous function such that $\lambda(v) > 0$ for $v > 0$. If for all $0 \leq t \leq T$, $h(t) \leq h(0) + \int_0^T w(s)\lambda(h(s))ds$, then for all $0 \leq t \leq T$,*

$$h(t) \leq H^{-1}(H(h_0) + \int_t^T w(s)ds),$$

such that $H(h_0) + \int_t^T w(s)ds \in Dom(H^{-1})$ where $H(q) = \int_t^q \frac{1}{\lambda(s)}ds$, $q \geq 0$ and H^{-1} is the inverse function of H.

Lemma 2.6. *Assume the assumptions of lemma 2.5 are satisfied and for $0 \leq t \leq T$, $w(t) \geq 0$. If for all $\epsilon > 0$, there exists $t_1 \geq 0$ such that for $0 \leq h_0 \leq \epsilon$, $\int_{t_1}^T w(s)ds \leq \int_{h_0}^T \frac{1}{\lambda(s)}ds$ holds, then for each $t_1 \leq t \leq T$*

$$h(t) \leq \epsilon,$$

holds.

3 Important results

In this section, we show some important lemmas. They will be used in the forth coming existence-and-uniqueness theorem. Let $X^0(t) = \zeta(0)$ for $t \in [0, T]$. Set $X^l(0) = \zeta$ for each $l = 1, 2, ...$, and define the following Picard iterations sequence,

$$\begin{aligned} X^l(t) = \zeta(0) + \int_0^t g(s, X_s^{l-1})ds + \int_0^t h(s, X_s^{l-1})d\langle B, B\rangle(s) \\ + \int_0^t w(s, X_s^{l-1})dB(s), \quad t \in [0, T]. \end{aligned} \tag{3.1}$$

First, we show that $X^l(.) \in M_G^2([-\theta, T]; \mathbb{R}^n)$.

Lemma 3.1. *Let assumptions A_i and A_{ii} hold. Then for all $l \geq 1$,*

$$\sup_{-\theta \leq t \leq T} E|X^l(t)|^2 \leq C,$$

where C is a positive constant.

Proof. Obviously, $X^0(.) \in M_G^2([-\theta, T]; \mathbb{R}^n)$. Using the basic inequality $|a+b+c+d|^2 \leq 4|a|^2 + 4|b|^2 + 4|c|^2 + 4|d|^2$, equation (3.1) yields

$$\begin{aligned}|X^l(t)|^2 \leq 4|\zeta(0)|^2 + 4|\int_0^t g(s, X_s^{l-1})ds|^2 + 4|\int_0^t h(s, X_s^{l-1})d\langle B, B\rangle(s)|^2 \\ + 4|\int_0^t w(s, X_s^{l-1})dB(s)|^2.\end{aligned}$$

Taking G-expectation on both sides, using the Burkholder-Davis-Gundy (BDG) inequalities [6] and Hölder inequality (lemma 2.3) we have

$$\begin{aligned}E|X^l(t)|^2 &\leq 4E|\zeta(0)|^2 + 4C_1E\int_0^t |g(s, X_s^{l-1})|^2 ds \\ &+ 4C_2E\int_0^t |h(s, X_s^{l-1})|^2 ds + 4C_3\int_0^t |w(s, X_s^{l-1})|^2 ds \\ &\leq 4E|\zeta(0)|^2 + 8C_1E\int_0^t (|g(s, X_s^{l-1}) - g(s,0)|^2 + |g(s,0)|^2)ds \\ &+ 8C_2E\int_0^t (|h(s, X_s^{l-1}) - h(s,0)|^2 + |h(s,0)|^2)ds \\ &+ 8C_3\int_0^t (|w(s, X_s^{l-1}) - w(s,0)|^2 + |w(s,0)|^2)d(s) \\ &\leq 4E|\zeta(0)|^2 + 8C_1E\int_0^t |g(s,0)|^2 ds + 8C_1E\int_0^t |g(s, X_s^{l-1}) - g(s,0)|^2 ds \\ &+ 8C_2E\int_0^t |h(s,0)|^2 d(s) + 8C_2E\int_0^t |h(s, X_s^{l-1}) - h(s,0)|^2 ds \\ &+ 8C_3\int_0^t |w(s,0)|^2 ds + 8C_3\int_0^t |w(s, X_s^{l-1}) - w(s,0)|^2 ds\end{aligned}$$

By assumptions A_i and A_{ii}, the above inequality yields

$$\begin{aligned}E|X^l(t)|^2 &\leq 4E|\zeta(0)|^2 + 8C_1KT + 8C_2KT + 8C_3KT \\ &+ 8C_1E\int_0^t \lambda(|X_s^{l-1})|^2)ds + 8C_2E\int_0^t \lambda(|X_s^{l-1})|^2)d(s) + 8C_3\int_0^t \lambda(|X_s^{l-1})|^2)d(s) \\ &= 4E|\zeta(0)|^2 + 8KT(C_1+C_2+C_3) + 8(C_1+C_2+C_3)E\int_0^t \lambda(|X_s^{l-1})|^2)ds \\ &\leq 4E|\zeta(0)|^2 + 8KT(C_1+C_2+C_3) + 8a(C_1+C_2+C_3)T \\ &+ 8b(C_1+C_2+C_3)E\int_0^t |X_s^{l-1})|^2 ds \\ &= K_1 + 8b(C_1+C_2+C_3)E\int_0^t |X_s^{l-1})|^2 ds,\end{aligned}$$

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where $K_1 = 4E|\zeta(0)|^2 + 8C_0KT + 8aC_0T$. and $C_0 = C_1 + C_2 + C_3$. Noting that

$$\sup_{0\le s\le t} |X_s^l|^2 \le \sup_{0\le s\le t}\sup_{-\theta\le u\le 0} |X^l(s+u)|^2 \le \sup_{-\theta\le q\le t} |X^l(q)|^2 \le |\zeta|^2 + \sup_{0\le q\le t} |X^l(q)|^2,$$

we have

$$\sup_{-\theta\le q\le t} E|X^l(q)|^2 \le E|\zeta|^2 + K_1 + 8b(C_1+C_2+C_3)E\int_0^t \sup_{-\theta\le q\le t} |X^{l-1}(q)|^2 ds.$$

Again noting that for any $j \ge 1$

$$\max_{1\le l\le j} E|X_s^{l-1}|^2 \le E|\zeta|^2 + \max_{1\le l\le j} E|X^l(q)|^2,$$

we obtain

$$\begin{aligned}\max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2 &\le E|\zeta|^2 + K_1 + 8b(C_1+C_2+C_3)\int_0^t [E|\zeta|^2 + \max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2]ds \\ &\le E|\zeta|^2 + K_1 + 8b(C_1+C_2+C_3)TE|\zeta|^2 + \int_0^t \max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2 ds \\ &= K_2 + 8b(C_1+C_2+C_3)\int_0^t \max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2 ds,\end{aligned}$$

where $K_2 = K_1 + (1+8bC_0T)E|\zeta|^2$. Now the Gronwall inequality (lemma 2.4) yields

$$\max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(t)|^2 \le C,$$

where $C = K_2e^{8bC_0T}$, but j is arbitrary, so

$$\sup_{-\theta\le t\le T} E|X^l(t)|^2 \le C.$$

The proof is complete. □

Lemma 3.2. *Under the assumptions A_i and A_{ii} there exists a positive constant C^* such that for all $l, d \ge 1$,*

$$\begin{aligned}E\sup_{-\theta\le s\le t} |X^{l+d}(s) - X^l(s)|^2 &\le \hat{C}\int_0^t \lambda(E\sup_{-\theta\le q\le s} |X^{l+d-1}(q) - X^{l-1}(q)|^2)ds \\ &\le C^* t.\end{aligned}$$

Proof. Using the basic inequality $|a+b+c|^2 \le 3|a|^2 + 3|b|^2 + 3|c|^2$, equation (3.1) yields

$$\begin{aligned}|X^{l+d}(t) - X^l(t)|^2 \le\ & 3|\int_0^t [g(s, X_s^{l+d-1}) - g(s, X_s^{l-1})]ds|^2 + 3|\int_0^t [h(s, X_s^{l+d-1}) - h(s, X_s^{l-1})]d\langle B, B\rangle(s)|^2 \\ & + 3|\int_{t_0}^t [w(s, X_s^{l+d-1}) - w(s, X_s^{l-1})]dB(s)|^2\end{aligned}$$

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Taking G-expectation on both sides, using the BDG inequalities [6], Jensen inequality $E(\lambda(x)) \leq \lambda(E(x))$, Holder inequality and assumptions A_i, A_i it gives

$$\begin{aligned} E[\sup_{-\theta\leq s\leq t}|X^{l+d}(s)-X^{l}(s)|^2] &\leq 3C_1\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds \\ &+ 3C_2\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds \\ &+ 3C_3\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds \\ &\leq 3(C_1+C_2+C_3)\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds. \end{aligned}$$

$$E[\sup_{-\theta\leq s\leq t}|X^{l+d}(s)-X^{l}(s)|^2] \leq \hat{C}\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds,$$

where $\hat{C} = 3C_0$. Finally, using lemma 3.1 it yields

$$E[\sup_{-\theta\leq s\leq t}|X^{l+d}(s)-X^{l}(s)|^2] \leq \hat{C}\lambda(4C)t = C^*t,$$

where $C^* = \hat{C}\lambda(4C)$. The proof is complete. □

4 Existence and uniqueness results for G-SFDEs

We introduce the following new notations to prepare a key lemma. Choose $T_1 \in [0,T]$ such that for all $t \in [0,T_1]$

$$\hat{C}\lambda(C^*t) \leq C^*. \tag{4.1}$$

For all $l, d \geq 1$, define the following recursive function

$$\phi_1(t) = C^*t. \tag{4.2}$$

$$\begin{aligned} \phi_{l+1}(t) &= \hat{C}\int_0^t \lambda(\phi_l(s))ds, \\ \phi_{l,d}(t) &= E[\sup_{-\theta\leq q\leq t}|X^{l+d}(q)-X^{l}(q)|^2]. \end{aligned} \tag{4.3}$$

Lemma 4.1. *Under the hypothesis A_i and A_{ii} for any $d \geq 1$ and all $l \geq 1$ there exists a positive $T_1 \in [0,T]$ such that*

$$0 \leq \phi_{l,d}(t) \leq \phi_l(t) \leq \phi_{l-1}(t) \leq ... \leq \phi_1(t), \tag{4.4}$$

for all $t \in [0,T_1]$.

Proof. We use mathematical induction to prove the inequality (4.4). Using the definition of function $\phi(.)$ and lemma 3.2, we have

$$\phi_{1,d}(t) = E[\sup_{-\theta \le q \le t} |X^{1+d}(q) - X^1(q)|^2] \le C^* t = \phi_1(t).$$

$$\begin{aligned}
\phi_{2,d}(t) &= E[\sup_{-\theta \le q \le t} |X^{2+d}(q) - X^2(q)|^2] \\
&\le \hat{C} \int_0^t \lambda(E[\sup_{-\theta \le q \le t} |X^{1+d}(q) - X^1(q)|^2]) ds \\
&\le \hat{C} \int_0^t \lambda(\phi_1(s)) ds = \phi_2(t).
\end{aligned}$$

Using (4.1), we have

$$\phi_2(t) = \hat{C} \int_0^t \lambda(\phi_1(s)) ds = \int_0^t \hat{C} \lambda(C^* t) ds \le C^* t = \phi_1(t).$$

Hence for all $t \in [0, T_1]$, we derive that $\phi_{2,d}(t) \le \phi_2(t) \le \phi_1(t)$. Next, suppose that the inequality (4.4) holds for some $l \ge 1$. We now show that lemma 4.1 is valid for $l+1$, as follows

$$\begin{aligned}
\phi_{l+1,d}(t) &= E[\sup_{-\theta \le q \le t} |X^{l+d+1}(q) - X^{l+1}(q)|^2] \\
&\le \hat{C} \int_0^t \lambda(E[\sup_{-\theta \le q \le s} |X^{l+d}(q) - X^l(q)|^2]) ds \\
&= \hat{C} \int_0^t \lambda(\phi_{l,d}(s)) ds \\
&\le \hat{C} \int_0^t \lambda(\phi_l(s)) ds \\
&= \phi_{l+1}(t).
\end{aligned}$$

Also

$$\phi_{l+1}(t) = \hat{C} \int_0^t \lambda(\phi_l(s)) ds \le \hat{C} \int_0^t \lambda(\phi_{l-1}(s)) ds = \phi_l(s).$$

Hence for all $t \in [0, T_1]$, we derive that $\phi_{l+1,d}(t) \le \phi_{l+1}(t) \le \phi_l(s)$, that is, lemma 4.1 holds for $l+1$. The proof is complete. □

Theorem 4.2. *Let assumptions A_i and A_{ii} hold. Then the stochastic system* (1.1) *with initial data* (1.2) *has a unique solution.*

Proof. We split the whole proof in two steps. First, we show uniqueness and then existence. Let system (1.1) with initial data (1.2) has two solutions $X(t)$ and $Y(t)$. Then we have

$$\begin{aligned}
|X(t) - Y(t)| \le& \int_0^t |g(s, X_s) - g(s, Y_s)| ds + \int_0^t |h(s, X_s) - h(s, Y_s)| d\langle B, B\rangle(s) \\
&+ \int_0^t |w(s, X_s) - w(s, Y_s)| dB(s).
\end{aligned}$$

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Taking G-expectation on both sides and using the basic inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, Hölder inequality and BDG inequalities [6], it follows

$$\begin{aligned} E|X(t)-Y(t)|^2 \leq 3C_1 \int_0^t E|g(s,X_s)-g(s,Y_s)|^2 ds + 3C_2 \int_0^t E|h(s,X_s)-h(s,Y_s)|^2 ds \\ + 3C_3 \int_0^t E|w(s,X_s)-w(s,Y_s)|ds. \end{aligned}$$

Using assumptions A_i and A_{ii} we have

$$E[\sup_{-\theta\leq q\leq t} |X(q)-Y(q)|^2] \leq 3(C_1+C_2+C_3) \int_0^t \lambda(E[\sup_{-\theta\leq q\leq s} |X(q)-Y(q)|^2])ds,$$

Then lemma 2.5 and lemma 2.6 gives $E[\sup_{-\theta\leq q\leq t} |X(q)-Y(q)|^2] = 0$, $t \in [0,T]$. The proof of uniqueness is complete.

Next we show existence. We note that on $t \in [0,T_1]$, $\phi_l(t)$ is continuous. For $l \geq 1$, it is decreasing on $t \in [0,T_1]$. By dominated convergence theorem, we define the function $\phi(t)$ as follows

$$\phi(t) = \lim_{l\to\infty} \phi_l(t) = \lim_{l\to\infty} \hat{C} \int_0^t \lambda(\phi_{l-1}(s))ds = \hat{C} \int_0^t \lambda(\phi(s))ds, \quad 0 \leq t \leq T_1.$$

So,

$$\phi(t) \leq \phi(0) + \hat{C} \int_0^t \lambda(\phi(s))ds.$$

Thus for all $0 \leq t \leq T_1$, lemma 2.5 and lemma 2.6 follow that $\phi(t) = 0$. From lemma 4.1 for all $t \in [0,T_1]$ we get $\phi_{l,d}(s) \leq \phi_l(s) \to 0$ as $l \to \infty$, which yields $E|X^{l+d}(t) - X^l(t)|^2 \to 0$ as $l \to \infty$. By the property of function $\lambda(.)$, assumptions A_i, A_{ii} and completeness of L^2, it follows that for all $t \in [0,T_1]$,

$$g(t,X_t^l) \to g(t,X_t), h(t,X_t^l) \to h(t,X_t), w(t,X_t^l) \to w(t,X_t) \ \ in\ L^2 \ \ as\ \ l \to \infty.$$

Hence for all $t \in [0,T_1]$,

$$\begin{aligned} \lim_{l\to\infty} X^l(t) = \zeta(0) + \lim_{l\to\infty} \int_0^t g(s,X_s^{l-1})ds \\ + \lim_{l\to\infty} \int_0^t h(s,X_s^{l-1})d\langle B,B\rangle(s) + \lim_{l\to\infty} \int_0^t w(s,X_s^{l-1})dB(s), \end{aligned}$$

that is,

$$X(t) = \zeta(0) + \int_0^t g(s,X_s)ds + \int_0^t h(s,X_s)d\langle B,B\rangle(s) + \int_0^t w(s,X_s)dB(s).$$

Thus $X(t)$ is a unique solution of stochastic system (1.1) with initial data (1.2) on $t \in [0,T_1]$. Thus by iteration, one can obtain that the system (1.1) has a unique solution on $t \in [0,T]$. The proof is complete. □

5 Dependence of solutions

In this section, we use lemma 2.5 and lemma 2.6 to give continuous dependence of solutions for stochastic system (1.1) with initial data (1.2).

Theorem 5.1. *Let assumptions A_i and A_{ii} hold. Assume $X(t)$ and $Y(t)$ be two solutions of dynamical system (1.1) with initial data ζ and ξ respectively. If for all $\epsilon > 0$ and $t \in [0, T]$ there exists $\delta(\epsilon) > 0$ such that $E|\zeta - \xi|^2 < \delta(\epsilon)$, then*

$$E|X(t) - Y(t)|^2 \leq \epsilon.$$

Proof. Since $X(t)$ and $Y(t)$ are any two solutions of system (1.1). It follows that for any $t \in [0, T]$,

$$X(t) = \zeta(0) + \int_0^t g(s, X_s)ds + \int_0^t h(s, X_s)d\langle B, B\rangle(s) + \int_0^t w(s, X_s)dB(s) \ \ q.s.$$

$$Y(t) = \xi(0) + \int_0^t g(s, Y_s)ds + \int_0^t h(s, Y_s)d\langle B, B\rangle(s) + \int_0^t w(s, Y_s)dB(s) \ \ q.s.$$

Then

$$\begin{aligned} X(t) - Y(t) = \zeta(0) - \xi(0) &+ \int_0^t [g(s, X_s) - g(s, Y_s)]ds + \int_0^t [h(s, X_s) - h(s, Y_s)]d\langle B, B\rangle(s) \\ &+ \int_0^t [w(s, X_s) - w(s, Y_s)]dB(s) \ \ q.s. \end{aligned}$$

Taking G-expectation on both sides, using the fundamental inequality $(a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)$, BDG inequalities [6] and Hölder inequality, it follows

$$E[\sup_{-\theta \leq r \leq t} |X(r) - Y(r)|^2] \leq 4E|\zeta(0) - \xi(0)|^2 + 4(C_1 + C_2 + C_3) \int_0^t \lambda(E[\sup_{-\theta \leq r \leq t} |X(r) - Y(r)|^2])ds.$$

Thus from lemma 2.5 and 2.6 we have

$$E[|X(t) - Y(t)|^2] \leq \epsilon,$$

for $t \in [0, T]$. The proof is complete. □

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Approximation of a kind of new Bernstein-Bézier type operators

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Abstract. In this paper, a kind of new Bernstein-Bézier type operators is introduced. The Korovkin type approximation theorem of these operators is investigated. The rates of convergence of these operators are studied by means of modulus of continuity. Then, by using the Ditzian-Totik modulus of smoothness, a direct theorem concerned with an approximation for these operators is also obtained.

Keywords: Bernstein-Bézier type operators; Korovich type approximation theorem; rate of convergence; direct theorem; modulus of smoothness

Mathematical subject classification: 41A10, 41A25, 41A36

1. Introduction

In view of the Bézier basis function, which was introduced by Bézier [1], in 1983, Chang [2] defined the generalized Bernstein-Bézier polynomials for any $\alpha > 0$, and a function f defined on $[0, 1]$ as follows:

$$B_{n,\alpha}(f;x) = \sum_{k=0}^{n} f(\frac{k}{n})[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)], \tag{1}$$

where $J_{n,n+1}(x) = 0$, and $J_{n,k}(x) = \sum\limits_{i=k}^{n} P_{n,i}(x)$, $k = 0, 1, ..., n$, $P_{n,i}(x) = \left(\begin{array}{c} n \\ i \end{array} \right) x^i(1-x)^{n-i}$. $J_{n,k}(x)$ is the Bézier basis function of degree n.

Obviously, when $\alpha = 1$, $B_{n,\alpha}(f;x)$ become the well-known Bernstein polynomials $B_n(f;x)$, and for any $x \in [0, 1]$, we have $1 = J_{n,0}(x) > J_{n,1}(x) > ... > J_{n,n}(x) = x^n, J_{n,k}(x) - J_{n,k+1}(x) = P_{n,k}(x)$.

During the last ten years, the Bézier basis function was extensively used for constructing various generalizations of many classical approximation processes. Some Bézier type operators, which are based on the Bézier basis function, have been introduced and studied (e.g., see [3-9]).

In 2013, Ren [10] introduced generalized Bernstein operators as follows:

$$E_{n,\beta}(f;x) = f(0)P_{n,0}(x) + \sum_{k=1}^{n-1} P_{n,k}(x)F_{n,k}^{(\beta)}(f) + f(1)P_{n,n}(x), \tag{2}$$

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where $f \in C[0,1]$, $x \in [0,1]$, $P_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array} \right) x^k(1-x)^{n-k}$, $k = 0,1,...,n$, and

$$F_{n,k}^{(\beta)}(f) = \frac{1}{B(nk, n(n-k))} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} f(\beta t + (1-\beta)\frac{k}{n})dt, \quad (3)$$

where $k = 1,...,n-1$, $\beta \in [0,1]$, $B(.,.)$ is the beta function.

The moments of the operators $E_{n,\beta}(f;x)$ were obtained as follows (see [10]).

Remark For $E_{n,\beta}(t^j;x)$, $j = 0,1,2$, we have

(i) $E_{n,\beta}(1;x) = 1$;
(ii) $E_{n,\beta}(t;x) = x$;
(iii) $E_{n,\beta}(t^2;x) = x^2 + \left[\frac{1}{n} + \frac{(n-1)\beta^2}{(n^2+1)n}\right] x(1-x)$.

In the present paper, we will study the Bézier variant of the generalized Bernstein operators $E_{n,\beta}(f;x)$ given by (2). We introduce Bernstein-Bézier type operators as follows:

$$E_{n,\beta}^{(\alpha)}(f;x) = f(0)Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(f) + f(1)Q_{n,n}^{(\alpha)}(x), \quad (4)$$

where $f \in C[0,1]$, $x \in [0,1]$, $\beta \in [0,1]$, $\alpha > 0$, $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{n,n+1}(x) = 0$, $J_{n,k}(x) = \sum_{i=k}^{n} P_{n,i}(x)$, $k = 0,1,...,n$, $P_{n,i}(x) = \left(\begin{array}{c} n \\ i \end{array} \right) x^i(1-x)^{n-i}$, and $F_{n,k}^{(\beta)}(f)$ is defined as above (3).

It is clear that $E_{n,\beta}^{(\alpha)}(f;x)$ are bounded and positive on C[0,1]. When $\alpha = 1$, $E_{n,\beta}^{(\alpha)}(f;x)$ become the operators $E_{n,\beta}(f;x)$. When $\beta = 0$, $E_{n,\beta}^{(\alpha)}(f;x)$ become the generalized Bernstein-Bézier operators $B_{n,\alpha}(f;x)$.

The goal of this paper is to study the approximation properties of these operators with the help of the Korovkin type approximation theorem. We also estimate the rates of convergence of these operators by using a modulus of continuity. Then we obtain the direct theorem concerned with an approximation for these operators by means of the Ditzian-Totik modulus of smoothness.

In the paper, for $f \in C[0,1]$, we denote $\|f\| = max\{|f(x)| : x \in [0,1]\}$. $\omega(f,\delta)$ $(\delta > 0)$ denotes the usual modulus of continuity of $f \in C[0,1]$.

2. Auxiliary results

In the sequel, we shall need the following auxiliary results.

Lemma 1 (see [2]) *Let* $\alpha > 0$. *We have*

(i) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} J_{n,k}^{\alpha}(x) = x$ *uniformly on* $[0,1]$;
(ii) $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} kJ_{n,k}^{\alpha}(x) = \frac{x^2}{2}$ *uniformly on* $[0,1]$.

Lemma 2 *Let $\alpha > 0$. We have*

$$\text{(i) } E_{n,\beta}^{(\alpha)}(1;x) = 1;$$

$$\text{(ii) } \lim_{n\to\infty} E_{n,\beta}^{(\alpha)}(t;x) = x \textit{ uniformly on } [0,1];$$

$$\text{(iii) } \lim_{n\to\infty} E_{n,\beta}^{(\alpha)}(t^2;x) = x^2 \textit{ uniformly on } [0,1].$$

Proof By simple calculation, we obtain $F_{n,k}^{(\beta)}(1) = 1$, $F_{n,k}^{(\beta)}(t) = \frac{k}{n}$, $F_{n,k}^{(\beta)}(t^2) = \frac{\beta^2}{n^2+1} \cdot \frac{k}{n} + (1 - \frac{\beta^2}{n^2+1})\frac{k^2}{n^2}$.

(i) Since $\sum\limits_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) = 1$, by (4) we can get $E_{n,\beta}^{(\alpha)}(1;x) = 1$.

(ii) By (4), we have

$$\begin{aligned}
&E_{n,\beta}^{(\alpha)}(t;x)\\
&= \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)\frac{k}{n} + Q_{n,n}^{(\alpha)}(x)\\
&= [J_{n,1}^{\alpha}(x) - J_{n,2}^{\alpha}(x)]\frac{1}{n} + ... + [J_{n,n-1}^{\alpha}(x) - J_{n,n}^{\alpha}(x)]\frac{n-1}{n} + J_{n,n}^{\alpha}(x)\frac{n}{n}\\
&= \frac{1}{n}\sum_{k=1}^{n} J_{n,k}^{\alpha}(x),
\end{aligned}$$

thus, by Lemma 1 (i), we have $\lim\limits_{n\to\infty} E_{n,\beta}^{(\alpha)}(t;x) = x$ uniformly on $[0,1]$.

(iii) By (4), we have

$$\begin{aligned}
&E_{n,\beta}^{(\alpha)}(t^2;x)\\
&= \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)\left[\frac{\beta^2}{n^2+1} \cdot \frac{k}{n} + (1 - \frac{\beta^2}{n^2+1})\frac{k^2}{n^2}\right] + Q_{n,n}^{(\alpha)}(x)\\
&= \frac{\beta^2}{n^2+1} \cdot \frac{1}{n}\sum_{k=1}^{n} kQ_{n,k}^{(\alpha)}(x) + (1 - \frac{\beta^2}{n^2+1}) \cdot \frac{1}{n^2}\sum_{k=1}^{n} k^2 Q_{n,k}^{(\alpha)}(x)\\
&= \frac{\beta^2}{n^2+1} \cdot \frac{1}{n}\sum_{k=1}^{n} J_{n,k}^{\alpha}(x) + (1 - \frac{\beta^2}{n^2+1}) \cdot \frac{1}{n^2}\sum_{k=1}^{n}(2k-1)J_{n,k}^{\alpha}(x),
\end{aligned}$$

thus, by Lemma 1, we have $\lim\limits_{n\to\infty} E_{n,\beta}^{(\alpha)}(t^2;x) = x^2$ uniformly on $[0,1]$.

Lemma 3 (see [11]) *For $x \in [0,1]$, $k = 0,1,...,n$, we have*

$$0 \le Q_{n,k}^{(\alpha)}(x) \le \begin{cases} \alpha P_{n,k}(x), & \alpha \ge 1;\\ P_{n,k}^{\alpha}(x), & 0 < \alpha < 1. \end{cases}$$

Lemma 4 (see [12]) *For $0 < \alpha < 1$, $\gamma > 0$, we have*

$$\sum_{k=0}^{n} |k - nx|^{\gamma} P_{n,k}^{\alpha}(x) \le (n+1)^{1-\alpha}(A_{\frac{\gamma}{\alpha}})^{\alpha} n^{\frac{\gamma}{2}},$$

where the constant A_s only depends on s.

Lemma 5 *For $\alpha \geq 1$, we have*

$$\text{(i) } E_{n,\beta}^{(\alpha)}((t-x)^2;x) \leq \frac{\alpha}{4}(1+\frac{\beta^2}{5})\cdot\frac{1}{n};$$

$$\text{(ii) } E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\cdot\sqrt{\frac{1}{n}}.$$

Proof Let $\alpha \geq 1$.

(i) By (4), Lemma 3 and Remark 1, we obtain

$$\begin{aligned}
&E_{n,\beta}^{(\alpha)}((t-x)^2;x)\\
&= x^2 Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2 Q_{n,n}^{(\alpha)}(x)\\
&\leq \alpha[x^2 P_{n,0}(x) + \sum_{k=1}^{n-1} P_{n,k}(x) F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2 P_{n,n}(x)]\\
&= \alpha E_{n,\beta}((t-x)^2;x)\\
&= \frac{\alpha}{n}\left(1+\frac{n-1}{n^2+1}\beta^2\right) x(1-x). \qquad (5)
\end{aligned}$$

Since $\max\limits_{0\leq x\leq 1} x(1-x) = \frac{1}{4}$, and for any $n \in N$, one can get $\frac{n-1}{n^2+1} \leq \frac{1}{5}$, so we have

$$E_{n,\beta}^{(\alpha)}((t-x)^2;x) \leq \frac{\alpha}{4}(1+\frac{\beta^2}{5})\cdot\frac{1}{n}.$$

(ii) In view of $E_{n,\beta}^{(\alpha)}(1;x) = 1$, by the Cauchy-Schwarz inequality, we have

$$E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)},$$

thus, we get $E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\cdot\sqrt{\frac{1}{n}}$.

Lemma 6 *For $0 < \alpha < 1$, we have*

$$\text{(i) } E_{n,\beta}^{(\alpha)}((t-x)^2;x) \leq M_\alpha^{(\beta)} n^{-\alpha};$$

$$\text{(ii) } E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{M_\alpha^{(\beta)}}\cdot n^{-\frac{\alpha}{2}}.$$

Where the constant $M_\alpha^{(\beta)}$ only depends on α, β.

Proof Let $0 < \alpha < 1$.

(i) In view of (4), Lemma 3 and $F_{n,k}^{(\beta)}((t-x)^2) = \frac{(k-nx)^2}{n^2} + \frac{\beta^2}{n^2+1}(\frac{k}{n} - \frac{k^2}{n^2})$,

we obtain

$$\begin{aligned}
&E_{n,\beta}^{(\alpha)}((t-x)^2;x)\\
&= x^2Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2Q_{n,n}^{(\alpha)}(x)\\
&\le x^2P_{n,0}^{\alpha}(x) + \sum_{k=1}^{n-1} P_{n,k}^{\alpha}(x)F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2P_{n,n}^{\alpha}(x)\\
&= \sum_{k=0}^{n} P_{n,k}^{\alpha}(x)\left[\frac{(k-nx)^2}{n^2} + \frac{\beta^2}{n^2+1}(\frac{k}{n}-\frac{k^2}{n^2})\right]\\
&= \frac{1}{n^2}\sum_{k=0}^{n}(k-nx)^2P_{n,k}^{\alpha}(x) + \frac{\beta^2}{n^2+1}\sum_{k=0}^{n}P_{n,k}^{\alpha}(x)(\frac{k}{n}-\frac{k^2}{n^2})\\
&:= I_1 + I_2.
\end{aligned}$$

By Lemma 4, we have $I_1 \le \frac{n+1}{n}(n+1)^{-\alpha}(A_{\frac{2}{\alpha}})^{\alpha} \le 2(A_{\frac{2}{\alpha}})^{\alpha}n^{-\alpha}$, where the constant $A_{\frac{2}{\alpha}}$ only depends on α.

Using the Hölder inequality, we have $\sum\limits_{k=0}^{n} P_{n,k}^{\alpha}(x) \le (n+1)^{1-\alpha}[\sum\limits_{k=0}^{n} P_{n,k}(x)]^{\alpha}$, and $(\frac{k}{n}-\frac{k^2}{n^2}) \le 1$, so we have

$$I_2 \le \frac{\beta^2}{n^2+1}(n+1)^{1-\alpha}[\sum_{k=0}^{n} P_{n,k}(x)]^{\alpha} = \frac{\beta^2}{n^2+1}(n+1)^{1-\alpha} \le \beta^2n^{-\alpha}.$$

Denote $M_{\alpha}^{(\beta)} = 2(A_{\frac{2}{\alpha}})^{\alpha} + \beta^2$, then we can get $E_{n,\beta}^{(\alpha)}((t-x)^2;x) \le M_{\alpha}^{(\beta)}n^{-\alpha}$.

(ii) Since

$$E_{n,\beta}^{(\alpha)}(|t-x|;x) \le \sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)},$$

thus, we get

$$E_{n,\beta}^{(\alpha)}(|t-x|;x) \le \sqrt{M_{\alpha}^{(\beta)}}\cdot n^{-\frac{\alpha}{2}}.$$

Lemma 7 *For* $f \in C[0,1],\ x \in [0,1]$ *and* $\alpha > 0,$ *we have*

$$\mid E_{n,\beta}^{(\alpha)}(f;x) \mid \le \parallel f \parallel .$$

Proof By (4) and Lemma 2 (i), we have

$$\mid E_{n,\beta}^{(\alpha)}(f;x) \mid \le \|f\|E_{n,\beta}^{(\alpha)}(1;x) = \|f\|.$$

3. Main results

First of all we give the following convergence theorem for the sequence $\{E_{n,\beta}^{(\alpha)}(f;x)\}$.

Theorem 1 *Let* $\alpha > 0.$ *Then the sequence* $\{E_{n,\beta}^{(\alpha)}(f;x)\}$ *converges to* f *uniformly on* $[0,1]$ *for any* $f \in C[0,1]$.

Proof Since $E_{n,\beta}^{(\alpha)}(f;x)$ is bounded and positive on $C[0,1]$, and by Lemma 2, we have $\lim\limits_{n\to\infty}\|E_{n,\beta}^{(\alpha)}(e_j;\cdot)-e_j\|=0$ for $e_j(t)=t^j$, $j=0,1,2$. So, according to the well-known Bohman-korovkin theorem ([13, P.40, Theorem 1.9]), we see that the sequence $\{E_{n,\beta}^{(\alpha)}(f;x)\}$ converges to f uniformly on [0,1] for any $f\in C[0,1]$.

Next we estimate the rates of convergence of the sequence $\{E_{n,\beta}^{(\alpha)}\}$ by means of the modulus of continuity.

Theorem 2 *Let* $f\in C[0,1]$, $x\in[0,1]$. *Then*

(i) *when* $\alpha\geq 1$, *we have* $\|E_{n,\beta}^{(\alpha)}(f;\cdot)-f\|\leq\left[1+\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\right]\omega(f,\frac{1}{\sqrt{n}})$;

(ii) *when* $0<\alpha<1$, *we have* $\|E_{n,\beta}^{(\alpha)}(f;\cdot)-f\|\leq(1+\sqrt{M_\alpha^{(\beta)}})\omega(f,n^{-\frac{\alpha}{2}})$.

Where the constant $M_\alpha^{(\beta)}$ *only depends on* α,β.

Proof (i) When $\alpha\geq 1$, by Lemma 2 (i), we have

$$
\begin{aligned}
&|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\\
&\leq|f(0)-f(x)|Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(|f(t)-f(x)|)+|f(1)-f(x)|Q_{n,n}^{(\alpha)}(x)\\
&\leq\omega(f,|0-x|)Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(\omega(f,|t-x|))+\omega(f,|1-x|)Q_{n,n}^{(\alpha)}(x)\\
&\leq(1+\sqrt{n}|0-x|)\omega(f,\frac{1}{\sqrt{n}})Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}((1+\sqrt{n}|t-x|)\omega(f,\frac{1}{\sqrt{n}}))\\
&\quad+(1+\sqrt{n}|1-x|)\omega(f,\frac{1}{\sqrt{n}})Q_{n,n}^{(\alpha)}(x)\\
&\leq\omega(f,\frac{1}{\sqrt{n}})+\sqrt{n}\omega(f,\frac{1}{\sqrt{n}})E_{n,\beta}^{(\alpha)}(|t-x|;x),
\end{aligned}
$$

so, by Lemma 5 (ii), we obtain

$$
|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\leq\left[1+\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\right]\omega(f,\frac{1}{\sqrt{n}}).
$$

The desired result follows immediately.

(ii) When $0<\alpha<1$, by Lemma 2 (i), we have

$$
\begin{aligned}
&|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\\
&\leq\omega(f,|0-x|)Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(\omega(f,|t-x|))+\omega(f,|1-x|)Q_{n,n}^{(\alpha)}(x)\\
&\leq(1+n^{\frac{\alpha}{2}}|0-x|)\omega(f,n^{-\frac{\alpha}{2}})Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(1+n^{\frac{\alpha}{2}}|t-x|)\omega(f,n^{-\frac{\alpha}{2}})\\
&\quad+(1+n^{\frac{\alpha}{2}}|1-x|)\omega(f,n^{-\frac{\alpha}{2}})Q_{n,n}^{(\alpha)}(x)\\
&=\omega(f,n^{-\frac{\alpha}{2}})+n^{\frac{\alpha}{2}}\omega(f,n^{-\frac{\alpha}{2}})E_{n,\beta}^{(\alpha)}(|t-x|;x),
\end{aligned}
$$

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so, by Lemma 6 (ii), we obtain

$$|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \leq (1 + \sqrt{M_\alpha^{(\beta)}})\omega(f, n^{-\frac{\alpha}{2}}).$$

The desired result follows immediately.

Theorem 3 *Let $f \in C^1[0,1]$, $x \in [0,1]$. Then*
(i) *when $\alpha \geq 1$, we have*

$$\begin{aligned}|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| &\leq \|f'\|\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})} \cdot \sqrt{\frac{1}{n}} \\ &+ \omega(f', \frac{1}{\sqrt{n}})\left[1 + \sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\right] \cdot \sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})} \cdot \sqrt{\frac{1}{n}};\end{aligned}$$

(ii) *when $0 < \alpha < 1$, we have*

$$|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \leq \|f'\|\sqrt{M_\alpha^{(\beta)} n^{-\alpha}} + \omega(f', n^{-\frac{\alpha}{2}})(1 + \sqrt{M_\alpha^{(\beta)}})\sqrt{M_\alpha^{(\beta)} n^{-\alpha}}.$$

Where the constant $M_\alpha^{(\beta)}$ only depends on α, β.

Proof Let $f \in C^1[0,1]$. For any $t, x \in [0,1]$, $\delta > 0$, we have

$$\begin{aligned}|f(t) - f(x) - f'(x)(t-x)| &\leq |\int_x^t |f'(u) - f'(x)|du| \\ &\leq \omega(f', |t-x|)|t-x| \\ &\leq \omega(f', \delta)(|t-x| + \delta^{-1}(t-x)^2),\end{aligned}$$

hence, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&|E_{n,\beta}^{(\alpha)}(f(t) - f(x) - f'(x)(t-x);x)| \\ &\leq \omega(f', \delta)\left(E_{n,\beta}^{(\alpha)}(|t-x|;x) + \delta^{-1}E_{n,\beta}^{(\alpha)}((t-x)^2;x)\right) \\ &\leq \omega(f', \delta)\left[\sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\right. \\ &\quad \left.+\delta^{-1}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}\right]\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}.\end{aligned}$$

So, we get

$$\begin{aligned}&|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \\ &\leq \|f'\|E_{n,\beta}^{(\alpha)}(|t-x|;x) \\ &+\omega(f', \delta)\left[1 + \delta^{-1}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}\right]\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}. \qquad (6)\end{aligned}$$

(i) When $\alpha \geq 1$, taking $\delta = \frac{1}{\sqrt{n}}$ in (6), by Lemma 5 and inequality (6), we obtain the desired result.
(ii)When $0 < \alpha < 1$, taking $\delta = n^{-\frac{\alpha}{2}}$ in (6), by Lemma 6 and inequality (6), we obtain the desired result.

Finally we study the direct theorem concerned with an approximation for the sequence $\{E_{n,\beta}^{(\alpha)}\}$ by means of the Ditzian-Totik modulus of smoothness. For the following theorem we shall use some notations.

For $f \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $0 \le \lambda \le 1$, $x \in [0,1]$, let

$$\omega_{\varphi^\lambda}(f,t) = \sup_{0<h\le t} \sup_{x\pm\frac{h\varphi^\lambda(x)}{2}\in[0,1]} |f(x+\frac{h\varphi^\lambda(x)}{2}) - f(x-\frac{h\varphi^\lambda(x)}{2})|$$

be the Ditzian-Totik modulus of first order, and let

$$K_{\varphi^\lambda}(f,t) = \inf_{g\in W_\lambda}\{\| f-g \| + t\|\varphi^\lambda g'\|\} \tag{7}$$

be the corresponding K-functional, where $W_\lambda = \{f | f \in AC_{loc}[0,1], \|\varphi^\lambda f'\| < \infty, \|f'\| < \infty\}$.

It is well known that (see [14])

$$K_{\varphi^\lambda}(f,t) \le C\omega_{\varphi^\lambda}(f,t), \tag{8}$$

for some absolute constant $C > 0$.

Now we state our following main result.

Theorem 4 *Let $f \in C[0,1]$, $\alpha \ge 1$, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$, $0 \le \beta, \lambda \le 1$. Then there exists an absolute constant $C > 0$ such that*

$$|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \le C\omega_{\varphi^\lambda}(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}).$$

Proof Let $g \in W_\lambda$, by Lemma 2 (i) and Lemma 7, we have

$$\begin{aligned}
&|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \\
&\le |E_{n,\beta}^{(\alpha)}(f-g;x)| + |f(x)-g(x)| + |E_{n,\beta}^{(\alpha)}(g;x) - g(x)| \\
&\le 2\|f-g\| + |E_{n,\beta}^{(\alpha)}(g;x) - g(x)|.
\end{aligned} \tag{9}$$

Since $g(t) = \int_x^t g'(u)du + g(x)$, $E_{n,\beta}^{(\alpha)}(1;x) = 1$, so, we have

$$\begin{aligned}
|E_{n,\beta}^{(\alpha)}(g;x) - g(x)| &\le |E_{n,\beta}^{(\alpha)}(\int_x^t |g'(u)|du;x)| \\
&\le \|\varphi^\lambda g'\| E_{n,\beta}^{(\alpha)}(|\int_x^t \varphi^{-\lambda}(u)du|;x).
\end{aligned} \tag{10}$$

By the Hölder inequality, we get

$$|\int_x^t \varphi^{-\lambda}(u)du| \le |\int_x^t \frac{1}{\sqrt{u(1-u)}}du|^\lambda |t-x|^{1-\lambda}, \tag{11}$$

also, in view of $1 \le \sqrt{u} + \sqrt{1-u} < 2$, $0 \le u \le 1$, we have

$$\begin{aligned}
|\int_x^t \frac{1}{\sqrt{u(1-u)}}du| &\le |\int_x^t (\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}})du| \\
&\le 2(|\sqrt{t}-\sqrt{x}| + |\sqrt{1-x}-\sqrt{1-t}|) \\
&\le 2(\frac{|t-x|}{\sqrt{t}+\sqrt{x}} + \frac{|t-x|}{\sqrt{1-t}+\sqrt{1-x}}) \\
&\le 2|t-x|(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}) \\
&\le 4|t-x|\varphi^{-1}(x),
\end{aligned} \tag{12}$$

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thus, by (11) and (12), we obtain

$$|\int_x^t \varphi^{-\lambda}(u)du| \leq C\varphi^{-\lambda}(x)|t-x|. \tag{13}$$

Also, by (10) and (13), we have

$$\begin{aligned} |E_{n,\beta}^{(\alpha)}(g;x) - g(x)| &\leq C\|\varphi^\lambda g'\| E_{n,\beta}^{(\alpha)}(\varphi^{-\lambda}(x)|t-x|;x) \\ &= C\|\varphi^\lambda g'\|\varphi^{-\lambda}(x) E_{n,\beta}^{(\alpha)}(|t-x|;x). \end{aligned} \tag{14}$$

In view of (5) and Lemma 2 (i), by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} E_{n,\beta}^{(\alpha)}(|t-x|;x) &\leq \sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)} \\ &\leq \sqrt{\frac{\alpha}{n}\left(1+\frac{n-1}{n^2+1}\beta^2\right)x(1-x)} \\ &\leq C\frac{\varphi(x)}{\sqrt{n}}, \end{aligned} \tag{15}$$

so, by (14) and (15), we obtain

$$|E_{n,\beta}^{(\alpha)}(g;x) - g(x)| \leq C\|\varphi^\lambda g'\|\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}, \tag{16}$$

thus, by (9) and (16), we have

$$|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \leq 2\|f-g\| + C\|\varphi^\lambda g'\|\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}.$$

Then, in view of (17), (7) and (8), we obtain

$$|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \leq CK_{\varphi^\lambda}(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}) \leq C\omega_{\varphi^\lambda}(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}),$$

where C is a positive constant, in different places, the value of C may be different.

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Approximation by complex Stancu type summation-integral operators in compact disks

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Abstract. In this paper we introduce a class of complex Stancu type summation-integral operators and study the approximation properties of these operators. We obtain a Voronovskaja-type result with quantitative estimate for these operators attached to analytic functions on compact disks. We also study the exact order of approximation. More important, our results show the overconvergence phenomenon for these complex operators.

Keywords: complex Stancu type summation-integral operators; Voronovskaja-type result; Exact order of approximation; Simultaneous approximation; Overconvergence

Mathematical subject classification: 30E10, 41A25 , 41A36

1. Introduction

In 1986, some approximation properties of complex Bernstein polynomials in compact disks were initially studied by Lorentz [11]. Very recently, the problem of the approximation of complex operators has been causing great concern, which is becoming a hot topic of research. A Voronovskaja-type result with quantitative estimate for complex Bernstein polynomials in compact disks was obtained by Gal [3]. Also, in [1-2, 4-10, 12-15] similar results for complex Bernstein-Kantorovich polynomials, Bernstein-Stancu polynomials, Kantorovich-Schurer polynomials, Kantorovich-Stancu polynomials, complex Favard-Szász-Mirakjan operators, complex Beta operators of first kind, complex Baskajov-Stancu operators, complex Bernstein-Durrmeyer operators based on Jacobi weights, complex genuine Durrmeyer Stancu polynomials, complex Schurer-Stancu operators, complex q-Szász-Mirakjan operators, complex q-Gamma operators, and complex q-Durrmeyer type operators were obtained.

The aim of the present article is to obtain approximation results for complex Stancu type summation-integral operators which are defined for $f : [0, 1] \to \mathbf{C}$ continuous on $[0, 1]$ by

$$M_n^{(\alpha,\beta)}(f; z) := p_{n,0}(z)f(\frac{\alpha}{n+\beta}) + \sum_{k=1}^{n-1} p_{n,k}(z)L_{n,k}^{(\alpha,\beta)}(f) + p_{n,n}(z)f(\frac{n+\alpha}{n+\beta}), \quad (1)$$

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where α, β are two given real parameters satisfying the condition $0 \leq \alpha \leq \beta$, $z \in \mathbf{C}, n \in \mathbf{N}$, $L_{n,k}^{(\alpha,\beta)}(f) = \frac{1}{B(n(n-k),nk)} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} f(\frac{nt+\alpha}{n+\beta})dt$ with $B(x,y)$ is Beta function, and $p_{n,k}(z) = \left(\begin{array}{c} n \\ k \end{array}\right) z^k(1-z)^{n-k}$.

Note that, for $\alpha = \beta = 0$, these operators become the complex summation-integral type operators $M_n(f;z) = M_n^{(0,0)}(f;z)$, this case has been investigated in [16].

2. Auxiliary results

In the sequel, we shall need the following auxiliary results.

Lemma 1 *Let $e_m(z) = z^m$, $m \in \mathbf{N} \cup \{0\}$, $z \in \mathbf{C}$, $n \in \mathbf{N}$, $0 \leq \alpha \leq \beta$, we have $M_n^{(\alpha,\beta)}(e_m;z)$ is a polynomial of degree less than or equal to min (m,n) and*

$$M_n^{(\alpha,\beta)}(e_m;z) = \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array}\right) \frac{n^j \alpha^{m-j}}{(n+\beta)^m} M_n(e_j;z).$$

Proof By the definition given by (1) , the proof is easy, here the proof is omitted.

Let $m = 0, 1, 2$, according to [16, Lemma 1] , by simple computation, we have

$$\begin{aligned} M_n^{(\alpha,\beta)}(e_0;z) &= 1; \\ M_n^{(\alpha,\beta)}(e_1;z) &= \frac{nz}{n+\beta} + \frac{\alpha}{n+\beta}; \\ M_n^{(\alpha,\beta)}(e_2;z) &= \frac{n^2}{(n+\beta)^2}\left[\frac{n(n-1)}{n^2+1}z^2 + \frac{n+1}{n^2+1}z\right] \\ &\quad + \frac{2n\alpha z}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}. \end{aligned}$$

Lemma 2 *Let $e_m(z) = z^m$, $m \in \mathbf{N} \cup \{0\}$, $z \in \mathbf{C}$, $n \in \mathbf{N}$, $0 \leq \alpha \leq \beta$, for all $|z| \leq r$, $r \geq 1$, we have $|M_n^{(\alpha,\beta)}(e_m;z)| \leq r^m$.*

Proof The proof follows directly Lemma 1 and [16, Corollary 1].

Lemma 3 *Let $e_m(z) = z^m$, $m, n \in$ N, $z \in \mathbf{C}$ and $0 \leq \alpha \leq \beta$, we have*

$$\begin{aligned} M_n^{(\alpha,\beta)}(e_{m+1};z) &= \frac{z(1-z)n^2}{(n+\beta)(n^2+m)}(M_n^{(\alpha,\beta)}(e_m;z))' \\ &\quad + \frac{(m+n^2z)n + \alpha(n^2+2m)}{(n+\beta)(n^2+m)} M_n^{(\alpha,\beta)}(e_m;z) \\ &\quad - \frac{\alpha m(n+\alpha)}{(n+\beta)^2(n^2+m)} M_n^{(\alpha,\beta)}(e_{m-1};z). \qquad (2) \end{aligned}$$

Proof Let

$$\widetilde{L}_{n,k}^{(\alpha,\beta)}(f) := \frac{1}{B(n(n-k),nk)} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} t f(\frac{nt+\alpha}{n+\beta})dt,$$

$$\widehat{L}_{n,k}^{(\alpha,\beta)}(f) := \frac{1}{B(n(n-k),nk)}\int_0^1 t^{nk-1}(1-t)^{n(n-k)-1}t^2 f(\frac{nt+\alpha}{n+\beta})dt,$$
$$E_n^{(\alpha,\beta)}(f;z) := \sum_{k=1}^{n-1} p_{n,k}(z)L_{n,k}^{(\alpha,\beta)}(f),$$

we have

$$M_n^{(\alpha,\beta)}(f;z) = p_{n,0}(z)f(\frac{\alpha}{n+\beta}) + E_n^{(\alpha,\beta)}(f;z) + p_{n,n}(z)f(\frac{n+\alpha}{n+\beta}),$$
$$\widetilde{L}_{n,k}^{(\alpha,\beta)}(e_m) = \frac{n+\beta}{n}L_{n,k}^{(\alpha,\beta)}(e_{m+1}) - \frac{\alpha}{n}L_{n,k}^{(\alpha,\beta)}(e_m),$$
$$\widehat{L}_{n,k}^{(\alpha,\beta)}(e_m) = (\frac{n+\beta}{n})^2 L_{n,k}^{(\alpha,\beta)}(e_{m+2}) - \frac{2\alpha(n+\beta)}{n^2}L_{n,k}^{(\alpha,\beta)}(e_{m+1}) + (\frac{\alpha}{n})^2 L_{n,k}^{(\alpha,\beta)}(e_m).$$

By simple calculation, we obtain

$$z(1-z)p'_{n,k}(z) = (k-nz)p_{n,k}(z),$$
$$t(1-t)[t^{nk-1}(1-t)^{n(n-k)-1}]' = [nk-1-(n^2-2)t]t^{nk-1}(1-t)^{n(n-k)-1},$$

it follows that

$$\begin{aligned}
&z(1-z)(E_n^{(\alpha,\beta)}(e_m;z))' \\
&= \sum_{k=1}^{n-1}(k-nz)\,p_{n,k}(z)L_{n,k}^{(\alpha,\beta)}(e_m) \\
&= \sum_{k=1}^{n-1} kp_{n,k}(z)\frac{1}{B(n(n-k),nk)}\int_0^1 t^{nk-1}(1-t)^{n(n-k)-1}(\frac{nt+\alpha}{n+\beta})^m dt - nzE_n^{(\alpha,\beta)}(e_m;z) \\
&= \frac{1}{n}\sum_{k=1}^{n-1} p_{n,k}(z)\frac{1}{B(n(n-k),nk)}\int_0^1 [nk-1-(n^2-2)t]t^{nk-1}(1-t)^{n(n-k)-1}(\frac{nt+\alpha}{n+\beta})^m dt \\
&\quad + \frac{1}{n}E_n^{(\alpha,\beta)}(e_m;z) + \frac{n^2-2}{n}\sum_{k=1}^{n-1} p_{n,k}(z)\widetilde{L}_{n,k}^{(\alpha,\beta)}(e_m) - nzE_n^{(\alpha,\beta)}(e_m;z),
\end{aligned}$$

where

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n-1} p_{n,k}(z)\frac{1}{B(n(n-k),nk)}\int_0^1 [nk-1-(n^2-2)t]t^{nk-1}(1-t)^{n(n-k)-1}(\frac{nt+\alpha}{n+\beta})^m dt \\
&= \frac{1}{n}\sum_{k=1}^{n-1} p_{n,k}(z)\frac{1}{B(n(n-k),nk)}\int_0^1 (t-t^2)[t^{nk-1}(1-t)^{n(n-k)-1}]'(\frac{nt+\alpha}{n+\beta})^m dt \\
&= -\frac{1}{n}E_n^{(\alpha,\beta)}(e_m;z) + \frac{2}{n}\sum_{k=1}^{n-1} p_{n,k}(z)\widetilde{L}_{n,k}^{(\alpha,\beta)}(e_m) - \frac{m}{n+\beta}\sum_{k=1}^{n-1} p_{n,k}(z)\widetilde{L}_{n,k}^{(\alpha,\beta)}(e_{m-1}) \\
&\quad + \frac{m}{n+\beta}\sum_{k=1}^{n-1} p_{n,k}(z)\widehat{L}_{n,k}^{(\alpha,\beta)}(e_{m-1}).
\end{aligned}$$

So, in conclusion, we have

$$\begin{aligned}
z(1-z)(E_n^{(\alpha,\beta)}(e_m;z))' = {} & \frac{(n+\beta)(n^2+m)}{n^2}E_n^{(\alpha,\beta)}(e_{m+1};z) \\
& - (\frac{\alpha n^2+mn+2\alpha m}{n^2} + nz)E_n^{(\alpha,\beta)}(e_m;z) \\
& + \frac{\alpha mn+\alpha^2 m}{n^2(n+\beta)}E_n^{(\alpha,\beta)}(e_{m-1};z),
\end{aligned}$$

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which implies the recurrence in the statement.

Lemma 4 *Let $n \in \mathbf{N}$, $m = 2, 3, \cdots$, $e_m(z) = z^m$, $S_{n,m}^{(\alpha,\beta)}(z) := M_n^{(\alpha,\beta)}(e_m; z) - z^m$, $z \in \mathbf{C}$ and $0 \le \alpha \le \beta$, we have*

$$
\begin{aligned}
S_{n,m}^{(\alpha,\beta)}(z) = & \frac{z(1-z)n^2}{(n+\beta)(n^2+m-1)}(M_n^{(\alpha,\beta)}(e_{m-1}; z))' \\
& + \frac{(m-1+n^2 z)n + \alpha(n^2+m-1)}{(n+\beta)(n^2+m-1)} S_{n,m-1}^{(\alpha,\beta)}(z) \\
& + \frac{\alpha(m-1)}{(n+\beta)(n^2+m-1)} M_n^{(\alpha,\beta)}(e_{m-1}; z) \\
& - \frac{\alpha(m-1)(n+\alpha)}{(n+\beta)^2(n^2+m-1)} M_n^{(\alpha,\beta)}(e_{m-2}; z) \\
& + \frac{(m-1+n^2 z)n + \alpha(n^2+m-1)}{(n+\beta)(n^2+m-1)} z^{m-1} - z^m. \qquad (3)
\end{aligned}
$$

Proof Using the recurrence formula (2), by simple calculation, we can easily get the recurrence (3), the proof is omitted.

3. Main results

The first main result is expressed by the following upper estimates.

Theorem 1 *Let $0 \le \alpha \le \beta$, $1 \le r \le R$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R, i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$.*

(i) for *all $|z| \le r$ and $n \in \mathbf{N}$, we have*

$$|M_n^{(\alpha,\beta)}(f; z) - f(z)| \le \frac{K_r^{(\alpha,\beta)}(f)}{n+\beta},$$

where $K_r^{(\alpha,\beta)}(f) = (1+r) \sum_{m=1}^{\infty} |c_m| m(m+1+\alpha+\beta) r^{m-1} < +\infty$.

(ii) (Simultaneous approximation) *If $1 \le r < r_1 < R$ are arbitrary fired, then for all $|z| \le r$ and $n, p \in \mathbf{N}$ we have*

$$|(M_n^{(\alpha,\beta)}(f; z))^{(p)} - f^{(p)}(z)| \le \frac{K_{r_1}^{(\alpha,\beta)}(f) p! r_1}{(n+\beta)(r_1 - r)^{p+1}},$$

where $K_{r_1}^{(\alpha,\beta)}(f)$ is defined as at the above point (i).

Proof Taking $e_m(z) = z^m$, by hypothesis that $f(z)$ is analytic in D_R, i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$, it is easy for us to obtain

$$M_n^{(\alpha,\beta)}(f; z) = \sum_{m=0}^{\infty} c_m M_n^{(\alpha,\beta)}(e_m; z),$$

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therefore, we get

$$\begin{aligned}|M_n^{(\alpha,\beta)}(f;z)-f(z)| &\leq \sum_{m=0}^{\infty}|c_m|\cdot|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)| \\ &= \sum_{m=1}^{\infty}|c_m|\cdot|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)|,\end{aligned}$$

as $M_n^{(\alpha,\beta)}(e_0;z)=e_0(z)=1$.

(i) For $m\in\mathbf{N}$, taking into account that $M_n^{(\alpha,\beta)}(e_{m-1};z)$ is a polynomial degree $\leq min(m-1,n)$, by the well-known Bernstein inequality and Lemma 2 we get

$$|(M_n^{(\alpha,\beta)}(e_{m-1};z))'|\leq\frac{m-1}{r}max\{|M_n^{(\alpha,\beta)}(e_{m-1};z)|:|z|\leq r\}\leq(m-1)r^{m-2}.$$

On the one hand, when $m=1$, for $|z|\leq r$, by Lemma 1, we have

$$|M_n^{(\alpha,\beta)}(e_1;z)-e_1(z)|=|\frac{nz}{n+\beta}+\frac{\alpha}{n+\beta}-z|\leq\frac{1+r}{n+\beta}(2+\alpha+\beta).$$

When $m\geq 2$, for $n\in\mathbf{N}$, $|z|\leq r$, $0\leq\alpha\leq\beta$, in view of $|(m-1+n^2z)n+\alpha(n^2+m-1)|\leq(n+\beta)(n^2+m-1)r$, using the recurrence formula (3) and the above inequality, we have

$$\begin{aligned}|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)| &= |S_{n,m}^{(\alpha,\beta)}(z)| \\ &\leq \frac{r(1+r)}{n+\beta}\cdot(m-1)r^{m-2}+r|S_{n,m-1}^{(\alpha,\beta)}(z)| \\ &\quad+\frac{\alpha}{n+\beta}r^{m-1}+\frac{\alpha}{n+\beta}r^{m-2}+\frac{m+1+\beta}{n+\beta}(1+r)r^{m-1} \\ &\leq \frac{m-1}{n+\beta}(1+r)r^{m-1}+r|S_{n,m-1}^{(\alpha,\beta)}(z)| \\ &\quad+\frac{\alpha}{n+\beta}(1+r)r^{m-1}+\frac{m+1+\beta}{n+\beta}(1+r)r^{m-1} \\ &= r|S_{n,m-1}^{(\alpha,\beta)}(z)|+\frac{2m+\alpha+\beta}{n+\beta}(1+r)r^{m-1}.\end{aligned}$$

By writing the last inequality, for $m=2,\cdots$, we easily obtain step by step the following

$$\begin{aligned}|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)| &\leq r\left(r|S_{n,m-2}^{(\alpha,\beta)}(z)|+\frac{2(m-1)+\alpha+\beta}{n+\beta}(1+r)r^{m-2}\right) \\ &\quad+\frac{2m+\alpha+\beta}{n+\beta}(1+r)r^{m-1} \\ &= r^2|S_{n,m-2}^{(\alpha,\beta)}(z))|+\frac{2(m-1+m)+2(\alpha+\beta)}{n+\beta}(1+r)r^{m-1} \\ &\leq\ldots\leq\frac{1+r}{n+\beta}m(m+1+\alpha+\beta)r^{m-1}.\end{aligned}$$

In conclusion, for any $m,n\in\mathbf{N}$, $|z|\leq r$, $0\leq\alpha\leq\beta$, we have

$$|M_{n+\beta}^{(\alpha,\beta)}(e_m;z)-e_m(z)|\leq\frac{1+r}{n+\beta}m(m+1+\alpha+\beta)r^{m-1},$$

it follows that

$$|M_n^{(\alpha,\beta)}(f;z)-f(z)|\le\frac{1+r}{n+\beta}\sum_{m=1}^{\infty}|c_m|m(m+1+\alpha+\beta)r^{m-1}.$$

By assuming that $f(z)$ is analytic in D_R, we have $f^{(2)}(z)=\sum\limits_{m=2}^{\infty}c_m m(m-1)z^{m-2}$ and the series is absolutely convergent in $|z|\le r$, so we get $\sum\limits_{m=2}^{\infty}|c_m|m(m-1)r^{m-2}<+\infty$, which implies $K_r^{(\alpha,\beta)}(f)=(1+r)\sum\limits_{m=1}^{\infty}|c_m|m(m+1+\alpha+\beta)r^{m-1}<+\infty$.

(ii) For the simultaneous approximation, denoting by Γ the circle of radius $r_1>r$ and center 0, since for any $|z|\le r$ and $\upsilon\in\Gamma$, we have $|\upsilon-z|\ge r_1-r$, by Cauchy's formulas it follows that for all $|z|\le r$ and $n\in\mathbf{N}$, we have

$$\begin{aligned}|(M_n^{(\alpha,\beta)}(f;z))^{(p)}-f^{(p)}(z)|&=\frac{p!}{2\pi}\left|\int_{\Gamma}\frac{M_n^{(\alpha,\beta)}(f;\upsilon)-f(\upsilon)}{(\upsilon-z)^{p+1}}d\upsilon\right|\\&\le\frac{K_{r_1}^{(\alpha,\beta)}(f)}{n+\beta}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}\\&=\frac{K_{r_1}^{(\alpha,\beta)}(f)}{n+\beta}\cdot\frac{p!r_1}{(r_1-r)^{p+1}},\end{aligned}$$

which proves the theorem.

Theorem 2 *Let $0\le\alpha\le\beta$, $R>1$, $D_R=\{z\in\mathbf{C}:|z|<R\}$. Suppose that $f:D_R\to\mathbf{C}$ is analytic in D_R, i.e. $f(z)=\sum\limits_{k=0}^{\infty}c_kz^k$ for all $z\in D_R$. For any fixed $r\in[1,R]$ and all $n\in\mathbf{N}$, $|z|\le r$, we have*

$$\begin{aligned}&\left|M_n^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-\beta z}{n+\beta}f'(z)-\frac{z(1-z)}{2(n+\beta)}f''(z)\right|\\&\le\frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)}+\frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2}+\frac{M_{r,2}(f)}{n^2},\end{aligned}\tag{4}$$

where $M_{r,2}(f)=M_r(f)+M_{r,1}(f)$, $M_r(f)=\sum\limits_{k=2}^{\infty}|c_k|(k-1)F_{k,r}r^k$ with $F_{k,r}=10k^3-30k^2+39k-16+4(k-2)(k-1)^2(1+r)$, $M_{r,1}(f)=\sum\limits_{k=2}^{\infty}|c_k|(\beta+1)k(k-1)(1+r)r^{k-1}$, $M_{r,1}^{(\alpha,\beta)}(f)=\sum\limits_{k=2}^{\infty}|c_k|[2k(k-1)^2\alpha+2k^3\beta r]r^{k-1}$, $M_{r,2}^{(\alpha,\beta)}(f)=\sum\limits_{k=2}^{\infty}|c_k|[\frac{k(k-1)(\alpha^2+\beta^2r^2)}{2}+k^2\alpha\beta r+k^2\beta^2r^2]r^{k-2}$.

Proof For all $z\in D_R$, we have

$$\begin{aligned}&M_n^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-\beta z}{n+\beta}f'(z)-\frac{z(1-z)}{2(n+\beta)}f''(z)\\&=\left[M_n(f;z)-f(z)-\frac{(n+1)z(1-z)}{2(n^2+1)}f''(z)\right]\\&\quad+\left[M_n^{(\alpha,\beta)}(f;z)-M_n(f;z)-\frac{\alpha-\beta z}{n+\beta}f'(z)+\frac{(\beta+1)n+(\beta-1)}{2(n+\beta)(n^2+1)}z(1-z)f''(z)\right]\\&:=I_1+I_2.\end{aligned}$$

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By [16, Theorem 2], we have $|I_1| \le \frac{M_r(f)}{n^2}$, where $M_r(f) = \sum_{k=2}^{\infty} |c_k|(k-1)F_{k,r}r^k$ and $F_{k,r} = 10k^3 - 30k^2 + 39k - 16 + 4(k-2)(k-1)^2(1+r)$.

Next let us to estimate $|I_2|$.

Denote $Q_{n,k}^{(\beta)}(z) = \frac{k(k-1)((\beta+1)n+(\beta-1))}{2(n+\beta)(n^2+1)} z^{k-1}(1-z)$. By f is analytic in D_R, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$, and $M_n^{(\alpha,\beta)}(e_1;z) = M_n(e_1;z) + \frac{\alpha-\beta z}{n+\beta}$, we have

$$\begin{aligned} |I_2| &= \left| \sum_{k=2}^{\infty} c_k \left[M_n^{(\alpha,\beta)}(e_k;z) - M_n(e_k;z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \right] \right| \\ &\le \sum_{k=2}^{\infty} |c_k| \left| M_n^{(\alpha,\beta)}(e_k;z) - M_n(e_k;z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \right|. \end{aligned}$$

When $k \ge 2$, since $\frac{n^k}{(n+\beta)^k} - 1 = -\sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k}$, by Lemma 1, we obtain

$$\begin{aligned} & M_n^{(\alpha,\beta)}(e_k;z) - M_n(e_k;z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j;z) + \left[\frac{n^k}{(n+\beta)^k} - 1 \right] M_n(e_k;z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} \\ &\quad + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j;z) + \frac{k n^{k-1}\alpha}{(n+\beta)^k} M_n(e_{k-1};z) \\ &\quad - \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} M_n(e_k;z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j;z) + \frac{k n^{k-1}\alpha}{(n+\beta)^k} \left[M_n(e_{k-1};z) - e_{k-1}(z) \right] \\ &\quad + \frac{k n^{k-1}\alpha}{(n+\beta)^k} z^{k-1} - \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} M_n(e_k;z) \\ &\quad - \frac{k n^{k-1}\beta}{(n+\beta)^k} \left[M_n(e_k;z) - e_k(z) \right] - \frac{k n^{k-1}\beta}{(n+\beta)^k} z^k - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j;z) + \frac{k n^{k-1}\alpha}{(n+\beta)^k} \left[M_n(e_{k-1};z) - e_{k-1}(z) \right] \\ &\quad - \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} M_n(e_k;z) - \frac{k n^{k-1}\beta}{(n+\beta)^k} \left[M_n(e_k;z) - e_k(z) \right] \\ &\quad - \left[\frac{1}{n+\beta} - \frac{n^{k-1}}{(n+\beta)^k} \right] k\alpha z^{k-1} + \left[\frac{1}{n+\beta} - \frac{n^{k-1}}{(n+\beta)^k} \right] k\beta z^k + Q_{n,k}^{(\beta)}(z). \end{aligned}$$

By the proof of the [16, Theorem 1], for any $k \in \mathbf{N}$, $|z| \le r$, $r \ge 1$, we have

$$|M_n(e_k;z)| \le r^k, \ |M_n(e_k;z) - e_k| \le \frac{2k^2}{n} r^k,$$

hence, for any $k \geq 2$, $|z| \leq r$, $r \geq 1$, we can get

$$\begin{aligned}
&\left|\sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j;z)\right| \\
&\leq \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} r^{k-2} \\
&= \sum_{j=0}^{k-2} \frac{k(k-1)}{(k-j)(k-j-1)} \binom{k-2}{j} \frac{n^j \alpha^{k-2-j}}{(n+\beta)^{k-2}} \cdot \frac{\alpha^2}{(n+\beta)^2} r^{k-2} \\
&\leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{n^j \alpha^{k-2-j}}{(n+\beta)^{k-2}} r^{k-2} \\
&\leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} r^{k-2}
\end{aligned}$$

and

$$\left|\frac{kn^{k-1}\alpha}{(n+\beta)^k}\left[M_n(e_{k-1};z) - e_{k-1}(z)\right]\right| \leq \frac{2k(k-1)^2\alpha}{n(n+\beta)} r^{k-1}.$$

Also, using

$$\frac{1}{n+\beta} - \frac{n^{k-1}}{(n+\beta)^k} = \frac{\sum_{j=0}^{k-2} \binom{k-1}{j} n^j \beta^{k-1-j}}{(n+\beta)^k} \leq \frac{(k-1)\beta}{(n+\beta)^2},$$

thus, for any $k \geq 2$, $|z| \leq r$, $r \geq 1$, we get

$$\begin{aligned}
&|M_n^{(\alpha,\beta)}(e_k;z) - M_n(e_k;z) - \frac{\alpha - \beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z)| \\
&\leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} r^{k-2} + \frac{2k(k-1)^2\alpha}{n(n+\beta)} r^{k-1} + \frac{k(k-1)}{2} \cdot \frac{\beta^2}{(n+\beta)^2} r^k \\
&\quad + \frac{2k^3\beta}{n(n+\beta)} r^k + \frac{k^2\alpha\beta}{(n+\beta)^2} r^{k-1} + \frac{k^2\beta^2}{(n+\beta)^2} r^k + \frac{(\beta+1)k(k-1)(1+r)r^{k-1}}{n^2} \\
&= \frac{r^{k-1}}{n(n+\beta)}\left[2k(k-1)^2\alpha + 2k^3\beta r\right] + \frac{(\beta+1)k(k-1)(1+r)r^{k-1}}{n^2} \\
&\quad + \frac{r^{k-2}}{(n+\beta)^2}\left[\frac{k(k-1)(\alpha^2+\beta^2r^2)}{2} + k^2\alpha\beta r + k^2\beta^2r^2\right].
\end{aligned}$$

Hence, we have

$$|I_2| \leq \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)} + \frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2} + \frac{M_{r,1}(f)}{n^2},$$

where

$$\begin{aligned}
M_{r,1}(f) &= \sum_{k=2}^{\infty} |c_k|(\beta+1)k(k-1)(1+r)r^{k-1}, \\
M_{r,1}^{(\alpha,\beta)}(f) &= \sum_{k=2}^{\infty} |c_k|[2k(k-1)^2\alpha + 2k^3\beta r]r^{k-1}, \\
M_{r,2}^{(\alpha,\beta)}(f) &= \sum_{k=2}^{\infty} |c_k|\left[\frac{k(k-1)(\alpha^2+\beta^2r^2)}{2} + k^2\alpha\beta r + k^2\beta^2r^2\right]r^{k-2}.
\end{aligned}$$

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In conclusion, we obtain

$$\left| M_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha-\beta z}{n+\beta} f'(z) - \frac{z(1-z)}{2(n+\beta)} f''(z) \right|$$

$$\leq |I_1| + |I_2| \leq \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)} + \frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2} + \frac{M_{r,2}(f)}{n^2},$$

where $M_{r,2}(f) = M_r(f) + M_{r,1}(f)$.

In the following theorem, we will obtain the exact order in approximation.

Theorem 3 *Let $0 < \alpha \leq \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. If f is not a polynomial of degree 0 , then for any $r \in [1, R)$ we have*

$$\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r \geq \frac{C_r^{(\alpha,\beta)}(f)}{n+\beta}, \ n \in \mathbf{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \leq r\}$ and the constant $C_r^{(\alpha,\beta)}(f) > 0$ depends on f, r and α, β but it is independent of n.

Proof Denote $e_1(z) = z$ and

$$H_n^{(\alpha,\beta)}(f;z) = M_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha-\beta z}{n+\beta} f'(z) - \frac{z(1-z)}{2(n+\beta)} f''(z).$$

For all $z \in D_R$ and $n \in \mathbf{N}$ we have

$$M_n^{(\alpha,\beta)}(f;z) - f(z)$$
$$= \frac{1}{n+\beta}\left\{(\alpha-\beta z)f'(z) + \frac{z(1-z)}{2} f''(z) + (n+\beta)H_n^{(\alpha,\beta)}(f;z)\right\}.$$

In view of the property: $\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r$, it follows

$$\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r$$
$$\geq \frac{1}{n+\beta}\left\{\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2} f''\|_r - (n+\beta)||H_n^{(\alpha,\beta)}(f;\cdot)||_r\right\}.$$

Considering the hypothesis that f is not a polynomial of degree 0 in D_R, we have $\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2} f''\|_r > 0$.

Indeed, supposing the contrary, it follows that

$$(\alpha-\beta z)f'(z) + \frac{z(1-z)}{2} f''(z) = 0, \text{ for all } z \in \overline{D_r}.$$

Denoting $y(z) = f'(z)$ and looking for the analytic function $y(z)$ under the form $y(z) = \sum_{k=0}^{\infty} a_k z^k$, after replacement in the differential equation, the identification of the coefficients method immediately leads to $a_k = 0$, for all $k \in \mathbb{N}\bigcup\{0\}$. This implies that $y(z) = 0$ for all $z \in \overline{D_r}$ and therefore f is constant on $\overline{D_r}$, a contradiction with the hypothesis.

Using the inequality (4), we get

$$\lim_{n\to\infty} (n+\beta)\|H_n^{(\alpha,\beta)}(f;\cdot)\|_r = 0, \tag{5}$$

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therefore, there exists an index n_0 depending only on f, r and α, β, such that for all $n \geq n_0$, we have

$$\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f''\|_r - (n+\beta)||H_n^{(\alpha,\beta)}(f;\cdot)||_r$$
$$\geq \frac{1}{2}\left\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f''\right\|_r,$$

which implies

$$\|M_n^{(\alpha,\beta)}(f;\cdot)-f\|_r \geq \frac{1}{2n}\left\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f''\right\|_r, \text{ for all } n \geq n_0.$$

For $n \in \{1,2,\cdots,n_0-1\}$, we have $\|M_n^{(\alpha,\beta)}(f;\cdot)-f\|_r \geq \frac{W_{r,n}^{(\alpha,\beta)}(f)}{n+\beta}$, where $W_{r,n}^{(\alpha,\beta)}(f) = (n+\beta)\|M_n^{(\alpha,\beta)}(f;\cdot)-f\|_r > 0$.

As a conclusion, we have $\|M_n^{(\alpha,\beta)}(f;\cdot)-f\|_r \geq \frac{C_r^{(\alpha,\beta)}(f)}{n+\beta}$, for all $n \in \mathbf{N}$, where

$$C_r^{(\alpha,\beta)}(f) = min\Big\{W_{r,1}^{(\alpha,\beta)}(f), W_{r,2}^{(\alpha,\beta)}(f), \ldots, W_{r,n_0-1}^{(\alpha,\beta)}(f),$$
$$\frac{1}{2}\|(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f''\|_r\Big\},$$

this complete the proof.

Combining Theorem 3 with Theorem 1, we get the following result.

Corollary *Let $0 \leq \alpha \leq \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. If f is not a polynomial of degree 0 , then for any $r \in [1, R)$ we have*

$$\|M_n^{(\alpha,\beta)}(f;\cdot)-f\|_r \asymp \frac{1}{n+\beta}, \quad n \in \mathbf{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \leq r\}$ and the constants in the equivalence depend on f, r and α, β but it is independent of n.

Theorem 4 *Let $0 \leq \alpha \leq \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. Also, let $1 \leq r < r_1 < R$ and $p \in \mathbf{N}$ be fixed. If f is not a polynomial of degree $\leq p-1$, then we have*

$$\|(M_n^{(\alpha,\beta)}(f;\cdot))^{(p)} - f^{(p)}\|_r \asymp \frac{1}{n+\beta}, \ n \in \mathbf{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \leq r\}$ and the constants in the equivalence depend on f, r, r_1, p, α and β, but it is independent of n.

Proof Taking into account that the upper estimate in Theorem 1 , it remains to prove the lower estimate only. Denoting by Γ the circle of radius $r_1 > r$ and center 0 , by the Cauchy's formula, it follows that for all $|z| \leq r$ and $n \in \mathbf{N}$, we have

$$(M_n^{(\alpha,\beta)}(f;z))^{(p)} - f^{(p)}(z) = \frac{p!}{2\pi i}\int_\Gamma \frac{M_n^{(\alpha,\beta)}(f;v)-f(v)}{(v-z)^{p+1}}dv.$$

Keeping the notation there for $H_n^{(\alpha,\beta)}(f;z)$, for all $n \in \mathbf{N}$, we have

$$M_n^{(\alpha,\beta)}(f;z) - f(z)$$
$$= \frac{1}{n+\beta}\left\{(\alpha-\beta z)f'(z) + \frac{z(1-z)}{2}f''(z) + (n+\beta)H_n^{(\alpha,\beta)}(f;z)\right\}.$$

by using Cauchy's formula, for all $v \in \Gamma$ we get

$$(M_n^{(\alpha,\beta)}(f;z))^{(p)} - f^{(p)}(z) = \frac{1}{n+\beta}\left\{\left[(\alpha-\beta z)f'(z) + \frac{z(1-z)}{2}f''(z)\right]^{(p)}\right.$$
$$\left. + \frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-z)^{p+1}}dv\right\},$$

passing now to $\|\cdot\|_r$ and denoting $e_1(z) = z$, it follows

$$\left\|(M_n^{(\alpha,\beta)}(f;\cdot))^{(p)} - f^{(p)}\right\|_r \geq \frac{1}{n+\beta}\left[\left\|\left[(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f''\right]^{(p)}\right\|_r\right.$$
$$\left. - \left\|\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r\right].$$

Since for any $|z| \leq r$ and $v \in \Gamma$, we have $|v-z| \geq r_1 - r$, so,

$$\left\|\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r \leq \frac{p!}{2\pi}\cdot\frac{2\pi r_1(n+\beta)\|H_n^{(\alpha,\beta)}(f;\cdot)\|_{r_1}}{(r_1-r)^{p+1}},$$

thus, by the inequality (5), we can get $\lim_{n\to\infty}\left\|\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r = 0$.

Taking into account the function f is analytic in D_R, by following exactly the lines in Gal [5], seeing also the book Gal [6, pp. 77-78], we have $\left\|[(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f'']^{(p)}\right\|_r > 0$,

In continuation, reasoning exactly as in the proof of Theorem 3, we can get the desired conclusion.

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On right multidimensional Riemann-Liouville fractional integral

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Abstract

Here we study some important properties of right multidimensional Riemann-Liouville fractional integral operator, such as of continuity and boundedness.

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1 Motivation

From [1] we have

Theorem 1 *Let $r > 0$, $F \in L_\infty (a, b)$, and*

$$G(s) = \int_s^b (t-s)^{r-1} F(t)\,dt,$$

all $s \in [a, b]$. Then $G \in AC([a, b])$ (absolutely continuous functions) for $r \geq 1$, and $G \in C([a, b])$, only for $r \in (0, 1)$.

2 Main Results

We give

Theorem 2 *Let $f \in L_\infty ([a, b] \times [c, d])$, $\alpha_1, \alpha_2 > 0$. Consider the function*

$$F(x_1, x_2) = \int_{x_1}^{b_1} \int_{x_2}^{b_2} (t_1 - x_1)^{\alpha_1 - 1} (t_2 - x_2)^{\alpha_2 - 1} f(t_1, t_2)\,dt_1 dt_2, \qquad (1)$$

where $x_1, b_1 \in [a, b]$, $x_2, b_2 \in [c, d] : x_1 \leq b_1$, $x_2 \leq b_2$.

Then F *is continuous on* $[a, b_1] \times [c, b_2]$.

Proof. (I) Let $a_1, a_1^*, b_1 \in [a, b]$ with $a_1 < a_1^* < b_1$, and $a_2, a_2^*, b_2 \in [c, d]$ with $a_2 < a_2^* < b_2$.

We observe that

$$F(a_1, a_2) - F(a_1^*, a_2^*) =$$

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2) \, dt_1 dt_2 -$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} f(t_1, t_2) \, dt_1 dt_2 = \tag{2}$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2) \, dt_1 dt_2 +$$

$$\int_{a_1^*}^{b_1} \int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2) \, dt_1 dt_2 +$$

$$\int_{a_1}^{a_1^*} \int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2) \, dt_1 dt_2 +$$

$$\int_{a_1}^{a_1^*} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2) \, dt_1 dt_2 - \tag{3}$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} f(t_1, t_2) \, dt_1 dt_2 =$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left[(t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} \right] f(t_1, t_2) \, dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2) \, dt_1 dt_2 + \tag{4}$$

$$\int_{a_1}^{a_1^*} \int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2) \, dt_1 dt_2 +$$

$$\int_{a_1}^{a_1^*} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2) \, dt_1 dt_2.$$

Call

$$I(a_1^*, a_2^*) =$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left| (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} \right| dt_1 dt_2. \tag{5}$$

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Thus

$$|F(a_1,a_2) - F(a_1^*,a_2^*)| \leq$$

$$\left\{ I(a_1^*,a_2^*) + \left(\frac{(b_1-a_1)^{\alpha_1} - (a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right)\frac{(a_2^*-a_2)^{\alpha_2}}{\alpha_2} + \right.$$

$$\left. \frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\frac{(a_2^*-a_2)^{\alpha_2}}{\alpha_2} + \frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\left(\frac{(b_2-a_2)^{\alpha_2} - (a_2^*-a_2)^{\alpha_2}}{\alpha_2}\right)\right\}\|f\|_\infty .$$

Hence, by the last inequality, it holds

$$\delta := \lim_{\substack{(a_1^*,a_2^*)\to(a_1,a_2) \\ \text{or} \\ (a_1,a_2)\to(a_1^*,a_2^*)}} |F(a_1,a_2) - F(a_1^*,a_2^*)| \leq$$

$$\left(\lim_{\substack{(a_1^*,a_2^*)\to(a_1,a_2) \\ \text{or} \\ (a_1,a_2)\to(a_1^*,a_2^*)}} I(a_1^*,a_2^*) \right) \|f\|_\infty =: \rho. \tag{6}$$

If $\alpha_1 = \alpha_2 = 1$, then $\rho = 0$, proving $\delta = 0$.

If $\alpha_1 = 1$, $\alpha_2 > 0$ we get

$$I(a_1^*,a_2^*) = (b_1 - a_1^*)\int_{a_2^*}^{b_2} \left|(t_2-a_2)^{\alpha_2-1} - (t_2-a_2^*)^{\alpha_2-1}\right| dt_2. \tag{7}$$

Assume $\alpha_2 > 1$, then $\alpha_2 - 1 > 0$. Hence

$$I(a_1^*,a_2^*) = (b_1 - a_1^*)\int_{a_2^*}^{b_2} \left((t_2-a_2)^{\alpha_2-1} - (t_2-a_2^*)^{\alpha_2-1}\right) dt_2$$

$$= (b_1 - a_1^*)\left\{\left(\frac{(b_2-a_2)^{\alpha_2} - (a_2^*-a_2)^{\alpha_2}}{\alpha_2}\right) - \frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}\right\}. \tag{8}$$

Clearly, then

$$\lim_{\substack{(a_1,a_2)\to(a_1^*,a_2^*) \\ \text{or} \\ (a_1^*,a_2^*)\to(a_1,a_2)}} I(a_1^*,a_2^*) = 0. \tag{9}$$

Similarly and symmetrically, we obtain that

$$\lim_{\substack{(a_1,a_2)\to(a_1^*,a_2^*) \\ \text{or} \\ (a_1^*,a_2^*)\to(a_1,a_2)}} I(a_1^*,a_2^*) = 0 \tag{10}$$

for the case of $\alpha_2 = 1$, $\alpha_1 > 1$.

3

If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I\left(a_1^*, a_2^*\right) = \left(b_1 - a_1^*\right) \int_{a_2^*}^{b_2} \left[\left(t_2 - a_2^*\right)^{\alpha_2 - 1} - \left(t_2 - a_2\right)^{\alpha_2 - 1}\right] dt_2 =$$

$$\left(b_1 - a_1^*\right) \left\{ \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} - \left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(a_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \right) \right\}. \tag{11}$$

Clearly, then

$$\lim_{\substack{a_2^* \to a_2 \\ \text{or} \\ a_2 \to a_2^*}} I\left(a_1^*, a_2^*\right) = 0. \tag{12}$$

Similarly and symmetrically, we derive that

$$\lim_{\substack{a_1^* \to a_1 \\ \text{or} \\ a_1 \to a_1^*}} I\left(a_1^*, a_2^*\right) = 0, \tag{13}$$

for the case of $\alpha_2 = 1$, $0 < \alpha_1 < 1$.

Case now of $\alpha_1, \alpha_2 > 1$, then

$$I\left(a_1^*, a_2^*\right) =$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left(\left(t_1 - a_1\right)^{\alpha_1 - 1} \left(t_2 - a_2\right)^{\alpha_2 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \right) dt_1 dt_2 =$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right) \left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(a_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \right)$$

$$- \frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2}. \tag{14}$$

That is

$$\lim_{\substack{\left(a_1^*, a_2^*\right) \to \left(a_1, a_2\right) \\ \text{or} \\ \left(a_1, a_2\right) \to \left(a_1^*, a_2^*\right)}} I\left(a_1^*, a_2^*\right) = 0. \tag{15}$$

Case now of $0 < \alpha_1, \alpha_2 < 1$, then

$$I\left(a_1^*, a_2^*\right) =$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left(\left(t_1 - a_1^*\right)^{\alpha_1 - 1} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} - \left(t_1 - a_1\right)^{\alpha_1 - 1} \left(t_2 - a_2\right)^{\alpha_2 - 1} \right) dt_1 dt_2 =$$

$$\frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} -$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right) \left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(a_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \right). \tag{16}$$

Hence, when $0 < \alpha_1, \alpha_2 < 1$, we get

$$\lim_{\substack{\left(a_1^*, a_2^*\right) \to (a_1, a_2) \\ (a_1, a_2) \to \left(a_1^*, a_2^*\right)}} I\left(a_1^*, a_2^*\right) = 0. \tag{17}$$

We observe that

$$I\left(a_1^*, a_2^*\right) \leq I^*\left(a_1^*, a_2^*\right) :=$$
$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left| (t_2 - a_2)^{\alpha_2 - 1} - \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \right| dt_1 dt_2$$
$$+ \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \left| (t_1 - a_1)^{\alpha_1 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1} \right| dt_1 dt_2. \tag{18}$$

Next we treat the case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$.

Therefore it holds

$$I^*\left(a_1^*, a_2^*\right) = \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left(\left(t_2 - a_2^*\right)^{\alpha_2 - 1} - (t_2 - a_2)^{\alpha_2 - 1} \right) dt_1 dt_2$$
$$+ \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \left((t_1 - a_1)^{\alpha_1 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1} \right) dt_1 dt_2 = \tag{19}$$
$$\left(\frac{(b_1 - a_1)^{\alpha_1} - \left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right) \left(\frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} - \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} + \frac{\left(a_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \right) +$$
$$\frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} \left(\frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} - \frac{\left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} - \frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \right).$$

Clearly then ($\alpha_1 > 1$, $0 < \alpha_2 < 1$)

$$\lim_{\substack{(a_1, a_2) \to \left(a_1^*, a_2^*\right) \\ \text{or} \\ \left(a_1^*, a_2^*\right) \to (a_1, a_2)}} I\left(a_1^*, a_2^*\right) = 0. \tag{20}$$

Finally, we prove the case of $\alpha_2 > 1$ and $0 < \alpha_1 < 1$. We have that

$$I^*\left(a_1^*, a_2^*\right) = \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left((t_2 - a_2)^{\alpha_2 - 1} - \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \right) dt_1 dt_2$$
$$+ \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \left(\left(t_1 - a_1^*\right)^{\alpha_1 - 1} - (t_1 - a_1)^{\alpha_1 - 1} \right) dt_1 dt_2 = \tag{21}$$
$$\left(\frac{(b_1 - a_1)^{\alpha_1} - \left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right) \left(-\frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} + \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} - \frac{\left(a_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \right) +$$
$$\frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} \left(-\frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} + \frac{\left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} + \frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \right).$$

Clearly then ($\alpha_2 > 1$, $0 < \alpha_1 < 1$)

$$\lim_{\substack{(a_1,a_2)\to(a_1^*,a_2^*) \\ \text{or} \\ (a_1^*,a_2^*)\to(a_1,a_2)}} I(a_1^*,a_2^*) = 0. \tag{22}$$

We proved $\rho = 0$, and $\delta = 0$ in all cases of this section.

The case of $a_1 > a_1^*$ and $a_2 > a_2^*$, as symmetric to the already treated one of $a_1 < a_1^*$ and $a_2 < a_2^*$, is omitted.

(II) The remaining cases are: let $a_1, a_1^*, b_1 \in [a,b]$; $a_2, a_2^*, b_2 \in [c,d]$, we can have

$$\begin{aligned} &(\mathrm{II}_1)\ a_1 > a_1^* \ \text{ and } \ a_2 < a_2^*, \\ &\text{or} \\ &(\mathrm{II}_2)\ a_1 < a_1^* \ \text{ and } \ a_2 > a_2^*. \end{aligned} \tag{23}$$

Notice that the subcases (II_1) and (II_2) are symmetric, and treated the same way. As such we treat only the case (II_2).

We observe again that

$$F(a_1,a_2) - F(a_1^*,a_2^*) = \tag{24}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} (t_1-a_1)^{\alpha_1-1}(t_2-a_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 -$$

$$\int_{a_1^*}^{b_1}\int_{a_2^*}^{b_2} (t_1-a_1^*)^{\alpha_1-1}(t_2-a_2^*)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 =$$

$$\int_{a_1}^{a_1^*}\int_{a_2}^{b_2} (t_1-a_1)^{\alpha_1-1}(t_2-a_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 +$$

$$\int_{a_1^*}^{b_1}\int_{a_2}^{b_2} (t_1-a_1)^{\alpha_1-1}(t_2-a_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 -$$

$$\int_{a_1^*}^{b_1}\int_{a_2^*}^{a_2} (t_1-a_1^*)^{\alpha_1-1}(t_2-a_2^*)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 - \tag{25}$$

$$\int_{a_1^*}^{b_1}\int_{a_2}^{b_2} (t_1-a_1^*)^{\alpha_1-1}(t_2-a_2^*)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 =$$

$$\int_{a_1^*}^{b_1}\int_{a_2}^{b_2} \left((t_1-a_1)^{\alpha_1-1}(t_2-a_2)^{\alpha_2-1} - (t_1-a_1^*)^{\alpha_1-1}(t_2-a_2^*)^{\alpha_2-1}\right) f(t_1,t_2)\,dt_1dt_2$$

$$+\int_{a_1}^{a_1^*}\int_{a_2}^{b_2} (t_1-a_1)^{\alpha_1-1}(t_2-a_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2 - \tag{26}$$

$$\int_{a_1^*}^{b_1}\int_{a_2^*}^{a_2} (t_1-a_1^*)^{\alpha_1-1}(t_2-a_2^*)^{\alpha_2-1} f(t_1,t_2)\,dt_1dt_2.$$

Call

$$I\left(a_1^*, a_2\right) :=$$

$$\int_{a_1^*}^{b_1} \int_{a_2}^{b_2} \left| \left(t_1 - a_1\right)^{\alpha_1 - 1} \left(t_2 - a_2\right)^{\alpha_2 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \right| dt_1 dt_2. \tag{27}$$

Hence, we have

$$\left|F\left(a_1, a_2\right) - F\left(a_1^*, a_2^*\right)\right| \leq$$

$$\left\{ I\left(a_1^*, a_2\right) + \frac{\left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \frac{\left(b_2 - a_2\right)^{\alpha_2}}{\alpha_2} + \frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \frac{\left(a_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} \right\} \|f\|_\infty . \tag{28}$$

Therefore it holds

$$\delta := \lim_{\substack{\left|a_1 - a_1^*\right| \to 0, \\ \left|a_2 - a_2^*\right| \to 0}} \left|F\left(a_1, a_2\right) - F\left(a_1^*, a_2^*\right)\right| \leq \left(\lim_{\substack{\left|a_1 - a_1^*\right| \to 0, \\ \left|a_2 - a_2^*\right| \to 0}} I\left(a_1^*, a_2\right) \right) \|f\|_\infty =: \theta. \tag{29}$$

We will prove that $\theta = 0$, hence $\delta = 0$, in all possible cases.

If $\alpha_1 = \alpha_2 = 1$, then $I\left(a_1^*, a_2\right) = 0$, hence $\theta = 0$.

If $\alpha_1 = 1$, $\alpha_2 > 0$ we get

$$I\left(a_1^*, a_2\right) = \left(b_1 - a_1^*\right) \int_{a_2}^{b_2} \left| \left(t_2 - a_2\right)^{\alpha_2 - 1} - \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \right| dt_2. \tag{30}$$

Assume $\alpha_2 > 1$, then $\alpha_2 - 1 > 0$. Hence

$$\begin{aligned} I\left(a_1^*, a_2\right) &= \left(b_1 - a_1^*\right) \int_{a_2}^{b_2} \left(\left(t_2 - a_2^*\right)^{\alpha_2 - 1} - \left(t_2 - a_2\right)^{\alpha_2 - 1} \right) dt_2 \\ &= \left(b_1 - a_1^*\right) \left\{ \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} - \frac{\left(a_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} - \frac{\left(b_2 - a_2\right)^{\alpha_2}}{\alpha_2} \right\}. \end{aligned} \tag{31}$$

Clearly, then

$$\lim_{\left|a_2 - a_2^*\right| \to 0,} I\left(a_1^*, a_2\right) = 0, \tag{32}$$

hence $\theta = 0$.

Let the case now of $\alpha_2 = 1$, $\alpha_1 > 1$. Then

$$\begin{aligned} I\left(a_1^*, a_2\right) &= \left(b_2 - a_2\right) \int_{a_1^*}^{b_1} \left| \left(t_1 - a_1\right)^{\alpha_1 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1} \right| dt_1 \\ &= \left(b_2 - a_2\right) \int_{a_1^*}^{b_1} \left(\left(t_1 - a_1\right)^{\alpha_1 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1} \right) dt_1 \\ &= \left(b_2 - a_2\right) \left(\frac{\left(b_1 - a_1\right)^{\alpha_1}}{\alpha_1} - \frac{\left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} - \frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \right). \end{aligned} \tag{33}$$

Then $\theta = 0$.

If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I(a_1^*, a_2) = (b_1 - a_1^*) \int_{a_2}^{b_2} \left((t_2 - a_2)^{\alpha_2 - 1} - (t_2 - a_2^*)^{\alpha_2 - 1} \right) dt_2 =$$

$$(b_1 - a_1^*) \left\{ \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} - \frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} + \frac{(a_2 - a_2^*)^{\alpha_2}}{\alpha_2} \right\}, \tag{34}$$

hence $\theta = 0$.

Let now $\alpha_2 = 1$, $0 < \alpha_1 < 1$. Then

$$I(a_1^*, a_2) = (b_2 - a_2) \int_{a_1^*}^{b_1} \left((t_1 - a_1^*)^{\alpha_1 - 1} - (t_1 - a_1)^{\alpha_1 - 1} \right) dt_1$$

$$= (b_2 - a_2) \left\{ \frac{(b_1 - a_1^*)^{\alpha_1}}{\alpha_1} - \frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} + \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} \right\}, \tag{35}$$

hence $\theta = 0$.

We observe that:

$$I(a_1^*, a_2) \leq \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left| (t_2 - a_2)^{\alpha_2 - 1} - (t_2 - a_2^*)^{\alpha_2 - 1} \right| dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} (t_2 - a_2^*)^{\alpha_2 - 1} \left| (t_1 - a_1)^{\alpha_1 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} \right| dt_1 dt_2 =: J(a_1^*, a_2). \tag{36}$$

I.e.

$$I(a_1^*, a_2) \leq J(a_1^*, a_2). \tag{37}$$

Case of $\alpha_1, \alpha_2 > 1$. Then

$$J(a_1^*, a_2) = \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left((t_2 - a_2^*)^{\alpha_2 - 1} - (t_2 - a_2)^{\alpha_2 - 1} \right) dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} (t_2 - a_2^*)^{\alpha_2 - 1} \left((t_1 - a_1)^{\alpha_1 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} \right) dt_1 dt_2 = \tag{38}$$

$$\left(\frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} - \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} \right) \left\{ \frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} - \frac{(a_2 - a_2^*)^{\alpha_2}}{\alpha_2} - \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} \right\}$$

$$+ \left(\frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} - \frac{(a_2 - a_2^*)^{\alpha_2}}{\alpha_2} \right) \left\{ \frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} - \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} - \frac{(b_1 - a_1^*)^{\alpha_1}}{\alpha_1} \right\}, \tag{39}$$

hence $\theta = 0$.

Case of $0 < \alpha_1, \alpha_2 < 1$, then

$$J(a_1^*, a_2) = \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left((t_2 - a_2)^{\alpha_2 - 1} - (t_2 - a_2^*)^{\alpha_2 - 1} \right) dt_1 dt_2$$

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$$+\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_2-a_2^*)^{\alpha_2-1}\left((t_1-a_1^*)^{\alpha_1-1}-(t_1-a_1)^{\alpha_1-1}\right)dt_1dt_2= \tag{40}$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}-\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right)\left\{\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}-\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}+\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right\}$$
$$+\left(\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right)\left\{\frac{(b_1-a_1^*)^{\alpha_1}}{\alpha_1}-\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}+\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right\}, \tag{41}$$

hence $\theta=0$.

Next case of $\alpha_1>1$, $0<\alpha_2<1$. We observe that

$$J(a_1^*,a_2)=\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}\left((t_2-a_2)^{\alpha_2-1}-(t_2-a_2^*)^{\alpha_2-1}\right)dt_1dt_2 \tag{42}$$
$$+\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_2-a_2^*)^{\alpha_2-1}\left((t_1-a_1)^{\alpha_1-1}-(t_1-a_1^*)^{\alpha_1-1}\right)dt_1dt_2=$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}-\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right)\left\{\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}-\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}+\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right\} \tag{43}$$
$$+\left(\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right)\left\{\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}-\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}-\frac{(b_1-a_1^*)^{\alpha_1}}{\alpha_1}\right\},$$

hence $\theta=0$.

Finally, we prove the case of $\alpha_2>1$ and $0<\alpha_1<1$. In that case it holds

$$J(a_1^*,a_2)=\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}\left((t_2-a_2^*)^{\alpha_2-1}-(t_2-a_2)^{\alpha_2-1}\right)dt_1dt_2$$

$$+\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_2-a_2^*)^{\alpha_2-1}\left((t_1-a_1^*)^{\alpha_1-1}-(t_1-a_1)^{\alpha_1-1}\right)dt_1dt_2= \tag{44}$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}-(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right)\left\{\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}\right\}$$
$$+\left(\frac{(b_2-a_2^*)^{\alpha_2}-(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right)\left\{\frac{(b_1-a_1^*)^{\alpha_1}}{\alpha_1}-\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}+\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right\}, \tag{45}$$

hence $\theta=0$.

We have proved that $\delta=0$, in all possible subcases of (II_2).

We have proved that F is a continuous function over $[a,b_1]\times[c,b_2]$. ■

Now we can state:

Theorem 3 *Let $f \in L_{\infty}\left(\prod_{i=1}^{k}[a_i, b_i]\right)$, $\alpha_i > 0$, $i = 1, ..., k \in \mathbb{N}$. Consider the function*

$$F(x_1, ..., x_k) = \int_{x_1}^{b_1^*} ... \int_{x_k}^{b_k^*} \prod_{i=1}^{k} (t_i - x_i)^{\alpha_i - 1} f(t_1, ..., t_k)\, dt_1 ... dt_k, \tag{46}$$

where $a_i \leq x_i \leq b_i^* \leq b_i$, $i = 1, ..., k$.

Then F is continuous on $\prod_{i=1}^{k}[a_i, b_i^*]$.

Remark 4 *In the setting of Theorem 3: Consider the right multidimensional Riemann-Liouville fractional integral of order $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_i > 0$, $i = 1, ..., k$:*

$$\left(I_{b_-^*}^{\alpha} f\right)(x) = \frac{1}{\prod_{i=1}^{k}\Gamma(\alpha_i)} \int_{x_1}^{b_1^*} ... \int_{x_k}^{b_k^*} \prod_{i=1}^{k} (t_i - x_i)^{\alpha_i - 1} f(t_1, ..., t_k)\, dt_1 ... dt_k, \tag{47}$$

where $a_i \leq x_i \leq b_i^* \leq b_i$, $i = 1, ..., k$, where $b^* = (b_1^*, ..., b_k^*)$, $x = (x_1, ..., x_k)$, Γ is the gamma function.

By Theorem 3 we get that $\left(I_{b_-^*}^{\alpha} f\right)$ is a continuous function for every $x \in \prod_{i=1}^{k}[a_i, b_i^*]$.

We notice that

$$\left|\left(I_{b_-^*}^{\alpha} f\right)(x)\right| \leq \frac{1}{\prod_{i=1}^{k}\Gamma(\alpha_i)} \left(\int_{x_1}^{b_1^*} ... \int_{x_k}^{b_k^*} \prod_{i=1}^{k} (t_i - x_i)^{\alpha_i - 1} dt_1 ... dt_k\right) \|f\|_{\infty} \tag{48}$$

$$= \frac{1}{\prod_{i=1}^{k}\Gamma(\alpha_i)} \prod_{i=1}^{k} \left(\int_{x_i}^{b_i^*} (t_i - x_i)^{\alpha_i - 1} dt_i\right) \|f\|_{\infty} =$$

$$\frac{1}{\prod_{i=1}^{k}\Gamma(\alpha_i)} \prod_{i=1}^{k} \frac{(b_i^* - x_i)^{\alpha_i}}{\alpha_i} \|f\|_{\infty} = \left(\prod_{i=1}^{k} \frac{(b_i^* - x_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right) \|f\|_{\infty}. \tag{49}$$

That is

$$\left|\left(I_{b_-^*}^{\alpha} f\right)(x)\right| \leq \left(\prod_{i=1}^{k} \frac{(b_i^* - x_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right) \|f\|_{\infty}. \tag{50}$$

In particular we get

$$\left(I_{b_-^*}^{\alpha} f\right)(b^*) = 0, \tag{51}$$

and

$$\left\|I_{b_-^*}^{\alpha} f\right\|_{\infty, \prod_{i=1}^{k}[a_i, b_i^*]} \leq \left(\prod_{i=1}^{k} \frac{(b_i^* - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right) \|f\|_{\infty}. \tag{52}$$

That is $I_{b_-^*}^{\alpha} f$ is a bounded linear operator, which here is also a positive operator.

10

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Dynamics of a difference equation with maximum

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Abstract The purpose of this work is to investigate the convergence of the solutions of the following max-type difference equation

$$z_n = \max\{\frac{1}{z_{n-s}}, \frac{P_n}{z_{n-t}^{\alpha_n}}\}, \quad n = 0, 1, 2, \cdots,$$

where $s, t \in \{1, 2, 3, \cdots\}$ with $s \neq t$, $\alpha_n \in (0, 1)$ is an s-periodic sequence, $\{P_n\}_{n=0}^{+\infty}$ is a constant sequence satisfying $P_n \in (0, 1]$ for every $n \geq 0$. We show that if $\{z_n\}_{n=-r}^{+\infty}$ ($r = \max\{s, t\}$) is a positive solution of the above equation with the initial conditions $z_{-r}, z_{-r+1}, \cdots, z_{-1} \in (0, +\infty)$, then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_{2sn+k}\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s - 1$. Further, we show that if P_n is a periodic sequence, $s = 1$ and t is even, then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_n\}_{n=-t}^{+\infty}$ is eventually periodic with period 2.

AMS Subject Classification: 39A10; 39A11.
Keywords: max-type equation, positive solution, eventual periodicity, monotonicity, periodic sequence.

1. Introduction

The max operator arises naturally in certain models in automatic control theory (see [6,7]). In the recent years, there has been a lot of interest in studying the convergence and boundedness of max-type difference equations (see [1,3,5,8-11]). In [2], Chen studied the second order max-type difference equation

$$z_{n+1} = \max\{\frac{1}{z_n}, \frac{A_n}{z_{n-1}}\}, \quad n = 0, 1, 2, \cdots, \tag{1.1}$$

and showed that every positive solution of (1.1) is eventually periodic with period 2 when $\{A_n\}_{n=0}^{+\infty}$ is a periodic sequence with period $k \geq 2$ and $A_n \in (0, 1)$ for all $n \geq 0$.

In [4], the authors studied the following non-autonomous max-type difference equation with two delays

$$z_n = \max\{\frac{f_n}{z_{n-m}^{\alpha}}, \frac{A}{z_{n-r}^{\beta}}\}, \quad n = 0, 1, 2, \cdots,$$

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where $\alpha, \beta \in \mathbb{R}$, $\{A_n\}_{n=0}^{+\infty}$ is a sequence of positive real numbers with a finite limit and $m, r \in \mathbb{N} \equiv \{1, 2, 3, \cdots\}$ with $m \neq r$.

In this paper, we study the periodicity, the boundedness and the convergence of the following max-type difference equation

$$z_n = \max\{\frac{1}{z_{n-s}}, \frac{P_n}{z_{n-t}^{\alpha_n}}\}, \quad n = 0, 1, 2, \cdots, \tag{1.2}$$

where $s, t \in \mathbb{N}$ with $s \neq t$, $\alpha_n \in (0, 1)$ is an s-periodic sequence, $\{P_n\}_{n=0}^{+\infty}$ is a constant sequence satisfying $P_n \in (0, 1]$ for every $n \geq 0$.

2. Some Propositions

In the following, suppose that $\{z_n\}_{n=-r}^{+\infty}$ is a positive solution of (1.2). To obtain the main results of this paper, we need the following propositions.

Proposition 2.1 (i) $z_n z_{n-s} \geq 1$ for all $n \geq 0$.

(ii) For any $n \geq r$, $z_n \leq \max\{z_{n-2s}, P_n z_{n-s-t}^{\alpha_{n-s}}\}$.

(iii) If $z_n = P_n / z_{n-t}^{\alpha_n} > 1/z_{n-s}$ for some $n \geq r$, then $z_n > z_{n-2s}$. If $z_n = 1/z_{n-s}$ for some $n \geq s$, then $z_n \leq z_{n-2s}$.

Proof (i) Since $z_n \geq 1/z_{n-s}$ for any $n \geq 0$, we have $z_n z_{n-s} \geq 1$.

(ii) According to (i), we get that for every $n \geq r$,

$$z_n = \max\left\{\frac{z_{n-2s}}{z_{n-s} z_{n-2s}}, \frac{P_n z_{n-s-t}^{\alpha_n}}{z_{n-s-t}^{\alpha_n} z_{n-t}^{\alpha_n}}\right\} \leq \max\{z_{n-2s}, P_n z_{n-s-t}^{\alpha_n}\}.$$

(iii) If $z_n = P_n / z_{n-t}^{\alpha_n} > 1/z_{n-s}$ for some $n \geq r$, then by (i) we obtain that

$$\begin{aligned} 1 &< z_n z_{n-s} = \max\{\frac{z_n}{z_{n-2s}}, \frac{z_n z_{n-t}^{\alpha_n} P_{n-s}}{z_{n-t-s}^{\alpha_{n-s}} z_{n-t}^{\alpha_n}}\} \\ &\leq \max\{\frac{z_n}{z_{n-2s}}, P_n P_{n-s}\} = \frac{z_n}{z_{n-2s}}. \end{aligned}$$

Which implies $z_n > z_{n-2s}$. If $z_n = 1/z_{n-s}$ for some $n \geq s$, then by (i) we obtain that

$$z_n = \frac{z_{n-2s}}{z_{n-s} z_{n-2s}} \leq z_{n-2s}.$$

The proof is complete.

Define

$$U_n = \max\{z_{n-1}, z_{n-2}, \cdots, z_{n-s-r}\} \quad (n \geq r). \tag{2.1}$$

According to Proposition 2.1 (i), we get $\max\{z_{n-1}, z_{n-s-1}\} \geq 1$, from which it follows $U_n \geq 1$ for any $n \geq r$.

Proposition 2.2 (i) Let U_n be as in (2.1). Then $z_n \leq U_n$ for any $n \geq r$ and $\{U_n\}_{n=r}^{+\infty}$ is a decreasing sequence.

(ii) There exist constants $R \geq R' > 0$ such that $R' \leq z_n \leq R$ for any $n \geq -r$.

Proof (i) If $z_{n-s-t} \leq 1$, then $z_{n-s-t}^{\alpha_n} \leq 1$. If $z_{n-s-t} \geq 1$, then $z_{n-s-t}^{\alpha_n} \leq z_{n-s-t}$. According to Proposition 2.1 (ii), we have that for any $n \geq r$,

$$z_n \leq \max\{z_{n-2s}, z_{n-s-t}^{\alpha_n}\} \leq \max\{z_{n-1}, z_{n-2}, \cdots, z_{n-s-r}\} = U_n.$$

Further, we get
$$U_{n+1} = \max\{z_n, z_{n-1}, \cdots, z_{n-s-r+1}\} \leq U_n.$$

(ii) Let $R = \max\{U_r, z_{r-1}, \cdots, z_{-r}\}$ and $R' = \min\{1/U_r, z_{r-1}, \cdots, z_{-r}\}$. Then $R' \leq z_n \leq R$ for any $n \geq -r$. The proof is complete.

Now we assume $\lim_{n\longrightarrow\infty} U_n = U$ and $\liminf_{n\longrightarrow\infty} U_n = u$. According to Proposition 2.2 (i), we obtain the following corollary.

Corollary 2.3 There exists a sequence $1 < n_1 < n_2 < \cdots < n_k < \cdots$ such that $z_{n_k} \geq U$ and $n_{k+1} - n_k \leq s + r$.

Proposition 2.4 The following statements hold:

(i) $U = \limsup_{n\longrightarrow\infty} z_n$.

(ii) Assume that $U > 1$. Then $\{n : U \leq z_n = P_n/z_{n-t}^{\alpha_n}\}$ is a finite set. Further, there exists $N \in \mathbb{N}$ such that:

i) $z_{N+2ks} \geq U$ and $z_{N+2ks} = 1/z_{N+(2k-1)s}$ for any $k \geq 0$, and z_{N+2ks} is decreasing.

ii) $\lim_{k\longrightarrow\infty} z_{N+(2k-1)s} = u = 1/U$.

Proof (i) According to (2.1), we see that U_n is a subsequence of z_n. Thus $U \leq \limsup_{n\longrightarrow\infty} z_n$. Further, since $z_n \leq U_n$ for all $n \geq r$, we obtain
$$\limsup_{n\longrightarrow\infty} z_n \leq \limsup_{n\longrightarrow\infty} U_n = U.$$

(ii) If $\{n : U \leq z_n = P_n/z_{n-t}^{\alpha_n}\}$ is an infinite set, then there exists a sequence $t < n_1 < n_2 < \cdots < n_k < \cdots$ such that
$$U \leq z_{n_k} = \frac{P_{n_k}}{z_{n_k-t}^{\alpha_{n_k}}} \leq P_{n_k} z_{n_k-t-s}^{\alpha_{n_k}} \leq z_{n_k-t-s}^{\alpha_{n_k}}.$$

Without loss of generality, suppose that $\lim_{k\longrightarrow\infty} z_{n_k-t-s} = u_1$ and $\lim_{k\longrightarrow\infty} \alpha_{n_k} = \alpha < 1$. Thus we obtain $U = \lim_{k\longrightarrow\infty} z_{n_k} \leq u_1^\alpha \leq U^\alpha < U$ since $U > 1$. A contradiction.

It follows from the above that there exists $M \in \mathbb{N}$ such that if $n \geq M$ and $z_n \geq U$, then $z_n = 1/z_{n-s}$. By Corollary 2.3 we see that there exists a sequence $1 < n_1 < n_2 < \cdots < n_k < \cdots$ such that $z_{n_k} \geq U$ and $\lim_{k\longrightarrow\infty} z_{n_k} = U$. Without loss of generality, suppose that $n_k = 2sr_k + \tau > M$ with $0 \leq \tau < 2s$ for all $k \in \mathbb{N}$. Then $z_{n_k} = 1/z_{n_k-s}$. Write $N = 2sr_1 + \tau$. By Proposition 2.1 (iii), we see that for any $k \geq 0$,
$$z_{N+2ks} \geq U \quad \text{and} \quad \frac{1}{z_{N+2ks-s}} = z_{N+2ks} \geq z_{N+2(k+1)s} = \frac{1}{z_{N+2(k+1)s-s}}.$$

Let $i_k \longrightarrow +\infty$ such that $z_{i_k} \longrightarrow u$ and $z_{i_k-s} \longrightarrow u_1$. Then
$$\frac{1}{U} = \lim_{k\longrightarrow\infty} \frac{1}{z_{N+2ks}} = \lim_{k\longrightarrow\infty} z_{N+(2k-1)s} \geq u = \lim_{k\longrightarrow\infty} z_{i_k} \geq \lim_{k\longrightarrow\infty} \frac{1}{z_{i_k-s}} = \frac{1}{u_1} \geq \frac{1}{U},$$

this implies $\lim_{k\longrightarrow\infty} z_{N+(2k-1)s} = u = 1/U$. The proof is complete.

Proposition 2.5 Let $N, p, q \in \mathbb{N}$ with $q \geq 2$ such that

(i) $\{z_{N+2ks}\}_{k=0}^{+\infty}$ is monotone.

(ii) $z_{N+2s(p+\lambda)+t} = P_{N+2s(p+\lambda)+t}/z_{N+2s(p+\lambda)}^{\alpha_{N+2s(p+\lambda)+t}} > 1/z_{N+2s(p+\lambda)+t-s}$ for every $\lambda \in \{0, q\}$.

(iii) $z_{N+2s(p+\lambda)+t} = 1/z_{N+2s(p+\lambda)+t-s}$ for every $1 \le \lambda \le q-1$.

Then $z_{N+2s(p+\lambda)+t} = z_{N+2s(p+\lambda+1)+t}$ for every $0 \le \lambda \le q-2$.

Proof There are two cases to be considered.

Case 1 $\{z_{N+2sk}\}_{k=0}^{+\infty}$ is decreasing. In this case, we claim that $z_{N+2s(p+\lambda)+t-s} = 1/z_{N+2s(p+\lambda-1)+t}$ for any $1 \le \lambda \le q-1$. Since, otherwise, if for some $1 \le \lambda \le q-1$,

$$z_{N+2s(p+\lambda)+t-s} = \frac{P_{N+2s(p+\lambda)+t-s}}{z_{N+2s(p+\lambda)-s}^{\alpha_{N+2s(p+\lambda)+t-s}}} > 1/z_{N+2s(p+\lambda-1)+t},$$

then by Proposition 2.1 (iii) it follows that

$$\begin{aligned} \frac{P_{N+2sp+t}}{z_{N+2s(p+\lambda)}^{\alpha_{N+2s(p+\lambda)+t}}} &\ge \frac{P_{N+2sp+t}}{z_{N+2sp}^{\alpha_{N+2sp+t}}} = z_{N+2sp+t} \ge z_{N+2s(p+\lambda-1)+t} \\ &> \frac{1}{z_{N+2s(p+\lambda)+t-s}} = \frac{z_{N+2s(p+\lambda)-s}^{\alpha_{N+2s(p+\lambda)+t-s}}}{P_{N+2s(p+\lambda)+t-s}}. \end{aligned}$$

This implies

$$1 \ge P_{N+2sp+t}P_{N+2s(p+\lambda)+t-s} > z_{N+2s(p+\lambda)}^{\alpha_{N+2s(p+\lambda)+t}} z_{N+2s(p+\lambda)-s}^{\alpha_{N+2s(p+\lambda)+t-s}} \ge 1.$$

A contradiction. From the above claim it follows that

$$z_{N+2s(p+\lambda)+t} = \frac{1}{z_{N+2s(p+\lambda)+t-s}} = z_{N+2s(p+\lambda-1)+t} \ge z_{N+2s(p+\lambda)+t}.$$

Thus $z_{N+2s(p+\lambda-1)+t} = z_{N+2s(p+\lambda)+t}$ for every $1 \le \lambda \le q-1$.

Case 2 $\{z_{N+2ks}\}_{k=0}^{+\infty}$ is increasing. In this case, it follows from Proposition 2.1 (iii) that

$$\begin{aligned} \frac{P_{N+2s(p+q)+t}}{z_{N+2s(p+q-1)}^{\alpha_{N+2s(p+q-1)+t}}} &\ge \frac{P_{N+2s(p+q)+t}}{z_{N+2s(p+q)}^{\alpha_{N+2s(p+q)+t}}} = z_{N+2s(p+q)+t} > z_{N+2s(p+q-1)+t} \\ &= \frac{1}{z_{N+2s(p+q-1)+t-s}} = \min\{z_{N+2s(p+q-2)+t}, \frac{z_{N+2s(p+q-1)-s}^{\alpha_{N+2s(p+q-1)+t-s}}}{P_{N+2s(p+q-1)+t-s}}\} \\ &= z_{N+2s(p+q-2)+t} \ge z_{N+2s(p+q-1)+t} \end{aligned}$$

since

$$P_{N+2s(p+q)+t}P_{N+2s(p+q-1)+r-s} \le 1 \quad \text{and} \quad z_{N+2s(p+q-1)}^{\alpha_{N+2s(p+q-1)+t}} z_{N+2s(p+q-1)-s}^{\alpha_{N+2s(p+q-1)+t-s}} \ge 1,$$

we have

$$z_{N+2s(p+q-1)+t} = z_{N+2s(p+q-2)+t}.$$

In a similar fashion, we may obtain that $z_{N+2s(p+q-1)+t} = z_{N+2s(p+\lambda)+t}$ for any $0 \le \lambda \le q-2$. The proof is complete.

Proposition 2.6 If there exists $N \in \mathbb{N}$ such that $\{z_{N+2ks}\}_{k=0}^{+\infty}$ is monotone, then $\{z_{N+t+2ks}\}_{k=0}^{+\infty}$ is eventually monotone.

Proof If there exists $K \in \mathbb{N}$ such that

$$z_{N+2ks+t} = 1/z_{N+2sk+t-s} \quad \text{for all} \quad k \ge K$$

or

$$z_{N+2ks+t} = P_{N+2ks+t}/z_{N+2ks}^{\alpha_{N+2ks+t}} > 1/z_{N+2ks+t-s} \quad \text{for all} \quad k \geq K,$$

then by Proposition 2.1 (iii) we obtain that $z_{N+2ks+t} \leq z_{N+2(k-1)s+t}$ for all $k \geq K$ (or $z_{N+2ks+t} > z_{N+2(k-1)s+t}$ for all $k \geq K$). Thus $\{z_{N+t+2ks}\}_{k=K}^{+\infty}$ is monotone.

If there exists a sequence $1 < p_1 < q_1 < p_2 < q_2 < \cdots < p_k < q_k < \cdots$ such that

$$z_{N+2rs+t} = \frac{P_{N+2rs+t}}{z_{N+2rs}^{\alpha_{N+2rs+t}}} > \frac{1}{z_{N+2rs+t-s}} \quad \text{for every} \quad p_i \leq r < q_i$$

and

$$z_{N+2rs+t} = \frac{1}{z_{N+2rs+t-s}} \quad \text{for every} \quad q_i \leq r < p_{i+1},$$

then by Proposition 2.1 (iii) and Proposition 2.5 it follows that $z_{N+2(r-1)s+t} < z_{N+2rs+t}$ for every $p_i \leq r < q_i$ and $z_{N+2(r-1)s+t} = z_{N+2rs+t}$ for every $q_i \leq r < p_{i+1}$, this follows that $\{z_{N+t+2rs}\}_{r=p_1}^{+\infty}$ is increasing. The proof is complete.

3. Main Results

In section, we state the main results of this paper.

Theorem 3.1 Let $\{z_n\}_{n=-r}^{+\infty}$ be a positive solution of (1.2). Then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_{2ns+k}\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s-1$.

Proof If $U = \limsup_{n\longrightarrow\infty} z_n = 1$, then let $i_k \longrightarrow +\infty$ such that $z_{i_k} \longrightarrow u = \liminf_{n\longrightarrow\infty} z_n$ and $z_{i_k-s} \longrightarrow u_1$. Thus

$$1 \geq u = \lim_{k\longrightarrow\infty} z_{i_k} \geq \lim_{k\longrightarrow\infty} \frac{1}{z_{i_k-s}} = \frac{1}{u_1} \geq \frac{1}{U} = 1.$$

Which implies $\lim_{n\longrightarrow\infty} z_n = 1$. Now assume that $U = \limsup_{n\longrightarrow\infty} z_n > 1$.

First we suppose that $\gcd(s,t) = 1$. Then by Proposition 2.4 (iii) we see that there exists $N \in \mathbb{N}$ such that the following statements hold:

(1) $z_{N+2ns}z_{N+(2n-1)s} = 1$ for any $n \geq 0$.

(2) z_{N+2ns} is decreasing ($n \geq 0$) and $\lim_{n\longrightarrow\infty} z_{N+2ns} = U$. $x_{N+(2n-1)s}$ is increasing ($n \geq 0$) and $\lim_{n\longrightarrow\infty} z_{N+(2n-1)s} = u = 1/U$.

Using Proposition 2.6 repeatedly, it follows that for every $1 \leq i \leq s-1$, $\{z_{N+2ns+it}\}_{n=0}^{+\infty}$ and $\{z_{N+(2n-1)s+it}\}_{n=0}^{+\infty}$ are eventually monotone. Since $\gcd(s,t) = 1$, it follows that for every $j \in \{0, 1, 2, \cdots, s-1\}$ there exist some $0 \leq i_j \leq s-1$ and integer λ_j such that $i_jt = \lambda_js + j$ and $i_jt - s = (\lambda_j - 1)s + j$. Thus $\{z_{N+2ns+\lambda_js+j}\}_{n=0}^{+\infty}$ and $\{z_{N+2ns+(\lambda_j-1)s+j}\}_{n=0}^{+\infty}$ are eventually monotone for every $j \in \{0, 1, 2, \cdots, s-1\}$, which implies that $\{z_{2ns+k}\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s-1$.

If $\gcd(s,t) = d > 1$, then we consider the max-type equation

$$z_n = \max\{\frac{1}{z_{n-ds_1}}, \frac{P_n}{z_{n-dt_1}^{\alpha_n}}\}, \quad n = 0, 1, 2, \cdots\cdots, \tag{3.1}$$

where $s = ds_1$ and $t = dt_1$ with $\gcd(s_1, t_1) = 1$. Write $y_n^i = z_{nd+i}$ for every $0 \leq i \leq d-1$ and $n = 0, 1, 2, \cdots$. Then (3.1) reduces to the equations

$$y_n^i = \max\{\frac{1}{y_{n-s_1}^i}, \frac{P_{nd+i}}{(y_{n-t_1}^i)^{\alpha_{nd+i}}}\}, \quad 0 \leq i \leq d-1, \quad n = 0, 1, 2, \cdots. \tag{3.2}$$

By an analogous way as in the above, we obtain that for every $0 \leq i \leq d-1$, y^i_n is a solution of equation

$$y^i_n = \max\{1/y^i_{n-s_1}, \frac{P_{nd+i}}{(y^i_{n-t_1})^{\alpha_{nd+i}}}\}.$$

Then $\{y^i_{2s_1n+k}\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s_1 - 1$. Thus for every $0 \leq k \leq 2s-1$, $\{z_{2ns+k}\}_{n=0}^{+\infty}$ is eventually monotone. The proof is complete.

Theorem 3.2 Assume that $s = 1$, and t is even, and P_n is a periodic sequence. Let $\{z_n\}_{n=-t}^{+\infty}$ be a positive solution of (1.2). Then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_n\}_{n=-t}^{+\infty}$ is eventually periodic with period 2.

Proof If $U = \limsup_{n\longrightarrow\infty} z_n = 1$, then using arguments similar to ones developed in the proof of Theorem 3.1 we can obtain $\lim_{n\longrightarrow\infty} z_n = 1$. Now assume that $U = \limsup_{n\longrightarrow\infty} z_n > 1$.

According to Proposition 2.4 (iii) and Theorem 3.1, we see that there exists $N \in \mathbb{N}$ such that the following statements hold:

(1) $z_{N+2n}z_{N+2n-1} = 1$ for any $n \geq 0$.

(2) z_{N+2n} is decreasing ($n \geq 0$) and $\lim_{n\longrightarrow\infty} z_{N+2n} = U$. z_{N+2n-1} is increasing ($n \geq 0$) and $\lim_{n\longrightarrow\infty} z_{N+2n-1} = u = 1/U$.

We claim that $z_{N+2n+1} = 1/z_{N+2n}$ eventually. In fact, if there exist $1 \leq k_1 < k_2 < \cdots < k_i < \cdots$ such that

$$z_{N+2k_i+1} = \frac{P_{N+2k_i+1}}{z^{\alpha_{N+2k_i+1}}_{N+2k_i+1-t}},$$

then by taking a subsequence we may assume that P_{N+2k_i+1} and α_{N+2k_i+1} are constant sequences since P_n and α_n are periodic sequences. Thus z_{N+2k_i+1} is decreasing since $z^{\alpha_{N+2k_i+1}}_{N+2k_i+1-t}$ is increasing. A contradiction. Which implies that $\{z_n\}_{n=-t}^{+\infty}$ is eventually periodic with period 2. The proof is complete.

Example 3.3 Assume that $s = 1$ and t is odd. Let $P_n = P \in (0, 1)$ and $\alpha_n = \alpha \in (0, 1)$ for any $n \geq 0$. Then there exists a positive solution $\{z_n\}_{n=-t}^{\infty}$ of (1.2) which is not eventually periodic such that $\lim_{n\longrightarrow\infty} z_n \neq 1$.

Proof Choose the initial values $z_{-t}, z_{1-t}, \cdots, z_{-1} \in (0, +\infty)$ satisfying

$$z_{-t} < z_{2-t} < \cdots < z_{-1} < z_{-t}/P, \quad z_{-t} < P^{2/(1-\alpha)}, \quad z_{k-t} = 1/z_{k-t-1} \quad k \in \{1, 3, \cdots, t-2\}.$$

Now we show that $z_{2k-1} < z_{2k+1}$ and $z_{2k} < z_{2k-2}$ for any $k \in \mathbb{N}$.

By $z_{-1} < z_{-t}/P$ and $z_{-t} < P^{2/(1-\alpha)}$, we have $z_{-1} < z_{-t}/P < Pz^{\alpha}_{-t}$. Which implies

$$\begin{aligned}
z_0 &= \max\{\frac{1}{z_{-1}}, \frac{P}{z^{\alpha}_{-t}}\} = \frac{1}{z_{-1}} < \frac{1}{z_{-3}} = z_{-2}. \\
z_1 &= \max\{\frac{1}{z_0}, \frac{P}{z^{\alpha}_{1-t}}\} = \max\{z_{-1}, Pz^{\alpha}_{-t}\} = Pz^{\alpha}_{-t} > z_{-1}. \\
z_2 &= \max\{\frac{1}{z_1}, \frac{P}{z^{\alpha}_{2-t}}\} = \max\{\frac{1}{Pz^{\alpha}_{-t}}, \frac{P}{z^{\alpha}_{2-t}}\} = \frac{1}{Pz^{\alpha}_{-t}} = \frac{1}{z_1} < \frac{1}{z_{-1}} = z_0. \\
z_3 &= \max\{\frac{1}{z_2}, \frac{P}{z^{\alpha}_{3-t}}\} = \max\{z_1, Pz^{\alpha}_{2-t}\} = \max\{Pz^{\alpha}_{-t}, Pz^{\alpha}_{2-t}\} = Pz^{\alpha}_{2-t} > \frac{1}{z_2} = z_1. \\
z_4 &= \max\{\frac{1}{z_3}, \frac{P}{z^{\alpha}_{4-t}}\} = \max\{\frac{1}{Pz^{\alpha}_{2-t}}, \frac{P}{z^{\alpha}_{4-t}}\} = \frac{1}{Pz^{\alpha}_{2-t}} = \frac{1}{z_3} < \frac{1}{z_1} = z_2.
\end{aligned}$$

Assume that there exists some $m \in \mathbb{N}$ such that

(1) $z_{2k-1} < z_{2k+1}$ and $z_{2k+2} < z_{2k}$ for any $(-t+1)/2 \leq k \leq m$.

(2) $z_{2k+1} = Pz^{\alpha}_{2k-t}$ for any $0 \leq k \leq m$ and $z_{2k+2}z_{2k+1} = 1$ for any $(-t+1)/2 \leq k \leq m$.

Then

$$\begin{aligned}
z_{2m+3} &= \max\{\frac{1}{z_{2m+2}}, \frac{P}{z^{\alpha}_{2m+3-t}}\} = \max\{z_{2m+1}, Pz^{\alpha}_{2m+2-t}\} \\
&= \max\{Pz^{\alpha}_{2m-t}, Pz^{\alpha}_{2m+2-t}\} = Pz^{\alpha}_{2m+2-t} > Pz^{\alpha}_{2m-t} = z_{2m+1}. \\
z_{2m+4} &= \max\{\frac{1}{z_{2m+3}}, \frac{P}{z^{\alpha}_{2m+4-t}}\} = \max\{\frac{1}{Pz^{\alpha}_{2m+2-t}}, \frac{P}{z^{\alpha}_{2m+4-t}}\} \\
&= \frac{1}{Pz^{\alpha}_{2m+2-t}} = \frac{1}{z_{2m+3}} < \frac{1}{z_{2m+1}} = z_{2m+2}.
\end{aligned}$$

Therefore $z_{2k-1} < z_{2k+1}$ and $z_{2k+2} < z_{2k}$ for any $k \geq (-t+1)/2$, which implies that $\{z_n\}^{\infty}_{n=-t}$ is not eventually periodic. Since $z_{2n+1} = Pz^{\alpha}_{2n-t}$ $(n \in \mathbb{N})$, we obtain $\lim_{n\longrightarrow\infty} z_n \neq 1$. The proof is complete.

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General properties of concave functions defined by the generalized Srivastava-Attiya operator

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Abstract

In this paper we introduce a class $\Im_{\mu,b}^{m,k}C_0(\alpha)$ of concave functions by using the generalized Srivastava-Attiya operator. Also, we get distortion bounds for this class.

Keywords: Hadamard product, concave functions, linear operator, distortion theorem, Hurwitz-Lerch Zeta functions, Srivastava-Attiya operator.

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1 Introduction

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The study of operators plays an important role in Geometric Function Theory in Complex Analysis and its related fields. Many derivative and integral operators can be written in terms of convolution of certain analytic functions. For functions

$$f_j(z) = \sum_{n=0}^{\infty} a_{n,j} z^n \quad (j = 1, 2)$$

analytic in U, we define the Hadamard product of f_1 and f_2 as

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{n,1,} a_{n,2} z^n = (f_2 * f_1)(z) \quad (z \in U). \tag{2}$$

In terms of the Hadamard product (or convolution), the Dziok-Srivastava linear convolution operator involving the generalized hypergeometric function was introduced and studied systematically by Dziok and Srivastava [9], [10] and (subsequently) by many other authors (see, for details, [11] and [20]).

We recall here a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined in [19] by

$$\Phi(z, s, a) := \sum_{n=2}^{\infty} \frac{z^n}{(n+a)^s}$$

$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C},\ \text{when } |z| < 1;\ Re(s) > 1 \text{ when } |z| = 1\big)$ where, as usual, $\mathbb{Z}_0^- := \mathbb{Z}\backslash\mathbb{N},$ and $\mathbb{N} := \{1, 2, 3, \ldots\}$). Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in [8], and the references stated there in (see also [16], [21], [22]). Srivastava and Attiya [21] (also see [4], [12]) introduced and investigated the linear operator.

$$\Im_b^\mu : A \to A$$

defined in terms of the Hadamard product by

$$\Im_b^\mu f\ (z) = (G_b^\mu * f)\ (z), \qquad \big(z \in U;\ b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \mu \in \mathbb{C};\ f \in A\big) \tag{3}$$

where, for convenience,

$$G_b^\mu(z) := (1+b)^\mu [\Phi(z, \mu, b) - b^{-\mu}] \quad (z \in U). \tag{4}$$

We recall here the following relationships which follow easily by using (1), (3) and (4)

$$\Im_b^\mu f\ (z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^\mu a_n z^n. \tag{5}$$

Motivated essentially by the Srivastava-Attiya operator, Murugusundaramoorthy [17] introduced the generalized integral operator $\Im_{\mu,b}^{m,k}$ given by

$$\Im_{\mu,b}^{m,k}\ f(z) = z + \sum_{n=2}^{\infty} C_n^m(b, \mu, k) a_n z^n \tag{6}$$

where

$$\Psi_n = C_n^m(b, \mu, k) = \left|\left(\frac{1+b}{n+b}\right)^\mu\right| \frac{m!(n+k-2)!}{(k-2)!(n+m-1)!} \tag{7}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C}\backslash\mathbb{Z}_0^-; \mu \in \mathbb{C}, k \geq 2$ and $m > -1$. It is of interest to note that $\Im_{\mu,b}^{1,2}$ is the Srivastava-Attiya operator and $\Im_{0,b}^{m,k}$ is the well-known Choi-Saigo-Srivastava operator (see [15]). Suitably specializing the parameters m, k, μ and b in $\Im_{\mu,b}^{m,k}\ f(z)$ we can get various integral operators introduced by Alexander [1] and Bernardi [5], Libera and Livingston [13], [14].

2 Preliminaries

Conformal maps of the unit disk onto convex domains are a classical topic. Recently Avkhadiev and Wirths [2] discovered that conformal maps onto concave domains (the complements of convex closed sets) have some novel properties.

A function $f : U \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

- f is analytic in U with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.
- f maps U conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
- The opening angle of $f(U)$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev et al. [3], Cruz and Pommerenke [7] and references there in.

In particular, the inequality

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0 \qquad (z \in U)$$

is used - sometimes also as a definition - for concave functions $f \in C_{0_O}$ (see e.g. [18] and others).

Bhowmik et al. [6] showed that an analytic function f maps U onto a concave domain of angle $\pi\alpha$, if and only if $ReP_f(z) > 0$, where

$$P_f(z) = \frac{2}{\alpha - 1}\left[\frac{\alpha + 1}{2}\frac{1 + z}{1 - z} - 1 - z\frac{f''(z)}{f'(z)}\right].$$

Definition 1 *Let $f(z) \in A$ and $\alpha \in (1, 2]$. Then $f(z) \in \Im^{m,k}_{\mu,b}C_0(\alpha)$ if and only if*

$$Re\frac{2}{\alpha - 1}\left[\frac{\alpha + 1}{2}\frac{1 + z}{1 - z} - 1 - z\frac{\left[\Im^{m,k}_{\mu,b}f(z)\right]''}{\left[\Im^{m,k}_{\mu,b}f(z)\right]'}\right] > 0 \qquad (z \in U).$$

3 Main results

Theorem 2 *If $f(z) \in A$ satisfies the inequality*

$$\sum_{n=2}^{\infty}\left[(\alpha - 1)n + 2n^2\right]|C^m_n(b, \mu, k)||a_n| < 3 - \alpha,$$

for some $\alpha \in (1, 2]$, $n \in \mathbb{N}$, then $f(z) \in \Im^{m,k}_{\mu,b}C_0(\alpha)$.

3

Proof. We want to prove that

$$Re\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-z\frac{\left[\Im_{\mu,b}^{m,k}f(z)\right]''}{\left[\Im_{\mu,b}^{m,k}f(z)\right]'}\right]>0.$$

By using the fact that

$$Re\frac{1}{w}>\frac{1}{2}\Leftrightarrow|w-1|<1,$$

it is enough to show that $|w|<1$.

$$\frac{1}{w}=\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-z\frac{g'(z)}{g(z)}\right] \tag{8}$$

where

$$g(z)=z\left(\Im_{\mu,b}^{m,k}f(z)\right)'=z\left\{1+\sum_{n=2}^{\infty}C_n^m(b,\mu,k)na_nz^{n-1}\right\} \tag{9}$$

and

$$g'(z)=1+\sum_{n=2}^{\infty}C_n^m(b,\mu,k)n^2a_nz^{n-1}. \tag{10}$$

Using (9) and (10), in (8) we obtain

$$|w|\leq\frac{\alpha-1}{2}\left|\frac{2(1-z)z\left[1+\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)na_nz^{n-1}\right]}{(\alpha+1)(1+z)z\left(1+\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)na_nz^{n-1}\right)-2(1-z)z\left(1+\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)n^2a_nz^{n-1}\right)}\right|.$$

Using triangle inequality and letting $z\to-1$, then

$$|w|<\frac{\alpha-1}{2}\left(\frac{1+\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n}{1-\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n^2}\right).$$

The last expression is bounded by 1, if

$$\frac{1+\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n}{1-\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n^2}<\frac{2}{\alpha-1}.$$

Finally, we can easily see that

$$\sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n|<3-\alpha. \tag{11}$$

■

4

4 Distortion Bounds

Theorem 3 *If* $f(z) \in \Im_{\mu,b}^{m,k} C_0(\alpha)$, *then*

$$|z| - \frac{3-\alpha}{2(3+\alpha)}|z|^2 \leq \left|\Im_{\mu,b}^{m,k} f(z)\right| \leq |z| + \frac{3-\alpha}{2(3+\alpha)}|z|^2.$$

Proof. From the Theorem 2, we have

$$2(3+\alpha)\sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n| < 3-\alpha,$$

That is

$$\sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n| \leq \frac{3-\alpha}{2(3+\alpha)}.$$

According to (11) we obtain

$$\begin{aligned} |\Im_{\mu,b}^{m,k} f(z)| &\leq |z| + \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^2 \\ &\leq |z| + \frac{3-\alpha}{2(3+\alpha)}|z|^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |\Im_{\mu,b}^{m,k} f(z)| &\geq |z| - \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^n \\ &\geq |z| - \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^2 \\ &\geq |z| - \frac{3-\alpha}{2(3+\alpha)}|z|^2. \end{aligned}$$

This completes the proof. ∎

Theorem 4 *If* $f(z) \in \Im_{\mu,b}^{m,k} C_0(\alpha)$, *then*

$$|z| - \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2 \leq |f(z)| \leq |z| + \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2.$$

Proof. According to the Theorem 2 we get that

$$2(3+\alpha)\left|\left(\frac{1+b}{2+b}\right)^{\mu}\right|\frac{k(k-1)}{m+1}\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n| < 3-\alpha.$$

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Thus we get

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right|.$$

Next from (1), we have

$$\begin{aligned}
|f(z)| &\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^n \\
&\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^2 \\
&\leq |z| + \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2.
\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}
|f(z)| &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^n \\
&\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^2 \\
&\geq |z| - \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2
\end{aligned}$$

This completes the proof. ■

Theorem 5 *If $f(z) \in \Im_{\mu,b}^{1,2}C_0(\alpha)$, then*

$$|z| - \frac{(3-\alpha)}{2(3+\alpha)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2 \leq |f(z)| \leq |z| + \frac{(3-\alpha)}{2(3+\alpha)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2.$$

Proof. According to the Theorem 2 we get that

$$2(3+\alpha)\sum_{n=2}^{\infty}C_n^1(b,\mu,2)|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n| < 3-\alpha,$$

or, equivalently

$$2(3+\alpha)\left|\left(\frac{1+b}{2+b}\right)^{\mu}\right|\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n| < 3-\alpha.$$

Thus we get

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(3-\alpha)}{2(3+\alpha)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right|.$$

6

Next from (1), we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty} |a_n||z|^2 \\ &\leq |z| + \frac{(3-\alpha)}{2(3+\alpha)} \left| \left(\frac{2+b}{1+b} \right)^{\mu} \right| |z|^2. \end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n||z|^n \\ &\geq |z| - \sum_{n=2}^{\infty} |a_n||z|^2 \\ &\geq |z| - \frac{(3-\alpha)}{2(3+\alpha)} \left| \left(\frac{2+b}{1+b} \right)^{\mu} \right| |z|^2. \end{aligned}$$

■

Theorem 6 *If* $f(z) \in \Im_{0,b}^{m,k} C_0(\alpha)$, *then*

$$|z| - \frac{(3-\alpha)}{2(3+\alpha)} \left| \frac{(k-2)!(n+m-1)!}{m!(n+k-2)!} \right| |z|^2 \leq |f(z)| \leq |z| + \frac{(3-\alpha)}{2(3+\alpha)} \left| \frac{(k-2)!(n+m-1)!}{m!(n+k-2)!} \right| |z|^2.$$

Proof. According to the Theorem 2 we get that

$$2(3+\alpha) \left| \frac{m!(n+k-2)!}{(k-2)!(n+m-1)!} \right| \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \left[(\alpha-1)n + 2n^2 \right] C_n^m(b,\mu,k) |a_n| \leq 3-\alpha.$$

Thus we get

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(3-\alpha)}{2(3+\alpha)} \left| \frac{(k-2)!(n+m-1)!}{m!(n+k-2)!} \right|.$$

Next from (1), we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty} |a_n||z|^2 \\ &\leq |z| + \frac{(3-\alpha)}{2(3+\alpha)} \left| \frac{(k-2)!(n+m-1)!}{m!(n+k-2)!} \right| |z|^2. \end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^n \\ &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^2 \\ &\geq |z| - \frac{(3-\alpha)}{2(3+\alpha)}\left|\frac{(k-2)!(n+m-1)!}{m!(n+k-2)!}\right||z|^2. \end{aligned}$$

■

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On the zeros of eigenfunctions of discontinuous Sturm-Liouville problems

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Abstract : In this paper, we prove analogues of the classical Sturm comparison and oscillation theorems for Sturm-Liouville problem together with boundary -transmission conditions on two disjoint intervals. We present a new version for Sturm's comparison and oscillation theorems. The obtained results generalizes the recently obtained oscillation and comparison theorems for regular Sturm-Liouville problem which contained transmission conditions.

Keywords : Sturm-Liouville problems, transmission conditions, Sturm comparison and oscillation theorems.

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1 Introduction

The oscillation theory for the solutions of differential equations is one of the traditional trends in the qualitative theory of differential equations. Its essence is to establish conditions for the existence of oscillating (nonoscillating) solutions, to study the laws of distribution of the zeros, to obtain estimates of the distance between the consecutive zeros and of the number of zeros in a given interval. The relationship between the oscillatory and other fundamental properties of the solutions of Sturm-Liouville type differential equations are of central importance in the theory of boundary value problems There are substantial literature on this subject. Many authors have expounded on various aspects of this theory, see [1, 9, 10] and the references cited therein. A considerable number of studies have been made on the oscillation and nonoscillation for a long time. Those results can be found in [14, 15] and the references contained therein. While the extensions and generalizations have much intrinsic interest, we believe their continued relevance is due in no small part to their important connection with problems of physical origin. Particularly the connections with the minimization problems of the calculus of variations and optimal control as well as the spectral theory of differential operators are important. Since the second order equations have applications in various problems in physics, biology, and economics (see for example [1, 5, 13], and the references cited therein) there is a permanent interest in obtaining new sufficient conditions for the oscillation or nonoscillation of solutions of various types of second order equations. In this study we investigated same aspects of comparison and oscillation properties for one discontinuous eigenvalue problem which consists of Sturm-Liouville

equation,

$$Ly := -y''(x) + q(x)y(x) = \lambda y(x) \tag{1.1}$$

to hold on two disjoint intervals $(-1, 0)$ and $(0, 1)$, where discontinuity in y and y' at the interior singular point $x = 0$ are prescribed by transmission conditions

$$y(0-) = \delta y(0+), \quad y'(0-) = \frac{1}{\delta} y'(0+), \tag{1.2}$$

together with the boundary conditions

$$y(-1) = y(1) = 0 \tag{1.3}$$

where the potential $q(x)$ is real-valued, continuous on $[-1, 0) \cup \in (0, 1]$ and has a finite limits $q(c\mp) = \lim\limits_{x \to 0\mp} q(x)$; λ is a complex eigenparameter; $\delta \neq 0$ any real number. Since various type transmission problems appear frequently in various fields of physics and technics, Sturm-Liouville problems with transmission conditions have been an important research topic in mathematical physics [2, 8, 11]. For the earlier developments about Sturm comparison and oscillation theory, we refer to [4, 5, 6, 9, 14, 15] and for recent developments, we refer to [1, 3, 7, 13, 16, 17].

2 Comparison Theorem for discontinuous Sturm-Liouville problems

At first we shall extend and generalize the classical Sturm-liouville comparison theorem.

Theorem 2.1. *Let $y = y_1(x)$ be solution of the equation*

$$L_1 y := -y'' + q_1(x)y = 0 \tag{2.1}$$

satisfying transmission conditions at the point of interaction $x = 0$ given by

$$y(0-) = \delta y(0+), \ y'(0-) = \frac{1}{\delta} y'(0+) \tag{2.2}$$

and let $y = y_2(x)$ be the solution of the equation

$$L_2 y := -y'' + q_2(x)y = 0 \tag{2.3}$$

satisfying the same transmission conditions (2.2) where $\delta \neq 0$ any real number if $q_1(x) > q_2(x)$ on $[-1, 0) \cup \in (0, 1]$, then between any two consecutive zeros of $y_1(x)$ there is at least one zero of $y_2(x)$.

Proof. Let x_1 and x_2 with $x_1 < x_2$ be consecutive zeroes of y_1. Suppose, it possible,that y_2 does not have a zero on (x_1, x_2). Lagrange's identity (see, [12]) gives

$$y_2 L_1 y_1 - y_1 L_2 y_2 = \frac{d}{dx}\{y_2' y_1 - y_1' y_2\} + \{q_1(x) - q_2(x)\} y_1 y_2 \tag{2.4}$$

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Hence

$$\frac{d}{dx}\{y_1'y_2 - y_2'y_1\} = \{q_1(x) - q_2(x)\}y_1y_2 \tag{2.5}$$

Case 1. Let $x_1 \in [-1, 0)$, $x_2 \in (0, 1]$ and $\delta > 0$. Integrating on both sides of the equation (2.9) over $[x_1, 0)$ and $(0, x_2]$ and then adding we get

$$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} (y_1'y_2 - y_2'y_1)|_{x_1}^{0-\epsilon_1} + \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} (y_1'y_2 - y_2'y_1)|_{0+\epsilon_2}^{x_2}$$

$$= \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} \int_{x_1}^{0-\epsilon_1} \{q_1(x) - q_2(x)\}y_1y_2dx + \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} \int_{0+\epsilon_2}^{x_2} \{q_1(x) - q_2(x)\}y_1y_2dx \tag{2.6}$$

Since $y_1(x_1) = y_1(x_2) = 0$ we get

$$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} W(y_1, y_2; 0 - \epsilon_1) - \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} W(y_1, y_2; 0 + \epsilon_2) - y_1'(x_1)y_2(x_1) + y_1'(x_2)y_2(x_2)$$

$$= \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} \int_{x_1}^{0-\epsilon_1} \{q_1(x) - q_2(x)\}y_1y_2dx + \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} \int_{0+\epsilon_2}^{x_2} \{q_1(x) - q_2(x)\}y_1y_2dx \tag{2.7}$$

Using the transmission conditions we obtain

$$-y_1'(x_1)y_2(x_1) + y_1'(x_2)y_2(x_2) = \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} \int_{x_1}^{0-\epsilon_1} \{q_1(x) - q_2(x)\}y_1y_2dx$$

$$+ \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} \int_{0+\epsilon_2}^{x_2} \{q_1(x) - q_2(x)\}y_1y_2dx \tag{2.8}$$

In this case with no restriction we can assume that $y_1(x) > 0$ and $y_2(x) > 0$ over $(x_1, 0)\cup(0, x_2)$. These conditions ensure that the integral on the right in (2.8) is positive. On the left, since $y_1(x) > 0$ by assumption, the function is increasing at the point x_1. Hence $y_1'(x_1) > 0$(it cannot vanish, because then it would follow from the uniqueness theorem for the solutions of (2.1) that $y_1(x) \equiv 0$, which is impossible). Similarly, $y_1'(x_2) < 0$. Thus, the left-hand side of the equation (2.8) is less or equal to zero, which is a contradiction.

Case 2. Let $x_1 \in [-1, 0)$, $x_2 \in (0, 1]$ and $\delta < 0$. In this case with no restriction it can be assumed that, $y_1(x) > 0$ over $(x_1, 0)$, $y_1(x) < 0$ over $(0, x_2)$, $y_2(x) > 0$ over $(x_1, 0)$ and $y_2(x) < 0$ over $(0, x_2)$. Since $y_1(x_1) = 0$ and $y_1(x_1) > 0$ over $(x_1, 0)$ $y_1'(x_1) > 0$. Further, since $y_2(x_2) = 0$ and $y_2(x_2) < 0$ immediately to left of x_2, $y_2'(x) < 0$. Hence, the left-hand side of (2.8) is is less

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or equal zero, but the right-hand side is positive which shows that (2.8) is impossible.
Case 3. Let $(x_1, x_2) \subset [-1, 0)$. Integrating on both sides of the equation (2.5) from x_1 to x_2, we get

$$(y_1' y_2 - y_2' y_1)|_{x_1}^{x_2} = \int_{x_1}^{x_2} \{q_1(x) - q_2(x)\} y_1 y_2 dx \tag{2.9}$$

Then with no restriction it can be assumed that $y_1(x) > 0$ and $y_2(x) > 0$ over (x_1, x_2). These conditions ensure that the integral on the right in (2.9) is positive. However, on the left, we have $y_1(x_1) = y_1(x_2) = 0$ with $y_1'(x_1) > 0$ and $y_1'(x_2) < 0$. The left-hand side therefore becomes

$$y_1'(x_2)y_2(x_2) - y_1'(x_1)y_2(x_1) \leq 0$$

which presents us with a contradiction: right-hand side > 0 and left-hand side < 0. Thus $y_2(x) = 0$ (at least once) between the zeros of $y_1(x)$. Since the conditions describing $y_1(x)$ are given, we conclude that $y_2(x)$ must change sign between $x = x_1$ and $x = x_2$.
Case 4. Let $(x_1, x_2) \subset (0, 1]$. This case is totaly similar to the previous case. □

3 On the zeros of eigenfunctions

In this section we examine the number of zeros of eigenfunctions.

Lemma 3.1. *There is an unique solution $y(x, \lambda)$ of the equation (1.1) satisfying the initial conditions*

$$y(x_0, \lambda) = \alpha(\lambda), \; y'(x_0, \lambda) = \beta(\lambda) \tag{3.1}$$

and the transmission conditions (1.2) where $\alpha(\lambda), \beta(\lambda)$ are given entire functions of $\lambda \in \mathbb{C}$ and $x_0 \in [-1, 0) \cup (0, 1]$. Moreover, $y(x, \lambda)$ is entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [-1, 0) \cup (0, 1]$.

Proof. The proof is totally similar to [**?**] and therefore is omitted. □

Theorem 3.2. *Let $\phi(x, \lambda_1) = \begin{cases} \phi_1(x, \lambda_1), & x \in [-1, 0) \\ \phi_2(x, \lambda_1), & x \in (0, 1] \end{cases}$ be solution of the equation (1.1), for $\lambda = \lambda_1$ satisfying the initial conditions*

$$\phi_1(-1, \lambda_1) = \alpha, \quad \phi_1'(-1, \lambda_1) = \beta \tag{3.2}$$

and the transmission conditions

$$\phi_2(0^+, \lambda_1) = \frac{1}{\delta}\phi_1(0^-, \lambda_1), \phi_2'(0^+, \lambda_1) = \delta\phi_1'(0^-, \lambda_1) \tag{3.3}$$

and $\varphi(x, \lambda_2) = \begin{cases} \varphi_1(x, \lambda_2), & x \in [-1, 0) \\ \varphi_2(x, \lambda_2), & x \in (0, 1] \end{cases}$ be solution of the equation (1.1), for $\lambda = \lambda_2$ satisfying the initial conditions

$$\varphi_1(-1, \lambda_2) = \alpha, \quad \varphi_1'(-1, \lambda_2) = \beta \tag{3.4}$$

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and the transmission conditions

$$\varphi_2(0^+,\lambda_2)=\frac{1}{\delta}\phi_2(0^-,\lambda_2),\varphi_2'(0^+,\lambda_2 1)=\delta\phi_1'(0^-,\lambda_1). \tag{3.5}$$

where δ,β,δ any real numbers with $\alpha^2+\beta^2\neq 0,\delta\neq 0$. Suppose that $\phi(x,\lambda_1)$ has a zeros in $[-1,0)\cup(0,1]$ and let $x_1(x_1\neq -1)$ be zero of the function $\phi(x,\lambda_1)$, nearest to $x=-1$. If $\lambda_2>\lambda_1$ then $\varphi(x_2,\lambda_2)$ has at least one zero in $[-1,x_1)$.

Proof. From the well-known Lagrange's identity (see,for example, [12]) we have

$$\frac{d}{dx}\{\phi_1'\varphi_1-\varphi_1'\phi_1\}=\{\lambda_2-\lambda_1\}\phi_1\varphi_1 \tag{3.6}$$

in the interval $(0,1)$.

$$\frac{d}{dx}\{\phi_2'\varphi_2-\varphi_2'\phi_2\}=\{\lambda_2-\lambda_1\}\phi_2\varphi_2 \tag{3.7}$$

Case 1. Let $x_1>0$ and $\delta>0$. Integrating on both sides of the equation (3.11) from -1 to x_1, we get

$$\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}(\phi_1'\varphi_1-\varphi_1'\phi_1)|_{-1}^{0-\epsilon_1}+\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}(\phi_2'\varphi_2-\varphi_2'\phi_2)|_{0+\epsilon_2}^{x_1}$$

$$=\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}\{\lambda_2-\lambda_1\}\int_{-1}^{0-\epsilon_1}\phi_1\varphi_1dx+\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}\{\lambda_2-\lambda_1\}\int_{0+\epsilon_2}^{x_1}\phi_2\varphi_2dx \tag{3.8}$$

Since $W(\phi_1,\varphi_1;-1)=0$ by (3.2) and (3.4) we get

$$\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}W(\phi_1,\varphi_1;0-\epsilon_1)-\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}W(\phi_2,\varphi_2;0+\epsilon_2)+\phi_2'(x_1,\lambda_1)\varphi_2(x_1,\lambda_2)$$

$$=\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}\{\lambda_2-\lambda_1\}\int_{-1}^{0-\epsilon_1}\phi_1\varphi_1dx+\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}\{\lambda_2-\lambda_1\}\int_{0+\epsilon_2}^{x_1}\phi_2\varphi_2dx \tag{3.9}$$

Using the transmission conditions we obtain

$$\begin{aligned}\phi_2'(x_1,\lambda_1)\varphi_2(x_1,\lambda_2) &= \lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}\{\lambda_2-\lambda_1\}\int_{-1}^{0-\epsilon_1}\phi_1\varphi_1dx\\ &+\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}\{\lambda_2-\lambda_1\}\int_{0+\epsilon_2}^{x_1}\phi_2\varphi_2dx\end{aligned} \tag{3.10}$$

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With no restriction it can be assumed that $\phi(x,\lambda_1) < 0$ and $\varphi(x,\lambda_2) < 0$ in $[-1,x_1)$. These conditions ensure that the integral on the right in (3.10) is positive. Since $\phi_2(x_1,\lambda_1) = 0$ and $\phi_2(x,\lambda_1) > 0$ immediately to the left of x_1 by assumption, the function is increasing at the point x_1. Hence $\phi_2'(x_1,\lambda_1) > 0$(it cannot vanish, because then it would follow from the uniqueness theorem for the solutions of (2.1) that $\phi_2(x,\lambda_1) \equiv 0$, which is impossible). Thus, the left-hand side of the equation (3.10) is less or equal to zero, but the right-hand side is positive, which is a contradiction.

Case 2. Let $x_1 > 0$ and $\delta < 0$. In this case with no restriction it can be assumed that $\phi(x,\lambda_1) > 0$ and $\varphi(x,\lambda_2) < 0$ in $[-1,0)$ but $\phi(x,\lambda_1) < 0$ and $\varphi(x,\lambda_2) > 0$ in $(0,x_1]$. As in the previous case, these conditions ensure that the integral on the right of (3.10) is negative, but left hand side of (3.10) is positive or is equal to zero, i.e. the equality (3.10)is impossible.

Case 3. Let $x_1 \in [-1,0)$. Integrating on both sides of the equation (2.5) from a to x_1, we get

$$(\phi_1'\varphi_1 - \varphi_1'\phi_1)|_{-1}^{x_1} = \int_{-1}^{x_1} \{\lambda_2 - \lambda_1\}\phi_1\varphi_1 dx \tag{3.11}$$

Since $\phi_1(x,\lambda_1) = 0$ by using the initial conditions $\phi_1(-1,\lambda_1) = 0, \phi_1'(-1,\lambda_1) = 0$ we get

$$\phi_1'(x_1)\varphi_1(x_1) = \int_{-1}^{x_1} \{\lambda_2 - \lambda_1\}\phi_1\varphi_1 dx \tag{3.12}$$

Let $x_1 < 0$. Without loss of generality, we can put $\phi(x,\lambda_1) > 0$ and $\varphi(x,\lambda_2) > 0$ in $[-1,x_1)$. Since, by assumption, $\phi_1(x,\lambda_1) > 0$ and $\varphi_1(x,\lambda_2) > 0$ in $[-1,x_1)$ and $\lambda_2 > \lambda_1$, the right-hand side of the equality (3.12) is positive. However, on the left-hand side, since $\phi_1(x_1,\lambda_1) = 0$ and $\phi_1(x,\lambda_1) > 0$ immediately to the left of x_1, the function $\phi_1(x,\lambda_1)$ is decreasing in the vicinity of the point x_1. Therefore, $\phi_1'(x_1,\lambda_1) \leq 0$(it cannot vanish, because then it would follow from the uniqueness theorem for the solutions of (1.1) that $\phi_1(x,\lambda_1) \equiv 0$, which is impossible). The left-hand side therefore becomes

$$\phi_1'(x_1,\lambda_1)\varphi_1(x_1,\lambda_1) \leq 0$$

which presents us with a contradiction: right-hand side > 0 and left-hand side ≤ 0. The proof is complete. □

Now we are ready to establish the main result.

Theorem 3.3. *Let $\psi_1(x)$ and $\psi_2(x)$ be two eigenfunction corresponding to the eigenvalues λ_1 and λ_2 of the problem (1.1)-(1.3) and let $\lambda_2 > \lambda_1$. Then if $\psi_1(x)$ has m zeros in $[-1,0) \cup (0,1]$, $\psi_2(x)$ has not fewer than m zeros in the same two-interval $[-1,0) \cup (0,1]$. Moreover, $n-th$ zero of $\psi_2(x)$ is less than the $n-th$ zero of $\psi_1(x)$.*

Proof. Let $x_1', x_2', ..., x_m'$ with $x_1' < x_2 <, ... < x_m'$ be zeros of the eigenfunctions $\psi_1(x)$. By virtue of the Theorem 3.2 $\psi_2(x)$ has at least one zero in $[-1,x_1')$. Moreover, by applying the Theorem 2.1 to the solutions ψ_1 and ψ_2 we see that $\psi_2(x)$ has at least one zero in each of the intervals $(x_1',x_2'), (x_2',x_3'), ..., (x_{m-1}',x_m')$. Consequently the number of zeros of $\psi_2(x)$ is not fewer than the number of zeros $\psi_1(x)$ and $n-th$ zero of $\psi_2(x)$ is less than $n-th$ zero of $\psi_1(x)$. The proof is complete. □

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Fuzzy stability of an additive-quadratic functional equation in matrix fuzzy normed spaces

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Abstract. A mapping $f : X \times X \to Y$ is called additive-quadratic if f satisfies the system of equations

$$\begin{cases} f(x+y,z) = f(x,z) + f(y,z), \\ f(x,y+z) + f(x,y-z) = 2f(x,y) + 2f(x,z). \end{cases}$$

In this paper, using the fixed point method, we prove the Hyers-Ulam stability in matrix fuzzy normed spaces associated to the following additive-quadratic functional equation

$$f(x+y,z+w) + f(x+y,z-w) = 2f(x,z) + 2f(x,w) + 2f(y,z) + 2f(y,w)$$

for all $x, y, z, w \in X$.

1. Introduction and preliminaries

A definition of fuzzy norm on a vector space, to construct a fuzzy vector topological structure, introduced by Katsaras [15]. During the last four decades some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 16, 32]. In particular, Bag and Samanta [1], following Cheng and Mordeson [6], presented an idea of a fuzzy norm in such a manner the corresponding fuzzy metric is of Kramosil and Michalek type [6]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [2].

We use the definition of fuzzy normed spaces given in [1, 19, 21] to investigate a fuzzy version of the Hyers-Ulam stability of an additive-quadratic additive functional equation in the fuzzy normed vector space setting.

Definition 1.1. *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a* fuzzy norm *on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

(N_1) $N(x,t) = 0$ *for* $t \leqslant 0$*;*
(N_2) $x = 0$ *if and only if* $N(x,t) = 1$ *for all* $t > 0$*;*
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ *if* $c \neq 0$
(N_4) $N(x+y, s+t \geqslant \min\{N(x,s), N(y,t)\}$*;*

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(N_5) *$N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;*
(N_6) *for $x \neq 0, N(x,.)$ is continuous on $\mathbb{R}$.*

The pair (X, N) is called a *fuzzy normed vector space*. To see more properties and examples of fuzzy normed vector spaces, we refer to [19, 20].

Definition 1.2. *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be* convergent *or* converge *if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the* limit *of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 1.3. *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called a* Cauchy sequence *if for each $\varepsilon > 0$ and each $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

It is well known that every convergent sequence in a fuzzy normed vector space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges $f(x_0)$. If $f : X \to Y$ continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [2]).

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in X;

$e_j \in M_{1,n}(\mathbb{C})$ is that the jth component is 1 and the other components are zero;

$E_{ij} \in M_n(\mathbb{C})$ is that the (i, j)-component is 1 and the other components are zero;

$E_{ij} \otimes x \in M_n(X)$ is that the (i, j)-component is x and the other components are zero.

For $x \in M_n(X), y \in M_k(X)$,

$$x \otimes y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Let $(X, \|\cdot\|)$ be a normed space. Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leqslant \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C}), x = (x_{ij}) \in M_n(X)$ and $B \in M_{n,k}(\mathbb{C})$, and that $(X, \{\|.\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

A matrix normed space $(X, \{\|\cdot\|_n\})$ is called an *L^∞-matrix normed space* if $\|x \oplus y\|_{n+k} = \max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

Throughout this paper, let $(X, \{\|\cdot\|_n\})$ be a matrix normed space and $(Y, \{\|\cdot\|_n\})$ be a matrix Banach space.

We introduce the concept of a matrix fuzzy normed space.

Definition 1.4. *Let (X, N) be a fuzzy normed space.*

(1) (X, N) *is called a* matrix fuzzy normed space *if for each positive integer* n, $(M_n(X), N_n)$ *is a fuzzy normed space and* $N_k(AxB, t) \geqslant N_n\left(x, \frac{t}{\|A\|\cdot\|B\|}\right)$ *for all* $t > 0, A \in M_{k,n}(\mathbb{R}), x = [x_{ij}] \in M_n(X)$ *and* $B \in M_{n,k}(\mathbb{R})$ *with* $\|A\| \neq 0.\|B\| \neq 0$.

(2) $(X, \{N_n\})$ *is called a* matrix fuzzy Banach space *if* (X, N) *is a fuzzy Banach space and* $(X, \{N_n\})$ *is a matrix fuzzy normed space.*

Example 1.5. *Let* $(X, \{\|\cdot\|_n\})$ *be a matrix normed space. Let* $N_n(x, t) := \frac{t}{t+\|x\|_n}$ *for all* $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$*. Then*

$$N_k(AxB, t) = \frac{t}{t + \|AxB\|_k} \geqslant \frac{t}{t + \|A\|.\|x\|_n.\|B\|} = \frac{\frac{t}{\|A\|.\|B\|}}{\frac{t}{\|A\|.\|B\|} + \|x\|_n}$$

for all $t > 0, A \in M_{k,n}(\mathbb{R}), x = [x_{ij}] \in M_n(X)$ *and* $B \in M_{n,k}(\mathbb{R})$ *with* $\|A\|.\|B\| \neq 0$*. So,* $(X, \{N_n\})$ *is a matrix fuzzy normed space.*

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [29] implies that quotients, mapping spaces, and various tensor product of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces have an increasingly significant effect on operator algebra theory(see [10]).

The proof given in [29] appealed to the theory of ordered operator spaces [7]. Effros and Ruan [11] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [26] and Effors [9].

The study of stability problems have been formulated by Ulam [31] in 1940: Under what condition does there exist a homomorphism near an approximate homomorphism? In the following year, Hyers [14] answered affirmatively the question of Ulam for Banach spaces, which was stated that if $\varepsilon > 0$ and $f : X \to Y$ is a mapping with X a normed space and Y is a Banach space such that

$$\|f(x + y) - f(x) - f(y)\| \leqslant \varepsilon \tag{1.1}$$

for all $x, y \in X$, then there exists a unique additive map $T : X \to Y$ such that

$$\|f(x + y) - f(x) - f(y)\| \leqslant \varepsilon$$

for all $x \in X$. A generalized version of the theorem of Hyers for approximately linear mappings presented by Rassias [27] in 1978 by considering the case when (1.1) is unbounded.

In 2003, Cădariu and Radu applied the fixed point method to the investigation of the Jensen functional equation [3]. They could present a short and a simple proof (different of the "direct method", initiated by Hyers in 1941) for the Hyers-Ulam stability of the Jensen functional equation [3] and forthe quadratic functional equation [4]. See [12, 22, 23, 24, 28, 30] for more information on functional equations.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.6. [8] *Let (Ω, d) be a complete generalized metric space and $J : \Omega \to \Omega$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given $x \in \Omega$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geqslant n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $\Lambda = \{y \in \Omega : d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ for all $y \in \Lambda$.*

Definition 1.7. *A mapping $f : X \times X \to Y$ is called additive-quadratic if f satisfies the system of equations*

$$\begin{cases} f(x+y, z) = f(x, z) + f(y, z), \\ f(x, y+z) + f(x, y-z) = 2f(x, y) + 2f(x, z). \end{cases} \tag{1.2}$$

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) := cxy^2$ is a solution of (1.2). In particular, letting $x = y$, we get a cubic function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) := f(x, x) = cx^3$.

For a mapping $f : X \times X \to Y$, consider the functional equation:

$$f(x+y, z+w) + f(x+y, z-w) = 2f(x, z) + 2f(x, w) + 2f(y, z) + 2f(y, w). \tag{1.3}$$

for all $x, y, z, w \in X$. The solution of (1.3) was discussed in [25].

In this paper, by using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.3) in matrix fuzzy normed spaces.

2. Fuzzy stability of the additive-quadratic functional equation (1.3)

In this section, using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.3) in matrix fuzzy normed space.

We need the following lemma.

Lemma 2.1. [17, Lemma 2.1] *Let $(X, \{N_n\})$ be a matrix fuzzy normed space.*

(1) *$N_n(E_{kl} \otimes x, t) = N(x, t)$ for all $t > 0$ and $x \in X$.*
(2) *for all $[x_{ij}] \in M_n(X)$ and $t = \sum_{i,j=1}^{n} t_{ij}$,*

$$N(x_{kl}, t) \geqslant N([x_{ij}], t) \geqslant \min\{N(x_{ij}, t_{ij}) : i, j = 1, 2, \cdots, n\},$$

$$N(x_{kl}, t) \geqslant N([x_{ij}], t) \geqslant \min\left\{N\Big(x_{ij}, \frac{t}{n^2}\Big) : i, j = 1, 2, \cdots, n\right\}$$

(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{ijn} = x_{ij}$ *for* $x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)$

Proof. (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1, N_n(E_{kl} \otimes x, t) \geqslant N(x,t)$. Since $e_k(E_{kl} \otimes x)e_l^* = x, N_n(E_{kl} \otimes x, t) \leqslant N(x,t)$. So $N(E_{kl} \otimes x, t)$=N(x,t).

(2) $N(x_{kl}, t) = N(e_k[x_{ij}]e_l^*, t) \geqslant N_n\left([x_{ij}], \frac{t}{\|e_k\| \cdot \|e_l\|}\right) = N_n([x_{ij}], t)$.

$$\begin{aligned} N_n([x_{ij}], t) &= N_n\Big(\sum_{i,j=1}^n E_{ij} \otimes x_{ij}, t\Big) \geqslant \min\{N_n(E_{ij} \otimes x_{ij}, t_{ij}) : i, j = 1, 2, \cdots, n\} \\ &= \min\{N(x_{ij}, t_{ij}) : i, j = 1, 2, \cdots, n\}, \end{aligned}$$

where $t = \sum_{i,j=1}^n t_{ij}$. So, $N_n([x_{ij}], t) \geqslant \min\{N(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, \cdots, n\}$.

(3) By $N(x_{kl}, t) \geqslant N_n([x_{ij}], t) \geqslant \min\{N(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, \cdots, n\}$, we obtain the result. This completes the proof. □

For a mapping $f : X \to Y$, define $Df : X^m \to Y$ and $Df_n : M_n(X^4) \to M_n(Y)$ by

$$\begin{aligned} Df(a, b, c, d) &:= f(a + b, c + d) + f(a + b, c - d) \\ &\quad - 2f(a, c) - 2f(a, d) - 2f(b, c) - 2f(b, d), \\ Df_n\Big([x_{ij}], [y_{ij}], [z_{ij}], [w_{ij}]\Big) &:= f_n\Big([x_{ij}] + [y_{ij}], [z_{ij}] + [w_{ij}]\Big) + f_n\Big([x_{ij}] + [y_{ij}], [z_{ij}] - [w_{ij}]\Big) \\ &\quad - 2f_n\Big([x_{ij}], [z_{ij}]\Big) - 2f_n\Big([x_{ij}], [w_{ij}]\Big) - 2f_n\Big([y_{ij}], [z_{ij}]\Big) - 2f_n\Big([y_{ij}], [w_{ij}]\Big) \end{aligned}$$

for all $a, b, c, d \in X$ and all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}], w = [w_{ij}] \in M_n(X)$.

Theorem 2.2. *Let $f : X \to Y$, with $f(x, 0) = 0$, be a mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$ such that*

$$N_n\Big(f_n([x_{ij}], [y_{ij}], [z_{ij}], [w_{ij}]), t\Big) \geqslant \frac{t}{t + \sum_{i,j=1}^n \varphi(x_{ij}, y_{ij}, z_{ij}, w_{ij})} \tag{2.1}$$

for all $t > 0$ and all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}], w = [w_{ij}] \in M_n(X)$. If there exists an $\alpha < 1$ such that

$$\varphi(a, b, c, d) \leqslant 8\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \frac{d}{2}\right) \tag{2.2}$$

for all $a, b, c, d \in X$, then there exists a unique additive-quadratic mapping $T : X \times X \to Y$ such that

$$N_n\left(f_n\left([x_{ij}], [y_{ij}]\right) - T_n\left([x_{ij}], [y_{ij}]\right), t\right) \geqslant \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2 \sum_{i,j=1}^n \varphi(x_{ij}, x_{ij}, y_{ij}, y_{ij})} \tag{2.3}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. Putting $n = 1$ in (2.1), we have

$$N(Df(x, y, z, w), t) \geqslant \frac{t}{t + \varphi(x, y, z, w)} \tag{2.4}$$

for all $t > 0$ and $x, y, z, w \in X$.

Letting $x = y$ and $z = w$ in (2.4), we obtain

$$N(f(2x,2z) - 8f(x,z), t) \geqslant \frac{t}{t + \varphi(x,x,z,z)} \tag{2.5}$$

and also

$$N\left(\frac{1}{8}f(2x,2z) - f(x,z), \frac{t}{8}\right) \geqslant \frac{t}{t + \varphi(x,x,z,z)}$$

for all $t > 0$ and $x, z \in X$. Also it can be written as

$$N\left(\frac{1}{8}f(2x,2y) - f(x,y), \frac{t}{8}\right) \geqslant \frac{t}{t + \varphi(x,x,y,y)} \tag{2.6}$$

for all $t > 0$ and $x, y \in X$.

By considering the set of

$$\Omega := \{g : X \to Y\},$$

we introduce the generalized metric on Ω as following:

$$d(g,h) = \inf\left\{k \in \mathbb{R}^+ : N(g(x,y) - h(x,y), kt) \geqslant \frac{t}{t + \varphi(x,x,y,y)}, \forall x, y \in X, \forall t > 0\right\}$$

where, as usual $\inf \emptyset = +\infty$. It is easy to show that (Ω, d) is complete (see [5, 18]).

Now we define $J : \Omega \to \Omega$ by

$$Jg(x,y) := \frac{1}{8}h(2x,2y)$$

for all $x, y \in X$.

Let $g, h \in \Omega$ be given such that $d(g,h) = c$. Then

$$\begin{aligned}
&N(g(x,y) - h(x,y), ct) \geqslant \frac{t}{t + \varphi(2x,2x,2y,2y)} \\
&\Rightarrow N\left(\frac{1}{8}g(2x,2y) - \frac{1}{8}h(2x,2y), \frac{c}{8}t\right) \geqslant \frac{t}{t + \varphi(2x,2x,2y,2y)} \\
&\Rightarrow N\left(\frac{1}{8}g(2x,2y) - \frac{1}{8}h(2x,2y), \frac{c}{8}t\right) \geqslant \frac{t}{t + 8\alpha\varphi(x,x,y,y)} \\
&\Rightarrow N\left(\frac{1}{8}g(2x,2y) - \frac{1}{8}h(2x,2y), \alpha ct\right) \geqslant \frac{t}{t + \varphi(x,x,y,y)} \\
&\Rightarrow d(Jg, Jh) \leqslant \alpha c
\end{aligned}$$

for all $x, y \in X$. Hence we get that

$$d(Jg, Jh) \leqslant \alpha d(g,h)$$

for all $g, h \in \Omega$. It follows from (2.6) that $d(f, Jf) \leqslant \frac{1}{8}$.

By Theorem 1.6, there exists a mapping $T : X \to Y$ satisfying the following:

(1) T is a fixed point of J, i.e., $T(2x,2y) = 8T(x,y)$ for all $x \in X$. The mapping T is a unique fixed point of J in the set $X = \{g \in \Omega : d(f,g) < \infty\}$.
(2) $d(J^k f, T) \to 0$ as $k \to \infty$. This implies the inequality $N - \lim_{k\to\infty} \frac{1}{8^k} f(2^k x, 2^k y) = T(x,y)$ for all $x, y \in X$.

(3) $d(f,T) \leqslant \frac{1}{1-\alpha} d(f,Jf)$, which implies the inequality

$$(f,T) \leqslant \frac{1}{8(1-\alpha)}. \tag{2.7}$$

By (2.2) and (2.4),

$$\begin{aligned} N\left(\frac{1}{8^k} Df(2^k x, 2^k y, 2^k z, 2^k w)\right) &\geqslant \frac{t}{t+\varphi(2^k x, 2^k y, 2^k z, 2^k w)} \\ &\geqslant \frac{8^k t}{8^k t + 8^k \alpha^k \varphi(x,y,z,w)} \end{aligned}$$

for all $x,y,z,w \in X$ and $t>0$. Since $\lim_{k\to\infty} \frac{8^k t}{8^k t+8^k\alpha^k\varphi(x,y,z,w)} = 1$ for all $x,y,z,w \in X$ and $t>0$,

$$N(DT(x,y,z,w),t) = 1$$

for all $x,y,z,w \in X$ and $t>0$. Therefore

$$T(x+y,z+w)+T(x+y,z-w) = 2T(x,z)+2T(x,w)+2T(y,z)+2T(y,w).$$

for all $x,y,z,w \in X$. Then, the mapping $T: X\times X \to Y$ is additive-quadratic.

It follows from Lemma 2.1 and (2.7) that

$$\begin{aligned} N_n\Big(f_n([x_{ij}],[y_{ij}]) - T_n([x_{ij}],[y_{ij}]),t\Big) &\geqslant \left\{N\left(f(x_{ij},y_{ij}) - T(x_{ij},y_{ij}), \frac{t}{n^2}\right) : i,j=1,2,\cdots,n\right\} \\ &\geqslant \min\left\{\frac{8(1-\alpha)t}{8(1-\alpha)t + n^2\varphi(x_{ij},x_{ij},y_{ij},y_{ij})} : i,j=1,2,\cdots,n\right\} \\ &\geqslant \frac{8(1-\alpha)t}{8(1-\alpha)t+n^2\sum_{i,j=1}^n \varphi(x_{ij},x_{ij},y_{ij},y_{ij})} \end{aligned}$$

for all $x=[x_{ij}] \in M_n(X)$. Therefore, we conclude that $T: X\times X\to Y$ is the unique mapping satisfying (2.3). $\square$

Corollary 2.3. *Let p,θ be positive real numbers $p<1$. Let $f: X\times X\to Y$, with $f(x,0)=0$, be a mapping satisfying*

$$N_n(Df_n([x_{ij}],[y_{ij}],[z_{ij}],[w_{ij}]),t) \geqslant \frac{t}{t+\sum_{i,j=1}^n \theta(\|x_{ij}\|^p+\|y_{ij}\|^p+\|z_{ij}\|^p+\|w_{ij}\|^p)} \tag{2.8}$$

for all $x=[x_{ij}], y=[y_{ij}], z=[z_{ij}], w=[w_{ij}] \in M_n(X)$ and $t>0$. Then $T(x,y) := N-\lim_{k\to\infty}\frac{1}{8^k} f(2^k x, 2^k y)$ exists for each $x,y\in X$ and defines an additive-quadratic mapping $T: X\times X\to Y$ such that

$$N_n\Big(f_n([x_{ij}],[y_{ij}]) - T_n([x_{ij}],[y_{ij}]),t\Big) \geqslant \frac{2(2-2^p)t}{2(2-2^p)t+n^2\sum_{i,j=1}^n \theta(\|x_{ij}\|^p+\|y_{ij}\|^p)}$$

for all $x=[x_{ij}], y=[y_{ij}] \in M_n(X)$ and $t>0$.

Proof. Putting $\varphi(a,b,c,z) := \theta\sum_{i=1}^m(\|a\|^p+\|b\|^p+\|c\|^p+\|d\|^p)$ for all $a,b,c,d\in X$ and letting $\alpha = 2^{p-1}$ in Theorem 2.2, we obtain the desired result. $\square$

Theorem 2.4. *Let $f : X \times X \to Y$, with $f(x, 0) = 0$, be a mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$ satisfying (2.1). If there exists an $\alpha < 1$ such that*

$$\varphi\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \frac{d}{2}\right) \leqslant \frac{\alpha}{8}\varphi(a, b, c, d)$$

for all $a, b, c, d \in X$, then there exists a unique additive-quadratic mapping $T : X \times X \to Y$ such that

$$N\Big(f_n([x_{ij}], [y_{ij}]) - T_n([x_{ij}], [y_{ij}]), t\Big) \geqslant \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2\alpha\sum_{i,j=1}^{n}\varphi(x_{ij}, x_{ij}, y_{ij}, y_{ij})}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. Let (Ω, d) be the generalized metric space defined in the proof of Theorem 2.2. Here, we define the linear mapping $J : \Omega \to \Omega$ such that

$$Jg(x, y) := 8g(\frac{x}{2}, \frac{y}{2})$$

for all $x, y \in X$.

It follows from (2.5) that $d(f, Jf) \leqslant \frac{\alpha}{8}$. Thus

$$d(f, T) \leqslant \frac{\alpha}{8(1-\alpha)}.$$

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let p, θ be positive real numbers with $p > 1$. Let $f : X \times X \to Y$, with $f(x, 0) = 0$, be a mapping satisfying (2.8). Then $T(x, y) := N - \lim_{k\to\infty} 8^k f(\frac{x}{2^k}, \frac{y}{2^k})$ exists for all $x \in X$ and defines an additive-quadratic mapping $T : X \times X \to Y$ such that*

$$N_n\Big(f_n([x_{ij}], [y_{ij}]) - T_n([x_{ij}], [y_{ij}]), t\Big) \geqslant \frac{4(2^p-2)t}{4(2^p-2)t + n^2 \cdot 2^p\sum_{i,j=1}^{n}\theta\left(\|x_{ij}\|^p + \|y_{ij}\|^p\right)}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and $t > 0$.

Proof. Putting $\varphi(a, b, c, d) := \theta(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$ for all $a, b, c, d \in X$ and letting $\alpha = 2^{1-p}$ in Theorem 2.4, we get the desired result. $\square$

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Closed Form Expressions of some systems of Nonlinear Partial Difference Equations

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Abstract

In this paper we give the closed form expressions of some two dimensional systems of nonlinear rational partial difference equations of second order.We shall use a new method to prove the results by using (odd-even) double mathematical induction. As a direct consequences , we investigate and drive the explicit solutions of some partial difference equations and some (systems of) ordinary difference equations .

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Key Words and Phrases: (partial)difference equations, solutions , double mathematical induction.

1 Introduction

While the study of (ordinary)difference equations has been widely treated in the past , partial difference equations (PΔEs) have not received the same full attention .Both of ordinary and partial difference equations may be found in the study of probability ,dynamics and other branches of mathematical physics .Moreover,partial difference equations arise in applications involving population dynamics with spatial migrations , chemical reactions and finite difference schemes . Indeed Laplace and Lagrange considered the solution of partial difference equations in their studies of dynamics and probability.
An example of a partial difference equation is the following well known relation

$$C_m^{(n)} = C_{m-1}^{(n-1)} + C_m^{(n-1)} \quad ,1 \leq m < n.$$

The solution of this equation is the celebrated binomial coefficient function $C_m^{(n)}$ defined by

$$C_m^{(n)} = \frac{n!}{m!(n-m)!} \quad ,0 \leq m < n.$$

An another example , the following PΔEs :

$$s_k^{(n+1)} = s_{k-1}^{(n)} - ns_k^{(n)} \quad ,1 \leq k < n.$$

$$S_k^{(n+1)} = S_{k-1}^{(n)} + kS_k^{(n)} \quad ,1 \leq k < n.$$

The solutions of these PΔEs are the stirling numbers of the first kind $s_k^{(n)}$ and the stirling numbers of the second kind $S_k^{(n)}$ respectively .

Some authors investigate the closed form solutions for certain Partial difference equations .
For instance , Heins [[9]] considered the solution of the partial difference equation

$$X_{n+1,m} + X_{n-1,m} = 2X_{n,m+1}$$

under some conditions .
In [[3]] Carlitz has studied a solution of the partial difference equation

$$X_{n,m} - X_{n,m-1} - X_{n-1,m} - X_{n,m-2} + 3X_{n-1,m-1} - X_{n-2,m} = 0$$

He used a power series expansion related to the Fibonacci numbers .

For more results about partial difference equations we refer to ([1],[2], [4]-[8],[10],[11]-[15]).

In this paper , we studied the closed form solutions of the following systems of partial difference equations

$$\alpha X_{n,m} + \beta X_{n,m}X_{n-2,m-2}Y_{n-1,m-1} - X_{n-2,m-2} = 0 \tag{1}$$

$$\gamma Y_{n,m} + \delta Y_{n,m}Y_{n-2,m-2}X_{n-1,m-1} - Y_{n-2,m-2} = 0 \tag{2}$$

where $n, m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$,$\alpha, \beta, \gamma, \delta \in \{1, -1\}$ and the initial values $X_{n,0}$,$X_{n,-1}$,$X_{0,m}$,$X_{-1,m}$, $Y_{n,0}$,$Y_{n,-1}$,$Y_{0,m}$,and $Y_{-1,m}$ are real numbers .

As a direct consequence , we can drive the explicit solutions of a family of partial difference equations in the following form

$$\alpha X_{n,m} + \beta X_{n,m}X_{n-2,m-2}X_{n-1,m-1} - X_{n-2,m-2} = 0$$

where $n, m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$,$\alpha, \beta \in \{1, -1\}$ and the initial values $X_{n,0}$,$X_{n,-1}$,$X_{0,m}$, ,and $X_{-1,m}$ are real numbers .
Moreover , we can derive the exact solution for the following systems of ordinary difference equations

$$\alpha X_n + \beta X_n X_{n-2} Y_{n-1} - X_{n-2} = 0$$

$$\gamma Y_n + \delta Y_n Y_{n-2} X_{n-1} - Y_{n-2} = 0$$

where $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$,$\alpha, \beta, \gamma, \delta \in \{1, -1\}$ and the initial values X_0,X_{-1},Y_0,and Y_{-1} are real numbers .

2 Forms of Solutions

In this section we shall give explicit forms of solutions of the system (1)-(2) for particular values of $\alpha, \beta, \gamma, \delta \in \{1, -1\}$. We can rewrite system (1)-(2) in the following form

$$X_{n,m} = \frac{X_{n-2,m-2}}{\alpha + \beta X_{n-2,m-2}Y_{n-1,m-1}} \quad , \quad Y_{n,m} = \frac{Y_{n-2,m-2}}{\gamma + \delta Y_{n-2,m-2}X_{n-1,m-1}} \tag{3}$$

2.1 Form of Solutions when $(\alpha, \beta) = (\gamma, \delta) = (1, -1)$

In this case we have the system

$$X_{n,m} = \frac{X_{n-2,m-2}}{1 - X_{n-2,m-2}Y_{n-1,m-1}} \quad , \quad Y_{n,m} = \frac{Y_{n-2,m-2}}{1 - Y_{n-2,m-2}X_{n-1,m-1}} \tag{4}$$

Theorem 1. *Let* $\{X_{n,m}, Y_{n,m}\}_{n,m=-k}^{\infty}$ *be a solution of system* (4) *with initial conditions*

$$X_{n,0}, X_{n,-1}, X_{0,m}, X_{-1,m}, Y_{n,0}, Y_{n,-1}, Y_{0,m}, Y_{-1,m}$$

where $n, m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. *Suppose* $X_{-1,m-2}Y_{0,m-1} \neq 1$,$X_{n-2,-1}Y_{n-1,0} \neq 1$,$Y_{-1,m-2}X_{0,m-1} \neq 1$,$Y_{n-2,-1}X_{n-1,0} \neq 1$. *Then, the form of solutions of system* (4) *,for* $n, m \geq 1$ *and* $n \geq m$ *, are as follows:*

$$X_{n,m} = \begin{cases} X_{n-m,0} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)X_{n-m,0}Y_{n-m-1,-1}}{-1+(2k+2)X_{n-m,0}Y_{n-m-1,-1}}, & m \quad even; \\ X_{n-m-1,-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{n-m-1,-1}Y_{n-m,0}}{1-(2k+1)X_{n-m-1,-1}Y_{n-m,0}}, & m \quad odd; \end{cases} \tag{5}$$

$$Y_{n,m} = \begin{cases} Y_{n-m,0} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)Y_{n-m,0}X_{n-m-1,-1}}{-1+(2k+2)Y_{n-m,0}X_{n-m-1,-1}}, & m \quad even; \\ Y_{n-m-1,-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)Y_{n-m-1,-1}X_{n-m,0}}{1-(2k+1)Y_{n-m-1,-1}X_{n-m,0}}, & m \quad odd; \end{cases} \tag{6}$$

$$X_{m,n} = \begin{cases} X_{-1,n-m-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{-1,n-m-1}Y_{0,n-m}}{1-(2k+1)X_{-1,n-m-1}Y_{0,n-m}}, & m \quad odd; \\ X_{0,n-m} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)X_{0,n-m}Y_{-1,n-m-1}}{-1+(2k+2)X_{0,n-m}Y_{-1,n-m-1}}, & m \quad even; \end{cases} \tag{7}$$

$$Y_{m,n} = \begin{cases} Y_{-1,n-m-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)Y_{-1,n-m-1}X_{0,n-m}}{1-(2k+1)Y_{-1,n-m-1}X_{0,n-m}}, & m \quad odd; \\ Y_{0,n-m} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)Y_{0,n-m}X_{-1,n-m-1}}{-1+(2k+2)Y_{0,n-m}X_{-1,n-m-1}}, & m \quad even; \end{cases} \tag{8}$$

Proof. We shall use the principle of (odd-even)double mathematical induction . Firstly , we shall prove that the relations (5)-(8) hold for $(n, m) = (1, 1)$. From equations in system (4)we can see

$$X_{1,1} = \frac{X_{-1,-1}}{1 - X_{-1,-1}Y_{0,0}} = X_{-1,-1} \prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)X_{-1,-1}Y_{0,0}}{1 - (2k+1)X_{-1,-1}Y_{0,0}}$$

$$Y_{1,1} = \frac{Y_{-1,-1}}{1 - Y_{-1,-1}X_{0,0}} = Y_{-1,-1} \prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)Y_{-1,-1}X_{0,0}}{1 - (2k+1)Y_{-1,-1}X_{0,0}}$$

Now , we shall prove that the relations (5)-(8)hold for $(n,m) = (2,2)$.

$$X_{2,2} = \frac{X_{0,0}}{1 - X_{0,0}Y_{1,1}} = \frac{X_{0,0}}{1 - X_{0,0}(\frac{Y_{-1,-1}}{1-Y_{-1,-1}X_{0,0}})} = X_{0,0}(\frac{1 - X_{0,0}Y_{-1,-1}}{1 - 2X_{0,0}Y_{-1,-1}})$$

$$= X_{0,0}\prod_{k=0}^{\frac{2-2}{2}} \frac{-1 + (2k+1)X_{0,0}Y_{-1,-1}}{-1 + (2k+2)X_{0,0}Y_{-1,-1}}$$

$$Y_{2,2} = \frac{Y_{0,0}}{1 - Y_{0,0}X_{1,1}} = \frac{Y_{0,0}}{1 - Y_{0,0}(\frac{X_{-1,-1}}{1-X_{-1,-1}Y_{0,0}})} = Y_{0,0}(\frac{1 - Y_{0,0}X_{-1,-1}}{1 - 2Y_{0,0}X_{-1,-1}})$$

$$= Y_{0,0}\prod_{k=0}^{\frac{2-2}{2}} \frac{-1 + (2k+1)Y_{0,0}X_{-1,-1}}{-1 + (2k+2)Y_{0,0}X_{-1,-1}}$$

Moreover ,We shall prove that the relations (5)-(8) hold for $(n,m) = (1,2)$ and $(n,m) = (2,1)$.

$$X_{1,2} = \frac{X_{-1,0}}{1 - X_{-1,0}Y_{0,1}} = X_{-1,0}\prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)X_{-1,0}Y_{0,1}}{1 - (2k+1)X_{-1,0}Y_{0,1}}$$

$$Y_{1,2} = \frac{Y_{-1,0}}{1 - Y_{-1,0}X_{0,1}} = Y_{-1,0}\prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)Y_{-1,0}X_{0,1}}{1 - (2k+1)Y_{-1,0}X_{0,1}}$$

$$X_{2,1} = \frac{X_{0,-1}}{1 - X_{0,-1}Y_{1,0}} = X_{0,-1}\prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)X_{0,-1}Y_{1,0}}{1 - (2k+1)X_{0,-1}Y_{1,0}}$$

$$Y_{2,1} = \frac{Y_{0,-1}}{1 - Y_{0,-1}X_{1,0}}$$

Now suppose that the relations (5)-(8) hold for $m = 1$ and $m = 2$ with $n \in \mathbb{N}$. So we have ,

$$X_{n,1} = X_{n-2,-1}\prod_{k=0}^{0} \frac{1 - (2k)X_{n-2,-1}Y_{n-1,0}}{1 - (2k+1)X_{n-2,-1}Y_{n-1,0}} = \frac{X_{n-2,-1}}{1 - X_{n-2,-1}Y_{n-1,0}}$$

$$Y_{n,1} = \frac{Y_{n-2,-1}}{1 - Y_{n-2,-1}X_{n-1,0}}$$

$$X_{n,2} = X_{n-2,0}(\frac{1 - X_{n-2,0}Y_{n-3,-1}}{1 - 2X_{n-2,0}Y_{n-3,-1}})$$

$$Y_{n,2} = Y_{n-2,0}(\frac{1 - Y_{n-2,0}X_{n-3,-1}}{1 - 2Y_{n-2,0}X_{n-3,-1}})$$

Now we try to prove that relations (5)-(8) hold for $m = 1$ with $n + 2$.

$$X_{n+2,1} = \frac{X_{n,-1}}{1 - X_{n,-1}Y_{n+1,0}} = X_{n,-1}\prod_{k=0}^{\frac{0}{2}} \frac{1-(2k)X_{n,-1}Y_{n+1,0}}{1-(2k+1)X_{n,-1}Y_{n+1,0}}$$

$$Y_{n+2,1} = \frac{Y_{n,-1}}{1 - Y_{n,-1}X_{n+1,0}} = Y_{n,-1}\prod_{k=0}^{\frac{1-1}{2}} \frac{1-(2k)Y_{n,-1}X_{n+1,0}}{1-(2k+1)Y_{n,-1}X_{n+1,0}}$$

Now we try to prove that relations (5)-(8) hold for $m = 2$ with $n + 2$.

$$X_{n+2,2} = \frac{X_{n,0}}{1 - X_{n,0}Y_{n+1,1}} = \frac{X_{n,0}}{1 - X_{n,0}(\frac{Y_{n-1,-1}}{1-Y_{n-1,-1}X_{n,0}})}$$

$$= \frac{X_{n,0}(1-Y_{n-1,-1}X_{n,0})}{1-2Y_{n-1,-1}X_{n,0}} = X_{n,0}\prod_{k=0}^{\frac{2-2}{2}} \frac{1-(2k+1)X_{n,0}Y_{n-1,-1}}{1-(2k+2)X_{n,0}Y_{n-1,-1}}$$

$$Y_{n+2,2} = \frac{Y_{n,0}}{1 - Y_{n,0}X_{n+1,1}} = Y_{n,0}\prod_{k=0}^{\frac{2-2}{2}} \frac{1-(2k+1)Y_{n,0}X_{n-1,-1}}{1-(2k+2)Y_{n,0}X_{n-1,-1}}$$

Finally , we suppose that relations (5)-(8) hold for $n, m \in \mathbb{N}$. We shall prove that relations (5)-(8) hold for $n, m+2 \in \mathbb{N}$.
From (4)we have

$$X_{n,m+2} = \frac{X_{n-2,m}}{1 - X_{n-2,m}Y_{n-1,m+1}} \tag{9}$$

There are four cases :

(1) If $n > m + 2$ and m even .

$$X_{n,m+2} = \frac{X_{n-2,m}}{1 - X_{n-2,m}Y_{n-1,m+1}}$$

$$= \frac{X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}}}{1-(X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}})(Y_{n-m-3,-1}\prod_{k=0}^{\frac{m}{2}}\frac{1-(2k)Y_{n-m-3,-1}X_{n-m-2,0}}{1-(2k+1)Y_{n-m-3,-1}X_{n-m-2,0}})}$$

$$= \frac{X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}}}{1-\frac{X_{n-m-2,0}Y_{n-m-3,-1}}{1-(m+1)X_{n-m-2,0}Y_{n-m-3,-1}}}$$

$$= X_{n-m-2,0}\prod_{k=0}^{\frac{m}{2}} \frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}}$$

(2) If $n > m+2$ and m odd

$$X_{n,m+2} = \frac{X_{n-2,m}}{1 - X_{n-2,m}Y_{n-1,m+1}}$$

$$= \frac{X_{n-m-3,-1}\prod_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}}}{1-(X_{n-m-3,-1}\prod_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}})(Y_{n-m-2,0}\prod_{k=0}^{\frac{m-1}{2}} \frac{1-(2k+1)Y_{n-m-2,0}X_{n-m-3,-1}}{1-(2k+2)Y_{n-m-2,0}X_{n-m-3,-1}})}$$

$$= \frac{X_{n-m-3,-1}\prod_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}}}{1-\frac{X_{n-m-3,-1}Y_{n-m-2,0}}{1-(m+1)X_{n-m-3,-1}Y_{n-m-2,0}}}$$

$$= X_{n-m-3,-1}\prod_{k=0}^{\frac{m+1}{2}} \frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}}$$

(3) If $n < m+2$ and m even
By symmetry ,using (7) and (8), we can prove it like part (1) .

(4) If $n < m+2$ and m odd
By symmetry ,using (7) and (8), we can prove it like part (2) ..

$$Y_{n,m+2} = \frac{Y_{n-2,m}}{1 - Y_{n-2,m}X_{n-1,m+1}}$$

We can do that by the same way in proving equation (9)

□

Proposition 1. *We have the following properties for the solutions of system (4) :*

(1) If m even and $X_{n-m,0} = 0$, then $X_{n,m} = 0$.

(2) If m odd and $X_{n-m,0} = 0$, then $Y_{n,m} = Y_{n-m-1,-1}$.

(3) If m even and $Y_{n-m,0} = 0$, then $Y_{n,m} = 0$.

(4) If m odd and $Y_{n-m,0} = 0$, then $X_{n,m} = X_{n-m-1,-1}$.

(5) If m even and $X_{n-m-1,-1} = 0$, then $Y_{n,m} = Y_{n-m,0}$.

(6) If m odd and $X_{n-m-1,-1} = 0$, then $X_{n,m} = 0$.

(7) If m even and $Y_{n-m-1,-1} = 0$, then $X_{n,m} = X_{n-m,0}$.

(8) If m odd and $Y_{n-m-1,-1} = 0$, then $Y_{n,m} = 0$.

Proposition 2. *We have the following properties for the solutions of system* (4) *:*

(1) If m *even and* $X_{0,n-m} = 0$ *, then* $X_{m,n} = 0$ *.*

(2) If m *odd and* $X_{0,n-m} = 0$ *, then* $Y_{m,n} = Y_{-1,n-m-1}$ *.*

(3) If m *even and* $Y_{0,n-m} = 0$ *, then* $Y_{m,n} = 0$ *.*

(4) If m *odd and* $Y_{0,n-m} = 0$ *, then* $X_{m,n} = X_{-1,n-m-1}$ *.*

(5) If m *even and* $X_{-1,n-m-1} = 0$ *, then* $Y_{m,n} = Y_{0,n-m}$ *.*

(6) If m *odd and* $X_{-1,n-m-1} = 0$*, then* $X_{m,n} = 0$ *.*

(7) If m *even and* $Y_{-1,n-m-1} = 0$ *, then* $X_{m,n} = X_{0,n-m}$ *.*

(8) If m *odd and* $Y_{-1,n-m-1} = 0$*, then* $Y_{m,n} = 0$ *.*

Remark 1. *If we take into account the one dimensional case of system* (4) *we have a partial difference equation in the form*

$$X_{n,m} = \frac{X_{n-2,m-2}}{1 - X_{n-2,m-2}X_{n-1,m-1}} \tag{10}$$

We can see that the closed form solution of equation(10) *is given ,from theorem*(1) *, by the following corollary .*

Corollary 2. *Let* $\{X_{n,m}\}_{n,m=-k}^{\infty}$ *be a solution of equation* (10) *with initial conditions* $X_{n,0}, X_{n,-1}, X_{0,m}, X_{-1,m}$*, where* $n, m \in \mathbb{N}_0$ *,* $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ *. Suppose* $X_{-1,m-2}X_{0,m-1} \neq 1$ *,*$X_{n-2,-1}X_{n-1,0} \neq 1$ *. Then, the form of solutions of equation* (10) *,for* $n, m \geq 1$ *and* $n \geq m$ *, are as follows:*

$$X_{n,m} = \begin{cases} X_{n-m,0} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)X_{n-m,0}X_{n-m-1,-1}}{-1+(2k+2)X_{n-m,0}X_{n-m-1,-1}}, & m \quad even; \\ X_{n-m-1,-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{n-m-1,-1}X_{n-m,0}}{1-(2k+1)X_{n-m-1,-1}X_{n-m,0}}, & m \quad odd; \end{cases}$$

$$X_{m,n} = \begin{cases} X_{-1,n-m-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{-1,n-m-1}X_{0,n-m}}{1-(2k+1)X_{-1,n-m-1}X_{0,n-m}}, & m \quad odd; \\ X_{0,n-m} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)X_{0,n-m}X_{-1,n-m-1}}{-1+(2k+2)X_{0,n-m}X_{-1,n-m-1}}, & m \quad even; \end{cases}$$

Proposition 3. *We have the following properties for the solutions of equation* (4)*:*

(1) If m even and $X_{n-m,0} = 0$, then $X_{n,m} = 0$.

(2) If m odd and $X_{n-m,0} = 0$, then $X_{n,m} = X_{n-m-1,-1}$.

(3) If m even and $X_{n-m-1,-1} = 0$, then $X_{n,m} = X_{n-m,0}$.

(4) If m odd and $X_{n-m-1,-1} = 0$, then $X_{n,m} = 0$.

(5) If m even and $X_{0,n-m} = 0$, then $X_{m,n} = 0$.

(6) If m odd and $X_{0,n-m} = 0$, then $X_{m,n} = X_{-1,n-m-1}$.

(7) If m even and $X_{-1,n-m-1} = 0$, then $X_{m,n} = X_{0,n-m}$.

(8) If m odd and $X_{-1,n-m-1} = 0$, then $X_{m,n} = 0$.

Remark 2. *If we put $n = m$ in system (4) we have a system of ordinary difference equations in the following form*

$$X_n = \frac{X_{n-2}}{1 - X_{n-2}Y_{n-1}}, \qquad Y_n = \frac{Y_{n-2}}{1 - Y_{n-2}X_{n-1}} \tag{11}$$

Corollary 3. *Let $\{X_n, Y_n\}_{n=-k}^{\infty}$ be a solution of system (11) with initial conditions X_0, X_{-1}, Y_0, Y_{-1} . Suppose $X_{-1}Y_0 \neq 1$,and $Y_{-1}X_0 \neq 1$,. Then, the form of solutions of system (11) ,for $n \geq 1$ are as follows:*

$$X_n = \begin{cases} X_0 \prod\limits_{k=0}^{\frac{n-2}{2}} \frac{-1+(2k+1)X_0Y_{-1}}{-1+(2k+2)X_0Y_{-1}}, n, even \\ X_{-1} \prod\limits_{k=0}^{\frac{n-1}{2}} \frac{1-(2k)X_{-1}Y_0}{1-(2k+1)X_{-1}Y_0}, n, odd \end{cases} \qquad Y_n = \begin{cases} Y_0 \prod\limits_{k=0}^{\frac{n-2}{2}} \frac{-1+(2k+1)Y_0X_{-1}}{-1+(2k+2)Y_0X_{-1}}, n, even \\ Y_{-1} \prod\limits_{k=0}^{\frac{n-1}{2}} \frac{1-(2k)Y_{-1}X_0}{1-(2k+1)Y_{-1}X_0}, n, odd \end{cases}$$

Remark 3. *If we put $X = Y$ in system(11) we get an ordinary difference equation in the form*

$$X_n = \frac{X_{n-2}}{1 - X_{n-2}X_{n-1}} \tag{12}$$

We can see that the closed form solution of equation(12) is given ,from corollary(3) , by the following

$$X_n = \begin{cases} X_0 \prod\limits_{k=0}^{\frac{n-2}{2}} \frac{-1+(2k+1)X_0X_{-1}}{-1+(2k+2)X_0X_{-1}}, \quad n \quad even; \\ X_{-1} \prod\limits_{k=0}^{\frac{n-1}{2}} \frac{1-(2k)X_{-1}X_0}{1-(2k+1)X_{-1}X_0}, \quad n \quad odd; \end{cases}$$

where $n \in \mathbb{N}$, and $X_{-1}X_0 \neq -1$ .We can easy see that if n even (or odd) and $X_0 = 0$ then $X_n = 0 (X_n = X_{-1})$. Also if n even (or odd) and $X_{-1} = 0$ then $X_n = X_0 (X_n = 0)$.

2.2 Form of Solutions when $(\alpha,\beta)=(1,1)\&(\gamma,\delta)=(1,-1)$

In this case we have the system

$$X_{n,m}=\frac{X_{n-2,m-2}}{1+X_{n-2,m-2}Y_{n-1,m-1}} \quad , \quad Y_{n,m}=\frac{Y_{n-2,m-2}}{1-Y_{n-2,m-2}X_{n-1,m-1}} \tag{13}$$

Theorem 4. *Let $\{X_{n,m},Y_{n,m}\}_{n,m=-k}^{\infty}$ be a solution of system (13) with initial conditions $X_{n,0},X_{n,-1},X_{0,m},X_{-1,m},Y_{n,0},Y_{n,-1},Y_{0,m},Y_{-1,m}$ where $n,m\in\mathbb{N}_0$, $\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$. Suppose $X_{-1,m-2}Y_{0,m-1}\neq-1$,$X_{n-2,-1}Y_{n-1,0}\neq-1$, $Y_{-1,m-2}X_{0,m-1}\neq1$,$Y_{n-2,-1}X_{n-1,0}\neq1$. Then, the form of solutions of system (13) ,for $n,m\geq1$ and $n\geq m$, are as follows:*

$$X_{n,m}=\begin{cases}\frac{X_{n-m-1,-1}}{(1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+1}{2}}}, & m \quad odd;\\ (-1)^{\frac{m}{2}}X_{n-m,0}(-1+X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

$$Y_{n,m}=\begin{cases}\frac{(-1)^{\frac{m+1}{2}}Y_{n-m-1,-1}}{(-1+Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{2}}}, & m \quad odd;\\ Y_{n-m,0}(1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

$$X_{m,n}=\begin{cases}\frac{X_{-1,n-m-1}}{(1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+1}{2}}}, & m \quad odd;\\ (-1)^{\frac{m}{2}}X_{0,n-m}(-1+X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

$$Y_{m,n}=\begin{cases}\frac{(-1)^{\frac{m+1}{2}}Y_{-1,n-m-1}}{(-1+Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{2}}}, & m \quad odd;\\ Y_{0,n-m}(1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

Proof. We can prove the theorem by odd-even double mathematical induction as in theorem (1). □

Remark 4. *We can see that both of proposition (1) and proposition (2) hold for the solutions of system (13) included in theorem(4) .*

Remark 5. *If we put $n=m$ in system (13) we have a system of ordinary difference equations in the following form*

$$X_n=\frac{X_{n-2}}{1+X_{n-2}Y_{n-1}}, \qquad Y_n=\frac{Y_{n-2}}{1-Y_{n-2}X_{n-1}} \tag{14}$$

We can drive the formulas for solutions from theorem(4) in the following corollary .

Corollary 5. *Let $\{X_n,Y_n\}_{n=-k}^{\infty}$ be a solution of system (14) with initial conditions X_0,X_{-1},Y_0,Y_{-1} . Suppose $X_{-1}Y_0\neq-1$,and $Y_{-1}X_0\neq1$,. Then, the form of solutions of system (14) ,for $n\geq1$ are as follows:*

$$X_n=\begin{cases}\frac{X_{-1}}{(1+X_{-1}Y_0)^{\frac{n+1}{2}}}; n, odd\\ (-1)^{\frac{n}{2}}X_0(-1+X_0Y_{-1})^{\frac{n}{2}}; n, even\end{cases} \qquad Y_n=\begin{cases}\frac{(-1)^{\frac{n+1}{2}}Y_{-1}}{(-1+Y_{-1}X_0)^{\frac{n+1}{2}}}; n, odd\\ Y_0(1+Y_0X_{-1})^{\frac{n}{2}}; n, even\end{cases}$$

.

2.3 Form of Solutions when $(\alpha,\beta)=(1,1)\&(\gamma,\delta)=(-1,1)$

In this case we have the system

$$X_{n,m}=\frac{X_{n-2,m-2}}{1+X_{n-2,m-2}Y_{n-1,m-1}} \quad , \quad Y_{n,m}=\frac{Y_{n-2,m-2}}{-1+Y_{n-2,m-2}X_{n-1,m-1}} \qquad (15)$$

Theorem 6. *Let $\{X_{n,m},Y_{n,m}\}_{n,m=-k}^{\infty}$ be a solution of system (15) with initial conditions $X_{n,0},X_{n,-1},X_{0,m},X_{-1,m},Y_{n,0},Y_{n,-1},Y_{0,m},Y_{-1,m}$ where $n,m\in\mathbb{N}_0$, $\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$. Suppose $X_{-1,m-2}Y_{0,m-1}\neq-1$,$X_{n-2,-1}Y_{n-1,0}\neq-1$, $Y_{-1,m-2}X_{0,m-1}\neq 1$,$Y_{n-2,-1}X_{n-1,0}\neq 1$. Then, the form of solutions of system (15) ,for $n,m\geq 1$ and $n\geq m$, are as follows:*

$$X_{n,m}=\begin{cases}\dfrac{(-1)^{\frac{m-1}{4}}X_{n-m-1,-1}}{(1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+3}{4}}(-1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m-1}{4}}}, & m=4K+1;\\ \dfrac{(-1)^{\frac{m-2}{4}}X_{n-m,0}(-1+X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{2}}}{(-1+2X_{n-m,0}Y_{n-m-1,-1})^{\frac{m+2}{4}}}, & m=4K+2;\\ \dfrac{(-1)^{\frac{m+1}{4}}X_{n-m-1,-1}}{(-1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+1}{4}}(1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+1}{4}}}, & m=4K+3;\\ \dfrac{(-1)^{\frac{m}{4}}X_{n-m,0}(-1+X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{2}}}{(-1+2X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{4}}}, & m=4K+4;\end{cases}$$

$$Y_{n,m}=\begin{cases}\dfrac{(-1)^{\frac{m-1}{4}}Y_{n-m-1,-1}(-1+2Y_{n-m-1,-1}X_{n-m,0})^{\frac{m-1}{4}}}{(-1+Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{2}}}, & m=4K+1;\\ (-1)^{\frac{m+2}{4}}Y_{n-m,0}(-1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m-2}{4}} & \\ \quad .(1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m+2}{4}}, & m=4K+2;\\ \dfrac{(-1)^{\frac{m+1}{4}}Y_{n-m-1,-1}(-1+2Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{4}}}{(-1+Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{2}}}, & m=4K+3;\\ (-1)^{\frac{m}{4}}Y_{n-m,0}(-1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m}{4}} & \\ \quad .(1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m}{4}}, & m=4K+4;\end{cases}$$

$$X_{m,n}=\begin{cases}\dfrac{(-1)^{\frac{m-1}{4}}X_{-1,n-m-1}}{(1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+3}{4}}(-1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m-1}{4}}}, & m=4K+1;\\ \dfrac{(-1)^{\frac{m-2}{4}}X_{0,n-m}(-1+X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{2}}}{(-1+2X_{0,n-m}Y_{-1,n-m-1})^{\frac{m+2}{4}}}, & m=4K+2;\\ \dfrac{(-1)^{\frac{m+1}{4}}X_{-1,n-m-1}}{(-1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+1}{4}}(1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+1}{4}}}, & m=4K+3;\\ \dfrac{(-1)^{\frac{m}{4}}X_{0,n-m}(-1+X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{2}}}{(-1+2X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{4}}}, & m=4K+4;\end{cases}$$

$$Y_{m,n}=\begin{cases}\dfrac{(-1)^{\frac{m-1}{4}}Y_{-1,n-m-1}(-1+2Y_{-1,n-m-1}X_{0,n-m})^{\frac{m-1}{4}}}{(-1+Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{2}}}, & m=4K+1;\\ (-1)^{\frac{m+2}{4}}Y_{0,n-m}(-1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m-2}{4}} & \\ \quad .(1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m+2}{4}}, & m=4K+2;\\ \dfrac{(-1)^{\frac{m+1}{4}}Y_{-1,n-m-1}(-1+2Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{4}}}{(-1+Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{2}}}, & m=4K+3;\\ (-1)^{\frac{m}{4}}Y_{0,n-m}(-1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m}{4}} & \\ \quad .(1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m}{4}}, & m=4K+4;\end{cases}$$

where $k = 0, 1, 2, 3......$.

Proof. We can prove the theorem by piecewise double mathematical induction as in theorem (1). ☐

Proposition 4. *We have the following properties for the solutions of system* (15) *:*

(1) *If* m *even and* $X_{n-m,0} = 0$, *then* $X_{n,m} = 0$.

(2) *If* m *odd and* $X_{n-m,0} = 0$, *then* $Y_{n,m} = \pm Y_{n-m-1,-1}$.

(3) *If* m *even and* $Y_{n-m,0} = 0$, *then* $Y_{n,m} = 0$.

(4) *If* m *odd and* $Y_{n-m,0} = 0$, *then* $X_{n,m} = X_{n-m-1,-1}$.

(5) *If* m *even and* $X_{n-m-1,-1} = 0$, *then* $Y_{n,m} = \pm Y_{n-m,0}$.

(6) *If* m *odd and* $X_{n-m-1,-1} = 0$, *then* $X_{n,m} = 0$.

(7) *If* m *even and* $Y_{n-m-1,-1} = 0$, *then* $X_{n,m} = \pm X_{n-m,0}$.

(8) *If* m *odd and* $Y_{n-m-1,-1} = 0$, *then* $Y_{n,m} = 0$.

Proposition 5. *We have the following properties for the solutions of system* (15) *:*

(1) *If* m *even and* $X_{0,n-m} = 0$, *then* $X_{m,n} = 0$.

(2) *If* m *odd and* $X_{0,n-m} = 0$, *then* $Y_{m,n} = \pm Y_{-1,n-m-1}$.

(3) *If* m *even and* $Y_{0,n-m} = 0$, *then* $Y_{m,n} = 0$.

(4) *If* m *odd and* $Y_{0,n-m} = 0$, *then* $X_{m,n} = X_{-1,n-m-1}$.

(5) *If* m *even and* $X_{-1,n-m-1} = 0$, *then* $Y_{m,n} = \pm Y_{0,n-m}$.

(6) *If* m *odd and* $X_{-1,n-m-1} = 0$, *then* $X_{m,n} = 0$.

(7) *If* m *even and* $Y_{-1,n-m-1} = 0$, *then* $X_{m,n} = \pm X_{0,n-m}$.

(8) *If* m *odd and* $Y_{-1,n-m-1} = 0$, *then* $Y_{m,n} = 0$.

Remark 6. *If we put* $n = m$ *in system* (15) *we have a system of ordinary difference equations in the following form*

$$X_n = \frac{X_{n-2}}{1 + X_{n-2}Y_{n-1}}, \qquad Y_n = \frac{Y_{n-2}}{-1 + Y_{n-2}X_{n-1}} \tag{16}$$

We can drive the formulas for solutions from theorem(6) *in the following corollary .*

Corollary 7. *Let $\{X_n, Y_n\}_{n=-k}^{\infty}$ be a solution of system (16) with initial conditions X_0, X_{-1}, Y_0, Y_{-1} . Suppose $X_{-1}Y_0 \neq -1$,and $Y_{-1}X_0 \neq 1$,. Then, the form of solutions of system (16) ,for $n \geq 1$ are as follows:*

$$X_n = \begin{cases} \frac{(-1)^{\frac{n-1}{4}} X_{-1}}{(1+X_{-1}Y_0)^{\frac{n+3}{4}}(-1+X_{-1}Y_0)^{\frac{n-1}{4}}}, & n = 4K+1; \\ \frac{(-1)^{\frac{n-2}{4}} X_0(-1+X_0Y_{-1})^{\frac{n}{2}}}{(-1+2X_0Y_{-1})^{\frac{n+2}{4}}}, & n = 4K+2; \\ \frac{(-1)^{\frac{n+1}{4}} X_{-1}}{(-1+X_{-1}Y_0)^{\frac{n+1}{4}}(1+X_{-1}Y_0)^{\frac{n+1}{4}}}, & n = 4K+3; \\ \frac{(-1)^{\frac{n}{4}} X_0(-1+X_0Y_{-1})^{\frac{n}{2}}}{(-1+2X_0Y_{-1})^{\frac{n}{4}}}, & n = 4K+4; \end{cases}$$

$$Y_n = \begin{cases} \frac{(-1)^{\frac{n-1}{4}} Y_{-1}(-1+2Y_{-1}X_0)^{\frac{n-1}{4}}}{(-1+Y_{-1}X_0)^{\frac{n+1}{2}}}, & n = 4K+1; \\ (-1)^{\frac{n+2}{4}} Y_0(-1+Y_0X_{-1})^{\frac{n-2}{4}}(1+Y_0X_{-1})^{\frac{n+2}{4}}, & n = 4K+2; \\ \frac{(-1)^{\frac{n+1}{4}} Y_{-1}(-1+2Y_{-1}X_0)^{\frac{n+1}{4}}}{(-1+Y_{-1}X_0)^{\frac{n+1}{2}}}, & n = 4K+3; \\ (-1)^{\frac{n}{4}} Y_0(-1+Y_0X_{-1})^{\frac{n}{4}}(1+Y_0X_{-1})^{\frac{n}{4}}, & n = 4K+4; \end{cases}$$

where $k = 0, 1, 2, 3......$.

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TWO-DIMENSIONAL CHLODOWSKY VARIANT OF q-BERNSTEIN-SCHURER-STANCU OPERATORS

MEHMET ALI ÖZARSLAN AND TUBA VEDI

ABSTRACT. In this paper, two-dimensional Chlodowsky variant q-based Bernstein-Schurer-Stancu operators are introduced. Korovkin-type approximation theorems in different function spaces are studied. The error of approximation by using full modulus of continuity and partial modulus of continuities are given. Moreover, we introduce a generalization of our operators and investigate its approximation in more general weighted space.

1. INTRODUCTION

It was Chlodowsky [3] who introduced the classical Bernstein-Chlodowsky operators as

$$C_n(f;x)=\sum_{r=0}^{n} f\left(\frac{r}{n}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}\right)^{r}\left(1-\frac{x}{b_n}\right)^{n-r},$$

where the function f is defined on $[0,\infty)$ and $\{b_n\}$ is a positive increasing sequence with $b_n\to\infty$ and $\frac{b_n}{n}\to 0$ as $n\to\infty$.
In 2008, the q-analogue of Chlodowsky operators were introduced and investigated by Karslı and Gupta [8] as

$$C_n(f;q;x)=\sum_{k=0}^{n+p} f\left(\frac{[k]}{[n]}b_n\right)\begin{bmatrix}n+p\\k\end{bmatrix}\left(\frac{x}{b_n}\right)^{k}\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right),\qquad 0\le x\le b_n$$

where $\{b_n\}$ has the same property of Bernstein-Chlodowsky operators.
On the other hand, the q-Bernstein-Schurer operators were defined by Muraru [9], for fixed $p\in\mathbb{N}_0$ and for all $x\in[0,1]$, by

$$B_n^p(f;q;x)=\sum_{k=0}^{n+p} f\left(\frac{[k]}{[n]}\right)\begin{bmatrix}n+p\\k\end{bmatrix}x^k\prod_{s=0}^{n+p-k-1}(1-q^s x).\tag{1.1}$$

Note that the case $q\to 1^-$ in (1.1) reduces to the operators considered by Schurer [12]. Then, some properties of the q-Bernstein-Schurer operators were given in [13]. In 2013, the q-analogue of Bernstein-Schurer-Stancu operators $S_{n,p}^{\alpha,\beta}:C[0,1+p]\to C[0,1]$ were introduced by Agrawal, et al in [4] by

$$S_{n,p}^{(\alpha,\beta)}(f;q;x)=\sum_{k=0}^{n+p} f\left(\frac{[k]+\alpha}{[n]+\beta}\right)\begin{bmatrix}n+p\\k\end{bmatrix}x^k\prod_{s=0}^{n+p-k-1}(1-q^s x),\tag{1.2}$$

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where α and β are non-negative numbers which satisfy $0 \leq \alpha \leq \beta$ and also p is a non-negative integer. Notice that, if we choose $\alpha = \beta = 0$ in (1.2), $S_{n,p}^{(\alpha,\beta)}(f;q;x)$ reduces to the classical q-Bernstein operator [10].

Recently, Chlodowsky variant of q-Bernstein-Schurer-Stancu operators were introduced by the authors in [14] as

$$(1.3)\quad C_{n,p}^{(\alpha,\beta)}(f;q;x) := \sum_{k=0}^{n+p} f\left(\frac{[k]+\alpha}{[n]+\beta}b_n\right)\begin{bmatrix} n+p \\ k \end{bmatrix}\left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right),$$

where $n \in \mathbb{N}$, $p \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $0 \leq \alpha \leq \beta$, $0 \leq x \leq b_n$ and $0 < q < 1$. If $\alpha = \beta = p = 0$ in (1.3), we get the operators $C_n(f;q;x)$ and if $q \to 1^-$ and $\alpha = \beta = p = 0$ in (1.3), we get the operators $C_n(f;x)$.

In 2009, Büyükyazıcı [1] defined the two-dimensional q-Bernstein-Chlodowsky polynomials as

$$\widetilde{B}_{n,m}^{q_n,q_m}(f;x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, \frac{[j]_{q_m}}{[m]_{q_m}}\beta_m\right)\Omega_{k,n,q_n}\left(\frac{x}{\alpha_n}\right)\Omega_{k,m,q_m}\left(\frac{y}{\beta_m}\right)$$

where $\Omega_{k,n,q_n}(u) = \begin{bmatrix} n \\ k \end{bmatrix} u^k \prod_{s=0}^{n-k-1}(1-q_n^s)$ and investigated its approximation properties on the rectangular unbounded domain.

On the other hand, Büyükyazıcı and Sharma [2] defined the two-dimensional q-Bernstein-Chlodowsky-Durrmeyer operators on the rectangular unbounded domain and derived the Korovkin type approximation properties. They also computed the order of convergence by means of the modulus of continuity and then examined the weighted approximation properties for these operators.

In the present paper we consider the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. Some of the results about the operators $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ defined in (1.3) will be useful in our investigations. For instance, the first three moments first three moments of the operator $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ are as follows [14]:

Lemma 1.1. *Let $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ defined. Then the first few moments of the operators are,*

(i) $C_{n,p}^{(\alpha,\beta)}(1;q;x) = 1,$

(ii) $C_{n,p}^{(\alpha,\beta)}(t;q;x) = \dfrac{[n+p]x+\alpha b_n}{[n]+\beta},$

(iii) $C_{n,p}^{(\alpha,\beta)}(\mathrm{t}^2;q;x) = \dfrac{1}{([n]+\beta)^2}\{[n+p-1][n+p]qx^2$

$+(2\alpha+1)[n+p]b_n x + \alpha^2 b_n^2\}.$

Before proceeding further let us recall that the some basic definitions of q-calculus. The q-integer of $k \in \mathbb{R}$ is [7]

$$[k]_q = \begin{cases} \left(1-q^k\right)/(1-q), & q \neq 1 \\ k &, \quad q = 1, \end{cases}$$

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the q-factorial is defined by

$$[k]_q! = \begin{cases} [k]_q [k-1]_q \dots [1]_q, & k = 1, 2, 3, \dots, \\ 1 & k = 0 \end{cases}$$

and q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! \, [k]_q!}$$

The organization of the paper as follows:
In section two, the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators is established and the first few moments of the operator is given. In section three, some Korovkin-type theorems in different function spaces are studied. In section four, we obtain the order of convergence of the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators by means of the first modulus of continuity and partial modulus of continuity. In section five, we study the generalization of the two-dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators and seek its approximation properties in more general weighted space.

2. Construction of the operators

Let $\{a_n\}$ and $\{b_m\}$ be increasing sequences of real numbers satisfying

$$\lim_{n\to\infty} a_n = \lim_{m\to\infty} b_m = \infty.$$

Let, D_{a_n,b_m} denotes

$$D_{a_n,b_m} = \{(x,y) : 0 \le x \le a_n, \, 0 \le y \le b_m\}. \tag{2.1}$$

For $(x,y) \in D_{a_n,b_m}$, we construct the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators as

$$\begin{aligned} & C_{n,m,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y) \\ := & \sum_{k=0}^{n+p} \sum_{j=0}^{m+p} f\left(\frac{[k]_{q_n} + \alpha}{[n]_{q_n} + \beta} a_n, \frac{[j]_{q_m} + \alpha}{[m]_{q_m} + \beta} b_m \right) \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right) \Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \end{aligned} \tag{2.2}$$

where $n \in \mathbb{N}$, $p \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $0 \le \alpha \le \beta$. $\Phi_{k,n,q_n}(z) = \begin{bmatrix} n+p \\ k \end{bmatrix}_{q_n} z^k \prod_{s=0}^{n+p-k-1} (1 - q_n^s z)$.
We also let $0 < q_n < 1$ $(n \in \mathbb{N})$ for the positivity of the operators. It is easy to show that $C_{n,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y)$ is a linear and positive operator.
Now, we start by giving the following lemma which will be used throughout the paper.

Lemma 2.1. *Let $C_{n,m,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y)$ be given in (2.2). Then the first few moments of the operators are,*

(i) $C_{n,m,p}^{(\alpha,\beta)}(1; q_n, q_m; x, y) = 1,$

(ii) $C_{n,m,p}^{(\alpha,\beta)}(t_1; q_n, q_m; x, y) = \dfrac{[n+p]_{q_n} x + \alpha a_n}{[n]_{q_n} + \beta},$

$$(iii)\ C_{n,m,p}^{(\alpha,\beta)}\left(t_2;q_n,q_m;x,y\right)=\frac{[m+p]_{q_m}y+\alpha b_m}{[m]_{q_m}+\beta}$$

$$(iv)\ C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;x,y\right)$$

$$=\frac{1}{\left([n]_{q_n}+\beta\right)^2}\left\{[n+p-1]_{q_n}[n+p]_{q_n}q_nx^2+(2\alpha+1)[n+p]_{q_n}a_nx+\alpha^2a_n^2\right\}$$

$$+\frac{1}{\left([m]_{q_m}+\beta\right)^2}\left\{[m+p-1]_{q_m}[m+p]_{q_m}q_my^2+(2\alpha+1)[m+p]_{q_m}b_my+\alpha^2b_m^2\right\}.$$

Proof. Using Lemma 1.1 and the linearity of the operators, the proof is easily obtained. □

3. Korovkin-type approximation theorems

In this section, Korovkin-type approximation theorems are given for the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. For fixed $\nu\geq 0$ consider the space C_{ρ^ν} which consists of all continuous functions f, satisfying the condition

$$|f(x,y)|\leq M_f\rho^\nu(x,y),\quad (x,y)\in[0,\infty)\times[0,\infty):=\mathbb{R}_+^2 \text{ and } \rho(x,y)=1+x^2+y^2.$$

Clearly, C_{ρ^ν} is a linear normed space with the following norm

$$\|f\|_{\rho^\nu}=\sup_{0\leq x,y<\infty}\frac{|f(x,y)|}{\rho^\nu(x,y)}.$$

The following theorem will be used in the investigation of approximation properties of $C_{n,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)$ in the weighted spaces.

Theorem 3.1. *Let the numbers A and B be any fixed positive real numbers. Let $D_{A,B}=\{(x,y):0\leq x\leq A,\ 0\leq y\leq B\}$, $q:=\{q_n\}$ with $0<q_n<1$, $\lim_{n\to\infty}q_n=1$ and $\{a_n\}$ and $\{b_m\}$ be increasing sequences of positive real numbers that satisfy the following properties:*

$$\lim_{n\to\infty}a_n=\lim_{m\to\infty}b_m=\infty \text{ and } \lim_{n\to\infty}\frac{a_n}{[n]_{q_n}}=\lim_{m\to\infty}\frac{b_m}{[m]_{q_m}}=0.$$

For all $f\in C(D_{A,B})$, we have

$$\lim_{n,m\to\infty}\max_{(x,y)\in D_{A,B}}\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|=0.$$

Proof. Using Lemma 2.1, we get

$$\left\| C_{n,m,p}^{(\alpha,\beta)}\left(1;q_n,q_m;\cdot,\cdot\right)-1\right\|_{C(D_{A,B})}=0$$

$$\left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_1;q_n,q_m;\cdot,\cdot\right)-x\right\|_{C(D_{A,B})}\leq A\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|+\frac{\alpha a_n}{[n]_{q_n}+\beta}$$

$$\left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_2;q_n,q_m;\cdot,\cdot\right)-y\right\|_{C(D_{A,B})}\leq B\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|+\frac{\alpha b_m}{[m]_{q_m}+\beta}.$$

And again using Lemma 2.1 we have

$$C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)=\frac{1}{\left([n]_{q_n}+\beta\right)^2}$$
$$\times\left\{\left([n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right)x^2+(2\alpha+1)[n+p]_{q_n}a_nx+\alpha^2a_n^2\right\}$$
$$+\frac{1}{\left([m]_{q_m}+\beta\right)^2}$$
$$\times\left\{\left([m+p+1]_{q_m}[m+p]_{q_m}q_m-\left([m]_{q_m}+\beta\right)^2\right)y^2+(2\alpha+1)[m+p]_{q_m}b_my+\alpha^2b_m^2\right\}.$$

Finally, from the above equality we obtain

$$\left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_{C(D_{A,B})}$$
$$\leq\frac{1}{\left([n]_{q_n}+\beta\right)^2}$$
$$\times\left\{\left|[n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right|A^2+(2\alpha+1)[n+p]_{q_n}a_nA+\alpha^2a_n^2\right\}$$
$$+\frac{1}{\left([m]_{q_m}+\beta\right)^2}$$
$$\times\left\{\left|[m+p+1]_{q_m}[m+p]_{q_m}q_m-\left([m]_{q_m}+\beta\right)^2\right|B^2+(2\alpha+1)[m+p]_{q_m}b_mB+\alpha^2b_m^2\right\}.$$

Therefore, from the hypothesis of the theorem, we have

$$\begin{aligned}\left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_1;q_n,q_m;\cdot,\cdot\right)-x\right\|_{C(D_{A,B})}&\to 0\\ \left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_2;q_n,q_m;\cdot,\cdot\right)-y\right\|_{C(D_{A,B})}&\to 0\\ \left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_{C(D_{A,B})}&\to 0\end{aligned}$$

when n and $m\to\infty$.

Hence, the proof is completed by the two dimensional Korovkin theorem. □

In studying Korovkin-type weighted approximation, the following theorem plays an important role.

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Theorem 3.2. *(See [6]) There exists a sequence of positive operators $T_{n,m}$, acting from $C_\rho\left(\mathbb{R}_+^2\right)$ to $C_\rho\left(\mathbb{R}_+^2\right)$, satisfying the conditions*

$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(1;\cdot,\cdot\right)-1\right\|_\rho=0$$
$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(t_1;\cdot,\cdot\right)-x\right\|_\rho=0$$
$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(t_2;\cdot,\cdot\right)-y\right\|_\rho=0$$
$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(t_1^2+t_2^2;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_\rho=0$$

and there exists a function $f^*\in C_\rho$ for which

$$\lim_{n,m\to\infty}\left\|T_{n,m}f^*-f^*\right\|_\rho\geq\frac{1}{4}$$

where $\rho=1+x^2+y^2$.

Now, consider the following operator

$$T_{n,m}\left(f;q_n,q_m;x,y\right)=\begin{cases} C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right), & (x,y)\in D_{a_n,b_n} \\ f\left(x,y\right), & \mathbb{R}_+^2\backslash D_{a_n,b_n}\end{cases}.$$

Theorem 3.3. *Let $f\in C_\rho\left(\mathbb{R}_+^2\right)$. Then for any $\gamma>0$*

$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(f;q_n,q_m;\cdot,\cdot\right)-f\left(\cdot\right)\right\|_{C_{\rho^{1+\gamma}}}=0$$

where $\{a_n\}$, $\{b_m\}$, $\{q_n\}$ and $\{q_m\}$have the same conditions as in Theorem 3.1.

Proof. For all $\varepsilon>0$, there exist sufficiently large positive real numbers A and B such that

$$\left(1+x^2+y^2\right)^{-\gamma}<\varepsilon \tag{3.1}$$

when $x>A$ and $y>B$.
Let n, m be sufficiently large so that $D_{A,B}\subset D_{a_n,b_m}$

$$\begin{aligned}&\left\|T_{n,m}\left(f;q_n,q_m;\cdot,\cdot\right)-f\left(\cdot\right)\right\|_{C_{\rho^{1+\gamma}}}\\ &\leq\sup_{(x,y)\in D_{A,B}}\frac{\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\gamma}}\\ &+\sup_{(x,y)\in D_{a_n,b_n}\backslash D_{A,B}}\frac{\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\gamma}}\\ &=y_{n,m}^{'}+y_{n,m}^{''}.\end{aligned}$$

By Theorem 3.1, $\lim_{n,m\to\infty}y_{n,m}^{'}=0$ and for the proof of the second term we have

$$y_{n,m}^{''}\leq\left(1+x^2+y^2\right)^{-\gamma}\left(\frac{\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)\right|}{1+x^2+y^2}+\frac{|f(x,y)|}{1+x^2+y^2}\right).$$

Finally, since $f\in C_\rho\left(\mathbb{R}_+^2\right)$, the term $\frac{|f(x,y)|}{1+x^2+y^2}$ is bounded. Furthermore, because of the fact that

$$\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)\right|\leq\left|C_{n,m,p}^{(\alpha,\beta)}\left(1+t_1^2+t_2^2;q_n,q_m;x,y\right)\right|,$$

using Lemma 2.1, the term $\frac{\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)\right|}{1+x^2+y^2}$ is bounded for sufficiently large n and m. Hence, we get by (3.1) that

$$y_{n,m}^{''} \leq \varepsilon(1+M)$$

Since $\varepsilon > 0$ is arbitrary, then $\lim\limits_{n,m\to\infty} y_{n,m}^{''} = 0$. This completes the proof. □

Now, consider the subspace C_ρ^0 of C_ρ which is defined by

$$C_\rho^0 := \left\{ f \in C_\rho : \lim_{x,y\to 0} \frac{|f(x,y)|}{1+x^2+y^2} = 0 \right\}.$$

Theorem 3.4. *Let the sequences $\{q_n\}$, $\{a_n\}$ and $\{b_m\}$ satisfy the same properties as in Theorem 3.1. Then for all $f \in C_\rho^0\left(\mathbb{R}_+^2\right)$, we obtain*

$$\lim_{n,m\to\infty} \left\| T_{n,m}(f;q_n,q_m;\cdot,\cdot) - f(\cdot) \right\|_{C_\rho} = 0.$$

Proof. For all $f \in C_\rho^0\left(\mathbb{R}_+^2\right)$, observe that

$$\lim_{x,y\to\infty} \frac{|f(x,y)|}{1+x^2+y^2} = 0, \qquad \lim_{n,m\to\infty} \frac{\left| f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta} b_m \right) \right|}{1+\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} a_n\right)^2 + \left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta} b_m\right)^2} = 0.$$

Therefore, for all $\varepsilon > 0$, we can find sufficiently large numbers A and B such that

$$|f(x,y)| < \varepsilon\left(1+x^2+y^2\right) \tag{3.2}$$

for $x > A$ and $y > B$ and there exists natural numbers n_0 and m_0 such that

$$\left| f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta} b_m \right) \right| < \varepsilon \left(1 + \left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} a_n \right)^2 + \left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta} b_m \right)^2 \right) \tag{3.3}$$

for all $n > n_0$ and $m > m_0$.
Hence, for large n and m, we have

$$\begin{aligned} &\left\| T_{n,m}(f;q_n,q_m;\cdot,\cdot) - f(\cdot) \right\|_{C_\rho} \\ &\leq \sup_{(x,y)\in D_{A,B}} \frac{\left| C_{n,m}^{\alpha,\beta}(f;q_n,q_m;x,y) - f(x,y) \right|}{1+x^2+y^2} \\ &+ \sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} \frac{\left| C_{n,m}^{\alpha,\beta}(f;q_n,q_m;x,y) - f(x,y) \right|}{1+x^2+y^2} = z_{n,m}^{'} + z_{n,m}^{''}. \end{aligned}$$

By Theorem 3.1 it is sufficient to show that $z_{n,m}^{''} \to 0$ as $n \to \infty$.
Using (3.2) and (3.3), we get

$$\begin{aligned} z_{n,m}^{''} &\leq \varepsilon + \sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} \frac{\left| C_{n,m}^{\alpha,\beta}(f;q_n,q_m;x,y)) \right|}{1+x^2+y^2} \\ &\leq \varepsilon + \varepsilon \sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} t_{n,m}(q_n,q_m;x;y) \\ &= \varepsilon\left(1 + \sup_{(x,y)\in D_{a_n,b_m}/D_{A,B}} t_{n,m}(q_n,q_m;x;y) \right) \end{aligned}$$

where $t_{n,m}(q_n,q_m;x;y) := \frac{C_{n,m}^{\alpha,\beta}(1;q_n,q_m;x,y))+C_{n,m}^{\alpha,\beta}\left(t_1^2;q_n,q_m;x,y)\right)+C_{n,m}^{\alpha,\beta}\left(t_2^2;q_n,q_m;x,y)\right)}{1+x^2+y^2}$.
By Lemma 2.1, it is clear that there exist K independent of n and m such that

$$\sup_{(x,y)\in D_{a_n,b_m}/D_{A,B}} t_{n,m}(q_n,q_m;x;y) \le K.$$

Therefore, for $n > n_0$ and $m > m_0$ we have

$$z''_{n,m} < (1+K)\varepsilon.$$

This completes the proof. □

4. Order of convergence

In this section, we compute the rate of convergence of the operators in terms of the the full modulus of continuity and partial modulus of continuities.

Let $f \in D_{A,B}$ and $x \ge 0$. Then the definition of the modulus of continuity of f is given by

$$\omega(f;\delta) = \max_{\substack{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\le\delta \\ x,y\in C(D_{A,B})}} |f(x_1,y_1)-f(x_2,y_2)|. \tag{4.1}$$

It is known that for any $\delta > 0$ we know that

$$|f(x_1,y_1)-f(x_2,y_2)| \le \omega(f,\delta)\left(\frac{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{\delta}+1\right) \tag{4.2}$$

and its partial modulus of continuies are defined by

$$\begin{aligned}
\omega^{(1)}(f;\delta) &= \max_{0\le y\le A}\max_{|x_1-x_2|\le\delta}|f(x_1,y)-f(x_2,y)| \\
\omega^{(2)}(f;\delta) &= \max_{0\le x\le B}\max_{|y_1-y_2|\le\delta}|f(x,y_1)-f(x,y_2)|.
\end{aligned}$$

Also, for any $\delta > 0$ we have

$$\begin{aligned}
|f(x_1,y_1)-f(x_2,y_2)| &\le \omega^{(1)}(f,\delta)\left(\frac{|x_1-x_2|}{\delta}+1\right), \\
|f(x_1,y_1)-f(x_2,y_2)| &\le \omega^{(2)}(f,\delta)\left(\frac{|y_1-y_2|}{\delta}+1\right).
\end{aligned}$$

Theorem 4.1. *For any $f \in C(D_{A,B})$, the following inequalities*

$$\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right| \le 2\left[\omega^{(1)}(f;\delta_m)+\omega^{(2)}(f;\delta_n)\right] \tag{4.3}$$

$$\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right| \le 2\omega\left(f;\sqrt{\delta_m^2+\delta_n^2}\right) \tag{4.4}$$

are satisfied where

(4.5)

$$\delta_n^2 := \frac{1}{\left([n]_{q_n}+\beta\right)^2} \times\left\{\left|[n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right|A^2+(2\alpha+1)[n+p]_{q_n}a_nA+\alpha^2a_n^2\right\}$$

and

(4.6)
$$\delta_m^2 := \frac{1}{\left([m]_{q_m}+\beta\right)^2} \times \left\{ \left| [m+p+1]_{q_m}[m+p]_{q_m} q_m - \left([m]_{q_m}+\beta\right)^2 \right| B^2 + (2\alpha+1)[m+p]_{q_m} b_m B + \alpha^2 b_m^2 \right\}.$$

Proof. We directly have,

$$\begin{aligned}
&C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y) - f(x,y) \\
&= \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left[f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) - f(x,y)\right] \\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&= \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left[f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) - f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y)\right. \\
&\left. + f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y) - f(x,y)\right]\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}$$

By linearity and positivity of the operators, we get

$$\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y) - f(x,y)\right| \\
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left| f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) - f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y)\right| \\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&+ \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left| f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y) - f(x,y)\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m - y\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&+ \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(1)}\left(f;\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n - x\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&= \Omega_1(x,y) + \Omega_2(x,y).
\end{aligned}$$

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Using Lemma 1.1 and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\Omega_1(x,y)\\
=\;& \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
=\;& \sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
\le\;& \omega^{(2)}(f;\delta_m)\left\{1+\frac{1}{\delta_m}\left[\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\right]^{1/2}\right\}.
\end{aligned}$$

Finally, using Lemma 2.1, we get

$$\Omega_1(x,y)\le 2\omega^{(2)}(f;\delta_m) \tag{4.7}$$

where we choose δ_m as in (4.6).
In the same way, we obtain

$$\Omega_2(x,y)\le 2\omega^{(1)}(f;\delta_n) \tag{4.8}$$

where δ_n is given in (4.5). Combining (4.7) and (4.8), we obtain (4.3) .
Now, by using linearity and the monotonicity of the operators, and taking into account (4.1), we have

$$\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|\\
&\le \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega\left(f;\sqrt{\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&\le \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m)-f(x,y)\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&\le 1+\frac{1}{\delta}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega(f;\sqrt{\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2})
\end{aligned}$$

(4.9)

$$\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)$$

Using (4.2) and the Cauchy-Schwartz inequality, we get (4.4). □

Theorem 4.2. *Let $f(x,y)$ have continuous partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$, let $\omega^1(f_x;.)$ and $\omega^2(f_y;.)$ denote the partial moduli of $\partial f/\partial x$ and $\partial f/\partial y$, respectively*

on $D_{A,B}$. Then the inequality

$$\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right| \\
&\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)+2\left[\delta_n\omega^{(1)}\left(\frac{\partial f}{\partial x};\delta_n\right)\right] \\
&+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)+2\left[\delta_m\omega^{(2)}\left(\frac{\partial f}{\partial y};\delta_m\right)\right].
\end{aligned}$$

where δ_n and δ_m are the same as in Theorem 4.1 and $\left|\frac{\partial f}{\partial x}\right| \leq N$, $\left|\frac{\partial f}{\partial y}\right| \leq M$ on $D_{A,B}$.

Proof. By the mean value theorem, we can write

$$\begin{aligned}
&f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)-f(x,y) \\
&=f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y\right)-f(x,y)+f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]+\alpha}{[m]+\beta}b_m\right) \\
&-f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y\right)
\end{aligned}$$

$$\begin{aligned}
&=\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\frac{\partial f(x,y)}{\partial x}+\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\left[\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right] \\
&+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\frac{\partial f(x,y)}{\partial y}+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right) \\
&\times\left[\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right]
\end{aligned} \tag{4.10}$$

for any fixed $y \in [0,B]$ and $x \in [0,A]$, where

$$x<\psi_1<\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n$$

and

$$y<\psi_2<\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m.$$

Applying the operator $C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)$ to (4.10)

$$\begin{aligned}
&C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\\
&=\frac{\partial f(x,y)}{\partial x}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\left[\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right]\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\frac{\partial f(x,y)}{\partial y}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\left[\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right]\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}$$

Hence, taking $\left|\frac{\partial f}{\partial x}\right|\leq N$ and $\left|\frac{\partial f}{\partial y}\right|\leq M$, we get

$$\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|\\
&\leq\left|\frac{\partial f(x,y)}{\partial x}\right|\left|C_{n,m,p}^{(\alpha,\beta)}(t_1-x;q_n,q_m;x,y)\right|\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|\left|\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right|
\end{aligned}$$

$$\begin{aligned}
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\left|\frac{\partial f(x,y)}{\partial y}\right|\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\left|\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right|\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq N\left|C_{n,m,p}^{(\alpha,\beta)}\left(t_1-x;q_n,q_m;x,y\right)\right| \\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|\left|\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right| \\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&+M\left|C_{n,m,p}^{(\alpha,\beta)}\left(t_2-x;q_n,q_m;x,y\right)\right| \\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\left|\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right| \\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}
$$

Then using the properties of partial modulus of continuities, we have

$$
\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right| \\
&\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right) \\
&+\omega^{(1)}\left(f_x;\delta_n\right)\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|\left(\frac{\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|}{\delta_n}+1\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right) \\
&+\omega^{(2)}\left(f_y;\delta_m\right)\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\left(\frac{\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|}{\delta_n}+1\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}
$$

since

$$
|\psi_1-x|\leq\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|,\quad |\psi_2-y|\leq\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|.
$$

Applying the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right| \\
&\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right) \\
&+\omega^{(1)}\left(f_x;\delta_n\right)\left(\sum_{k=0}^{n+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\right)^{1/2} \\
&+\frac{\omega^{(1)}(f_x;\delta_n)}{\delta_n}\sum_{k=0}^{n+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)
\end{aligned}
$$

$$+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)$$
$$+\omega^{(2)}(f_y;\delta_m)\left(\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\right)^{1/2}$$
$$+\frac{\omega^{(2)}(f_y;\delta_m)}{\delta_m}\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).$$

Therefore

$$\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)$$
$$+\omega^{(1)}(f_x;\delta_n)\left(\left(\sqrt{C_{n,m,p}^{(\alpha,\beta)}\left((t_1-x)^2;q_n,q_m;x,y\right)}\right)\right.$$
$$\left.+\frac{\omega^{(1)}(f_x;\delta_n)}{\delta_n}\left(C_{n,m,p}^{(\alpha,\beta)}\left((t_1-x)^2;q_n,q_m;x,y\right)\right)\right)$$
$$+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)$$
$$+\omega^{(2)}(f_y;\delta_m)\sqrt{C_{n,m,p}^{(\alpha,\beta)}\left((t_2-y)^2;q_n,q_m;x,y\right)}$$
$$+\frac{\omega^{(2)}(f_y;\delta_m)}{\delta_m}C_{n,m,p}^{(\alpha,\beta)}\left(C_{n,m,p}^{(\alpha,\beta)}\left((t_2-y)^2;q_n,q_m;x,y\right)\right).$$

Now using Lemma 2.1 and choosing δ_n and δ_m as in (4.5) and (4.6), respectively, we get

$$\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|$$
$$\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)+2\left[\delta_n\omega^{(1)}\left(\frac{\partial f}{\partial x};\delta_n\right)\right]$$
$$+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)+2\left[\delta_m\omega^{(2)}\left(\frac{\partial f}{\partial y};\delta_m\right)\right].$$

Whence the result. □

5. Generalization of the Two Dimensional of Chlodowsky Variant of q-Bernstein-Schurer-Stancu Operators

In this section, we introduce generalization of Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. The generalized operators help us to approximate continuous functions defined on more general weighted spaces. Note that this kind of generalization was considered earlier for the Chlodowsky-Bernstein polynomials [5]. For $x\geq 0$, consider any continuous function $\omega(x,y)\geq 1$ and define

$$G_f(t,s)=f(t,s)\frac{1+t^2+s^2}{w(t,s)}.$$

Let us consider the generalization of the $C_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)$ as follows

(5.1)
$$L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)=\begin{cases} \frac{w(x,y)}{1+x^2+y^2}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}G_f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) \\ \times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right), & (x,y)\in D_{a_n,b_n} \\ f(x,y), & \mathbb{R}_+^2\backslash D_{a_n,b_n}\end{cases}$$

where $(x,y)\in D_{a_nb_m}$ and $\{a_n\}$ and $\{b_m\}$ have the same properties of two dimensional of Chlodowsky variant of q-Bernstein-Schurer-Stancu operators.

Theorem 5.1. *For all continuous functions f satisfying $|f(x,y)|\le M_fw(x,y)$, $x,y\ge 0$, and $\lim\limits_{x,y\to\infty}\frac{f(x,y)}{w(x,y)}=0$, we have*
$$\lim_{n,m\to\infty}\left\|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;\cdot,\cdot)-f(\cdot,\cdot)\right\|_w=0$$
where $\rho(x,y)=1+x^2+y^2$.

Proof. Clearly,
$$\begin{aligned}&\left|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)-f(x,y)\right|\\ =\;&\frac{w(x,y)}{1+x^2+y^2}\left|\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}G_f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)\right.\\ &\left.\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)-G_f(x,y)\right|,\end{aligned}$$
thus
$$\begin{aligned}&\left\|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;\cdot,\cdot)-f(\cdot,\cdot)\right\|_w\\ =\;&\sup_{x,y\in\mathbb{R}_+^2}\frac{\left|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)-f(x,y)\right|}{w(x,y)}=\sup_{x,y\in\mathbb{R}_+^2}\frac{\left|T_{n,p}(G_f;q_n,q_m;x,y)-G_f(x,y)\right|}{1+x^2+y^2}.\end{aligned}$$
Since $|f(x,y)|\le M_fw(x,y)$, then $|G_f(x,y)|\le M_f\rho(x,y)$ for $x,y\ge 0$ and $G_f(x,y)$ is continuous function on R_+^2. Furthermore, from $\lim\limits_{x,y\to\infty}\frac{f(x,y)}{w(x,y)}=0$, we have
$$\lim_{x,y\to\infty}\frac{G_f(x,y)}{\rho(x,y)}=0.$$
Thus, from Theorem 3.4 we get the result. □

Finally, note that, taking $w(x,y)=1+x^2+y^2$, then the operators $L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)$ reduces $T_{n,p}^{\alpha,\beta}(G_f;q_n,q_m;x,y)$.

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Global stability in stochastic difference equations for predator-prey models

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Abstract

There are many publications on theoretical analysis of deterministic difference equations and stochastic differential equations. However, relatively few theoretical papers are published to consider the positivity of solutions of discrete-time stochastic difference equations (DSDEs), and no theoretical papers investigate the global stability of nontrivial solutions of DSDEs with nonlinear terms. In this paper, we consider a DSDE model that is a generalization of two-dimensional nonlinear models of stochastic predator-prey interactions, and show the positivity and global stability of the nontrivial solutions by using our new discretized version of the Itô formula. In addition, our results are compared with those of continuous-time stochastic differential equations and discrete-time deterministic difference equations. Numerical simulations are introduced to support the results.

Key words: Discrete-time stochastic difference equations, Positivity, Global stability.

1. Introduction

Many predator-prey models have been studied to describe the dynamics of biological systems in which two species interact, one as a predator and the other as a prey. A classic predator-prey model is given by

$$\frac{dx}{dt} = x(r_1 - a_{11}x - a_{12}y), \ \frac{dy}{dt} = y(r_2 + a_{21}x - a_{22}y), \tag{1}$$

where $x(t)$ and $y(t)$ denote the population density of the prey and predator at time t, respectively. In the model (1), r_1 is the intrinsic growth rate of the prey in the absence of the predator, $-r_2$ is the death rate of the predator in the absence of the prey, the coefficients $a_{ij}(i \neq j)$ give the strength of the interaction between the two species, and $a_{ii}(i = 1, 2)$ measure the inhibiting effect of environment on the two species.

In the model (1), the predator consumes the prey with functional response of type $a_{12}x(t)y(t)$. However the rate of prey capture is saturated when the population of the prey is relatively large. Such phenomena are described by nonlinear functions including Holling types [1–5], Beddington-DeAngelis type [6–8], Crowley-Martin type [9–11], and

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Ivlev-type of functional responses [12–14]. Other types of nonlinear functions have been applied to express the Allee effect [15–19], which describes a positive relation between the population density and the per capita growth rate of a species. There have been also models to take into account of diffusion of species ([15] and [20–22]).

On the other hand, the population is inevitably affected by environmental noise in nature, so that the reproduction rates can change randomly. In order to be more realistic, stochastic models should be considered. Stochastic differential equation (SDE) models have been increasingly used in a range of application areas, including biology, chemistry, mechanics, economics, and finance. The SDE models have been studied to understand extinction, stochastic permanence and stationary distributions of the stochastic systems. In particular, many authors have taken stochastic perturbation into deterministic predator prey models with Beddington-DeAngelis and Holling types of functional responses [23–33]. For example, putting noise into the deterministic model (1) gives the SDE model

$$
\begin{aligned}
dx(t) &= x(t)\{r_1 - a_{11}x(t) - a_{12}y(t)\}dt + \sigma_1 x(t)dW_1(t),\\
dy(t) &= y(t)\{r_2 + a_{21}x(t) - a_{22}y(t)\}dt + \sigma_2 y(t)dW_2(t),
\end{aligned}
\tag{2}
$$

which is a special model studied in [25] with zero-time delays. Here the positive coefficients σ_1 and σ_2 measure the intensity of environmental perturbations on the underlying growth rate of the prey and the death rate of the predator, respectively. The processes W_i are independent and real valued Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In general, the exact solutions of SDEs are not known, so one has to numerically solve these SDEs. This leads us to consider and analyze discrete-time stochastic difference equations (DSDEs), which can be also viewed as stochastically perturbed versions of deterministic difference equations (DDEs) (see [34], [35] and references therein). There are many publications on estimations of the difference between solutions of SDEs and DSDEs. The global asymptotic stability of the trivial solution of DSDEs has been also widely addressed (see [36], [37], [38] and references therein). However, relatively few theoretical studies consider the positivity of solutions of DSDEs that are scalar equations on a finite time interval (see [39] references therein). In particular, to the best of our knowledge, there is no paper that theoretically deals with the global stability of nontrivial solutions of DSDEs. Therefore, to investigate the positivity and global stability, we consider the DSDE model for (2)

$$
x_{k+1}^i = x_k^i \left\{1 + h\left(r_i + \sum\nolimits_{j=1}^{i-1} a_{ij}x_k^j - \sum\nolimits_{j=i}^{2} a_{ij}x_k^j\right) + h^{0.5}\sigma_i \xi_{k+1}^i\right\}, \tag{3}
$$

where $1 \le i \le 2$, $k \ge 0$, $x_0^i > 0$ and $0 < h < 1$. Although $r_1 > 0$, $r_2 < 0$ and $a_{ij} > 0$ in the SDE model (2) and the DDE model (3) with $\sigma_i = 0$ (see [34] and [35]), we weaken the conditions on the parameters and use the following conditions in the DSDE model (3): for $1 \le i, j \le 2$ and $i \ne j$

$$
r_i \in \mathbb{R}, a_{ii} > 0, a_{ij} \ge 0, \sigma_i > 0. \tag{4}
$$

The discrete Wiener processes $W_i(t_{k+1}) - W_i(t_k)$ are $h^{0.5}\xi_{k+1}^i$ with a mutually independent and identically distributed sequence $(\xi_k^1, \xi_k^2)_{k=1}^\infty$ of the standard normal random variables. The solutions of (3) are defined with respect to a complete, filtered probability space $(\Omega_h, \mathcal{F}_h, \{\mathcal{F}_k\}_{k=1}^\infty, \mathbb{P}_h)$, where $\{\mathcal{F}_k\}_{k=1}^\infty$ is the natural filtration generated by the stochastic sequence $(\xi_k^1, \xi_k^2)_{k=1}^\infty$, i.e., $\mathcal{F}_k = \sigma(\xi_1^1, \xi_1^2, \cdots, \xi_k^1, \xi_k^2)$ for $k \ge 1$. Therefore $(x_k^1, x_k^2)_{k=1}^\infty$ is

adapted to the filtration for any initial vector (x_0^1, x_0^2), which is supposed to be non-random.

The positivity of solutions of the SDEs (2) is obtained in the infinite time interval $[0, \infty)$ without boundedness of the noises $W_i(t)$ by using the concept of explosion time (see [25] and [40]). However, to the best of our knowledge, there is no method for applying the concept of explosion time to DSDEs. Then for obtaining the positivity of solutions of the DSDE model (3) in the infinite time interval, we restrict the noises to bounded noises, which means that $\xi_k^i (1 \le i \le 2, k \ge 1)$ are assumed to be doubly truncated standard normal random variables with support $[-\varsigma, \varsigma]$ for a positive constant ς

$$-\varsigma \le \xi_k^i \le \varsigma \tag{5}$$

and the probability density function

$$\psi(x) = \begin{cases} q(x)\left\{\Phi(\varsigma) - \Phi(-\varsigma)\right\}^{-1} & \text{if } x \in [-\varsigma, \varsigma], \\ 0 & \text{if } x \notin [-\varsigma, \varsigma], \end{cases} \tag{6}$$

where q and Φ are the probability density and cumulative distribution functions of the standard normal random variable, respectively. Denoting $\eta_\varsigma = 2\varsigma q(\varsigma)\left\{\Phi(\varsigma) - \Phi(-\varsigma)\right\}^{-1}$ gives that for $1 \le i \le 2$ and $k \ge 1$

$$E(\xi_k^i) = 0, \; E\left((\xi_k^i)^2\right) = 1 - \eta_\varsigma, \tag{7}$$

in which the positive value η_ς can be assumed to be sufficiently close to 0. For example, when $\varsigma = 20$, we have $0 < \eta_\varsigma < 10^{-85}$. The truncation constant ς will be first used in (12) for the positivity of the solutions x_k^i of the DSDE model (3).

The paper is organized as follows. Section 2 gives the positivity and boundedness of solutions of the model (3). In Section 3, we develop a new discrete Itô formula for (3) by using a known discrete Itô formula for DSDEs (see [41], [42] and [43]). The new discrete Itô formula is the main tool for finding conditions for the global stability of solutions of (3). Section 4 introduces auxiliary equations, the solutions of which are used for the upper bounds of solutions of (3). In Section 5, we present sufficient conditions for extinction and non-extinction of solutions of (3). Our results are compared with those for the DDEs in [35] and the SDEs in [25]. Section 6 gives simulation results to confirm the theoretical analysis obtained in this paper.

2. Positivity and boundedness of solutions of DSDEs

In this section, we show the positivity and boundedness of solutions of the DSDE model (3) by applying the approach used in the DDE model (3) with $\sigma_1 = \sigma_2 = 0$ (see [34] and [35]).

Notation 1. For simplicity, we use the symbols $\tilde{a}$ and $\hat{a}$ for every constant a to denote

$$\tilde{a} = a \cdot h^{0.5}, \; \hat{a} = a \cdot h$$

and the symbols $\varkappa_k^1$ and $\varkappa_k^2$ for a vector $\varkappa_k = (x_k^1, x_k^2)$ to denote

$$\varkappa_k^1 = x_k^2, \; \varkappa_k^2 = x_k^1.$$

Write the model (3) as

$$x_{k+1}^i = F_{k,\varkappa_k^i}^i(x_k^i),$$

where

$$\begin{aligned} F_{k,y}^1(x) = &\ x\left(1 + \hat{r}_1 - \hat{a}_{11}x - \hat{a}_{12}y + \tilde{\sigma}_1\xi_{k+1}^1\right), \\ F_{k,x}^2(y) = &\ y\left(1 + \hat{r}_2 + \hat{a}_{21}x - \hat{a}_{22}y + \tilde{\sigma}_2\xi_{k+1}^2\right). \end{aligned} \tag{8}$$

Note that for a vector $\boldsymbol{\zeta}_k = (\zeta_k^1, \zeta_k^2)$ of real numbers ζ_k^1 and ζ_k^2,

$$F_{k,\zeta_k^i}^i(\tau) \text{ is strictly increasing on } 0 \le \tau < V_k^i(\boldsymbol{\zeta}_k), \tag{9}$$

in which

$$V_k^i(\boldsymbol{\zeta}_k) = (2\hat{a}_{ii})^{-1}\left(1 + \hat{r}_i + \sum\nolimits_{j=1}^{i-1} \hat{a}_{ij}\zeta_k^j - \sum\nolimits_{j=i+1}^{2} \hat{a}_{ij}\zeta_k^j + \tilde{\sigma}_i\xi_{k+1}^i\right). \tag{10}$$

Denote that for $1 \le i \le 2$

$$\chi_i = \hat{a}_{ii}^{-1}\left(\hat{r}_i + \sum\nolimits_{j=1}^{i-1} \hat{a}_{ij}\chi_j + \tilde{\sigma}_i\varsigma_*\right), \tag{11}$$

where ς_* is a constant satisfying

$$\varsigma_* > \varsigma, \tag{12}$$

$$\chi_i \le (2\hat{a}_{ii})^{-1}\left(1 + \hat{r}_i - \sum\nolimits_{j=i+1}^{2} \hat{a}_{ij}\chi_j - \tilde{\sigma}_i\varsigma_*\right), \tag{13}$$

$$\hat{r}_i + \sum\nolimits_{j=1}^{i-1} \hat{a}_{ij}\chi_j + \tilde{\sigma}_i\varsigma_* < 1. \tag{14}$$

The relation (12) will be first used in (69) to find upper solutions of the model (3). The initial condition of the model (3) is assumed to satisfy

$$(x_0^1, x_0^2) \in (0, \chi_1) \times (0, \chi_2). \tag{15}$$

Remark 1. The definition (11) gives that $\chi_1 = \frac{\hat{r}_1 + \tilde{\sigma}_1\varsigma_*}{\hat{a}_{11}}$ and $\chi_2 = \hat{a}_{22}^{-1}(\hat{r}_2 + \hat{a}_{21}\chi_1 + \tilde{\sigma}_2\varsigma_*)$. Letting h in (3) be small, we can choose ς_* satisfying the two conditions (13) and (14). For example, let $h = 0.0001$, $\varsigma_* = 20$, $r_1 = 2$, $r_2 = a_{ij} = 1$ and $\sigma_i = 0.1$ $(1 \le i, j \le 2)$. Denoting by R_i and L_i the right and left-hand sides of (13) and (14), respectively, gives

$$(\chi_1, R_1, L_1) = (202, 4699.5, 0.3848), (\chi_2, R_2, L_2) = (403, 4900.5, 0.3518),$$

which show that the conditions (13) and (14) are satisfied.

Theorem 1. *Let x_k^i be the solutions of (3) and χ_i be defined in (11). Assume that (5), (12), (13), (14) and (15) hold. Then*

$$(x_k^1, x_k^2) \in (0, \chi_1) \times (0, \chi_2), \ \ k \ge 0.$$

Proof. The proof is divided into the following three steps.
Step 1. We prove the positivity: $x_1^i > 0$ for $1 \le i \le 2$.
Note that for $\varkappa_0 = (x_0^1, x_0^2)$

$$0 < x_0^i < \chi_i \le (2\hat{a}_{ii})^{-1}\left(1+\hat{r}_i - \sum_{j=i+1}^{2}\hat{a}_{ij}\chi_j - \tilde{\sigma}_i\varsigma_*\right) < V_0^i(\varkappa_0),$$

where the first two inequalities are obtained from (15), the third from (13) and the last from (10), (15), (5) and (12). Then using (9) with $\boldsymbol{\zeta}_0 = \varkappa_0$ and (15), we have the positivity

$$x_1^i = F^i_{0,\varkappa_0^i}(x_0^i) > F^i_{0,\varkappa_0^i}(0) = 0.$$

Step 2. We prove the upper-bound property: $x_1^i < \chi_i$ for $1 \le i \le 2$.
Let $\omega \in \Omega_h$. If $\hat{r}_i + \sum_{j=1}^{i-1}\hat{a}_{ij}x_0^j - \sum_{j=i+1}^{2}\hat{a}_{ij}x_0^j + \tilde{\sigma}_i\xi_1^i(\omega) \le 0$, then

$$x_1^i(\omega) = F^i_{0,\varkappa_0^i}(x_0^i)(\omega) \le x_0^i < \chi_i.$$

Otherwise, we have $0 < x_0^i < f_{0,i}(\varkappa_0^i)(\omega)$ with

$$f_{0,i}(\varkappa_0^i) = \hat{a}_{ii}^{-1}\left(\hat{r}_i + \sum_{j=1}^{i-1}\hat{a}_{ij}x_0^j - \sum_{j=i+1}^{2}\hat{a}_{ij}x_0^j + \tilde{\sigma}_i\xi_1^i\right).$$

Since $0 < f_{0,i}(\varkappa_0^i) < V_0^i(\varkappa_0)$ by (14), we get

$$0 < x_0^i < f_{0,i}(\varkappa_0^i)(\omega) < V_0^i(\varkappa_0)(\omega)$$

and further

$$x_1^i(\omega) = F^i_{0,\varkappa_0^i}(x_0^i)(\omega) < F^i_{0,\varkappa_0^i}(f_{0,i}(\varkappa_0^i))(\omega) = f_{0,i}(\varkappa_0^i)(\omega) < \chi_i,$$

where the first inequality is obtained from (9) with $\boldsymbol{\zeta}_0 = \varkappa_0$ and the last inequality from (11) and (15).
Step 3. We prove the boundedness: $(x_k^1, x_k^2) \in (0,\chi_1)\times(0,\chi_2)$ for $k \ge 0$.
Since Step1 and 2 give that

$$\text{if } (x_0^1, x_0^2) \in (0,\chi_1)\times(0,\chi_2), \text{ then } (x_1^1, x_1^2) \in (0,\chi_1)\times(0,\chi_2),$$

we can obtain the desired result by both applying mathematical induction and replacing $\left(x_0, \xi_1^i, \varkappa_0, \boldsymbol{\zeta}_0, V_0^i, F^i_{0,\varkappa_0^i}, f_{0,i}\right)$ with $\left(x_k, \xi_{k+1}^i, \varkappa_k, \boldsymbol{\zeta}_k, V_k^i, F^i_{k,\varkappa_k^i}, f_{k,i}\right)$ in Step 1 and 2. Here the function $f_{k,i}$ is defined as $f_{k,i}(\varkappa_k^i) = \hat{a}_{ii}^{-1}\left(\hat{r}_i + \sum_{j=1}^{i-1}\hat{a}_{ij}x_k^j - \sum_{j=i+1}^{2}\hat{a}_{ij}x_k^j + \tilde{\sigma}_i\xi_{k+1}^i\right)$. □

Remark 2. For simplicity, from now on we assume that the conditions (5), (12), (13), (14) and (15) used in Theorem 1 hold. Then we will not write the conditions explicitly in later sections when we need the positivity and boundedness of the solutions x_k^i.

3. A new discretized version of the Itô formula

In order to find conditions for the stability of (3), we need a discretized form of the Itô formula. Although there are discretized versions of the Itô formula (see [41], [42] and [43]), we need to formulate a variant which is suitable for our model (3). The proof of our new discrete Itô formula is almost the same as that of the discrete Itô formula in [42] and [43]. For the completeness of this paper, we reproduce the proof in the Appendix.

We write $q_1(h) = O(q_2(h))$ (or $q_1(h) = O(q_2(h))$ for $h \to 0$ to be more precise) if there exist positive constants C and h_0 such that $|q_1(h)| \le C|q_2(h)|$ for all h with $0 < h \le h_0$.

We make the two assumptions about the noise ξ: First, the noise ξ satisfies that for some constants M_1 and μ with $0 < \mu < 1$

$$E(\xi) = 0,\ E\left(\xi^2\right) = 1 - \mu,\ E\left(|\xi|^{\ell}\right) \le M_1\ (\ell = 1, 3). \tag{16}$$

Second, the probability density function p of the noise ξ exists with the property that for some constant M_2 and all sufficiently large $|x|$

$$|x|^3 p(x) \le M_2 |x|^{-1}. \tag{17}$$

Using $\mu = \eta_\varsigma$ in (7) and the probability density function $p(x) = \psi(x)$ in (6), one can obtain that the truncated standard normal random variables ξ_k^i satisfy the two assumptions (16) and (17). Let the symbol $\mathbb{R}$ denote the set of all real numbers and $C^3(\mathbb{R})$ denote the set of all functions defined on $\mathbb{R}$ that are continuously differentiable up to the order 3.

Lemma 1. *Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}_h$. Consider functions $\phi,\ \varphi : \mathbb{R} \to \mathbb{R}$ satisfying that for some $\delta > 0$,*

(i) $\varphi = \phi$ *on* $[1 - \delta, 1 + \delta]$
(ii) $\varphi \in C^3(\mathbb{R})$ *and* $|\varphi'''(x)| \le M_3$ *for some constant* M_3 *and all* $x \in \mathbb{R}$
(iii) $\int_{\mathbb{R}} |\varphi(x) - \phi(x)| dx < M_4$ *for some constant* M_4

and ϕ is almost everywhere continuous. Let f and g be $\mathcal{G}$-measurable random variables satisfying that for some positive constants ε and M_5,

$$\max\{h|f|, h^{0.5}|g|\} \le M_5 h^{\varepsilon}. \tag{18}$$

Let ξ be a $\mathcal{G}$-independent random variable satisfying (16) and (17). Then the conditional expectation of the random variable $\phi\left(1 + hf + h^{0.5} g\xi\right)$ with respect to the σ-algebra $\mathcal{G}$ becomes

$$\begin{aligned} &E\left[\phi\left(1 + hf + h^{0.5} g\xi\right) | \mathcal{G}\right] \\ &\quad = \phi(1) + \phi'(1) hf + 2^{-1}\phi''(1) hg^2 \cdot (1 - \mu) + hfO\left(h^{\varepsilon}\right) + hg^2 O\left(h^{\varepsilon}\right), \end{aligned}$$

where the first big O denotes

$$2^{-1}\varphi''(1) M_5 h^{\varepsilon} + 6^{-1} M_3 \left(M_5 h^{\varepsilon}\right)^2 \{1 + 3(1 - \mu)\}$$

and the last denotes

$$\left(M_1 M_5 + M_4 M_2 M_5 \delta_1\right) h^{\varepsilon}$$

for some positive constant δ_1 less than δ. Here M_1 and M_2 are defined in (16) and (17).

Proof. See the Appendix. □

Remark 3. Differently from the discretized Itô formulas in [43], [41] and [42], our discretized Itô formula in Lemma 1 does not require that the upper bounds of f and g are independent of h. Let $\mathcal{G} = \mathcal{F}_k$ and

$$f = r_i + \sum\nolimits_{j=1}^{i-1} a_{ij} x_k^j - \sum\nolimits_{j=i}^{2} a_{ij} x_k^j,\ g = \sigma_i,\ \xi = \xi_{k+1}^i \tag{19}$$

for the solutions x_k^i of (3) with $1 \le i \le 2$. Then f and g are $\mathcal{F}_k$-measurable and satisfy (18) with $\varepsilon = 0.5$ by applying the upper bound $\chi_i = O(h^{-0.5})$ of x_k^i to the definition of f. In addition, $\xi = \xi_{k+1}^i$ is an $\mathcal{F}_k$-independent random variable satisfying (16) and (17).

Remark 4. In order to construct φ in Lemma 1 corresponding to the function

$$\phi(x) = \begin{cases} \ln|x| & (|x| > 0), \\ 0 & (x = 0), \end{cases}$$

we modify the function φ used in [37]. Define the function φ as follows.

$$\varphi(x) = \begin{cases} \ln|x| & (|x| \geq e^{-1}), \\ -4^{-1}e^4x^4 + e^2x^2 - 4^{-1}7 + 6^{-1}e^6(x - e^{-1})^3(x + e^{-1})^3 & (|x| \leq e^{-1}). \end{cases}$$

Then ϕ and φ satisfy all the conditions in Lemma 1 with $\delta = 1 - e^{-1}$.

Notation 2. For simplicity, we use the notations

$$\overline{E}(x_k^i) = k^{-1}\sum_{s=0}^{k-1} E\left(x_s^i\right) \tag{20}$$

and

$$\mathring{a} = a \cdot \left\{1 + O(h^{0.5})\right\}, \ a_\eta = a \cdot (1 - \eta_\varsigma), \ r_{i\sigma} = r_i - 0.5\sigma_{i\eta}^2$$

for $k > 0$, $1 \leq i \leq 2$, constants a and η_ς in (7). Here $\sigma_{i\eta}^2$ is equal to $\{\sigma_i \cdot (1 - \eta_\varsigma)\}^2$.

Remark 5. Since the solutions x_k^i of (3) are positive by Theorem 1, we can take logarithm of (3), which gives

$$E\left[\ln x_{k+1}^i \middle| \mathcal{F}_k\right] = E\left[\ln x_k^i \middle| \mathcal{F}_k\right] + E\left[\ln\left(1 + hf + h^{0.5}g\xi_{k+1}^i\right) \middle| \mathcal{F}_k\right], \tag{21}$$

where f and g are defined in (19). In order to simplify the equation (21), applying $\mathcal{F}_k$-independence of ξ_{k+1}, $\mathcal{F}_k$-measurability of x_k^i and Lemma 1 with Remarks 3 and 4 to the three expectation terms in (21), respectively, we have

$$\begin{aligned} E(\ln x_{k+1}^i) &= \ln x_k^i + hf - \frac{1}{2}hg^2 \cdot (1 - \eta_\varsigma) + hfO\left(h^{0.5}\right) + hg^2O\left(h^{0.5}\right) \\ &= \ln x_k^i + \mathring{h}\left(r_i - \frac{1}{2}\sigma_{i\eta}^2 + \sum_{j=1}^{i-1} a_{ij}x_k^j - \sum_{j=i}^{2} a_{ij}x_k^j\right). \end{aligned} \tag{22}$$

Taking expectation of (22) and adding the result, we obtain

$$E(\ln x_k^i) = E(\ln x_0^i) + k\mathring{h}\left\{r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\overline{E}(x_k^j) - \sum_{j=i}^{2} a_{ij}\overline{E}(x_k^j)\right\}. \tag{23}$$

4. Auxiliary equations

In order to find upper bounds of x_k^i, we consider the auxiliary equations

$$z_{k+1}^i = z_k^i\left(1 + \hat{r}_i + \sum_{j=1}^{i-1} \hat{a}_{ij}z_k^j - \hat{a}_{ii}z_k^i + \tilde{\sigma}_i\xi_{k+1}^i\right), \ z_0^i = x_0^i \tag{24}$$

for $1 \leq i \leq 2$ and $k \geq 0$. Since (24) is the system (3) with $a_{12} = 0$, Theorem 1 with (4) gives that for $k \geq 0$

$$(z_k^1, z_k^2) \in (0, \chi_1) \times (0, \chi_2). \tag{25}$$

Let β_i be the solutions of the equations

$$r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j - a_{ii}\beta_i = 0 \tag{26}$$

for $1 \leq i \leq 2$. Note that (22) and (23) with $a_{12} = 0$ become

$$\begin{aligned} E\left(\ln z_{k+1}^1\right) &= \ln z_k^1 + \mathring{h}\left(r_{1\sigma} - a_{11}z_k^1\right), &(27)\\ E\left(\ln z_k^1\right) &= E\left(\ln z_0^1\right) + k\mathring{h}\left\{r_{1\sigma} - a_{11}\overline{E}\left(z_k^1\right)\right\} \\ &= E\left(\ln z_0^1\right) + k\mathring{h}a_{11}\left\{\beta_1 - k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right)\right\} &(28) \end{aligned}$$

due to (20) and $\beta_1 = a_{11}^{-1} r_{1\sigma}$ in (26). Similarly, we have

$$\begin{aligned} E\left(\ln z_{k+1}^2\right) &= \ln z_k^2 + \mathring{h}\left(r_{2\sigma} + a_{21}z_k^1 - a_{22}z_k^2\right), &(29)\\ E\left(\ln z_k^2\right) &= E\left(\ln z_0^2\right) + k\mathring{h}\left\{r_{2\sigma} + a_{21}\overline{E}(z_k^1) - a_{22}\overline{E}(z_k^2)\right\} \\ &= E\left(\ln z_0^2\right) + k\mathring{h}a_{22}\left\{\frac{r_{2\sigma}}{a_{22}} + \frac{a_{21}}{a_{22}}\overline{E}(z_k^1) - k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right)\right\}. &(30) \end{aligned}$$

Lemma 2. *Let z_k^1 and β_1 be the solutions of (24) and (26), respectively. If $\beta_1 \geq 0$, then for $\epsilon > 0$ and all sufficiently large k*

$$k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) \leq \beta_1 + \epsilon.$$

Proof. Suppose, on the contrary, that the theorem is false, which means that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ satisfying both for all k_m

$$k_m^{-1}\sum_{s=0}^{k_m-1} E\left(z_s^1\right) > \beta_1 + \varepsilon_0 \tag{31}$$

and for all k with $k \neq k_m$

$$k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) \leq \beta_1 + \varepsilon_0. \tag{32}$$

Combining (31) and (28), we have

$$\lim_{m\to\infty} E\left(\ln z_{k_m}^1\right) = -\infty. \tag{33}$$

Substituting (33) and the boundedness of z_k^1 into (27) gives

$$\lim_{m\to\infty} \ln z_{k_m-1}^1 = -\infty \quad a.s.$$

and then

$$\lim_{m\to\infty} z_{k_m-1}^1 = 0 \quad a.s. \tag{34}$$

Thus the dominated convergence theorem with (25) leads to

$$\lim_{m\to\infty} E(z_{k_m-1}^1) = 0. \tag{35}$$

In order to obtain a contraction we follow the two steps:

Step 1. If there exists $k = k_m - 1$ satisfying (32), then the system of (31) and (32) becomes

$$\begin{aligned} \sum\nolimits_{s=0}^{k_m-1} E\left(z_s^1\right) &> k_m\left(\beta_1+\varepsilon_0\right), \\ \sum\nolimits_{s=0}^{k_m-2} E\left(z_s^1\right) &\leq \left(k_m-1\right)\left(\beta_1+\varepsilon_0\right), \end{aligned}$$

which gives

$$E(z_{k_m-1}^1) > \beta_1 + \varepsilon_0, \tag{36}$$

and hence there exist finitely many k satisfying (32) due to (35) and (36). Therefore for all sufficiently large k

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^1\right) > \beta_1 + \varepsilon_0. \tag{37}$$

Step 2. As (31) implies (35), the equation (37) implies

$$\lim\nolimits_{k\to\infty} E(z_k^1) = 0,$$

which is contradictory to (37) due to $\beta_1 + \varepsilon_0 > 0$ and so the proof is completed. □

Lemma 3. *Let (z_k^1, z_k^2) and (β_1, β_2) be the solutions of (24) and (26), respectively.*

(a) *Assume $r_{1\sigma} < 0$. Then $\lim_{k\to\infty} z_k^1 = 0$ a.s.*
  (i) *If $r_{1\sigma} < 0$ and $r_{2\sigma} < 0$, then $\lim_{k\to\infty} z_k^2 = 0$ a.s.*
  (ii) *If $r_{1\sigma} < 0$ and $r_{2\sigma} \geq 0$, then $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) = a_{22}^{-1} r_{2\sigma}$.*

(b) *Assume $r_{1\sigma} \geq 0$. Then $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) = \beta_1$.*
  (i) *If $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$, then $\lim_{k\to\infty} z_k^2 = 0$ a.s.*
  (ii) *If $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}\beta_1 \geq 0$, then $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) = \beta_2$.*

Proof. (a) Since $r_{1\sigma} < 0$ is equivalent to $\beta_1 = a_{11}^{-1} r_{1\sigma} < 0$, it follows from (28) and the positivity of z_k^1 in (25) that if $r_{1\sigma} < 0$, then $\lim_{k\to\infty} E\left(\ln z_k^1\right) = -\infty$, and further

$$\lim\nolimits_{k\to\infty} z_k^1 = 0 \quad a.s. \tag{38}$$

as (33) implies (34).
(a)-(i) Assume that $r_{1\sigma} < 0$ and $r_{2\sigma} < 0$.
As (34) implies (35), the equation (38) yields $\lim_{m\to\infty} E(z_k^1) = 0$ and then

$$\lim\nolimits_{k\to\infty} \overline{E}(z_k^1) = 0. \tag{39}$$

Combining (39) and (30) with $r_{2\sigma} < 0$ and using $z_k^2 > 0$, we have from (30) that

$$\lim\nolimits_{k\to\infty} E\left(\ln z_k^2\right) = -\infty. \tag{40}$$

Therefore, as (33) implies (34), the equation (40) gives

$$\lim\nolimits_{k\to\infty} z_k^2 = 0 \quad a.s.$$

(a)-(ii) Assume that $r_{1\sigma} < 0$ and $r_{2\sigma} \geq 0$.
Using $(z_k^2, a_{22}^{-1} r_{2\sigma})$, (29) and (30) instead of (z_k^1, β_1), (27) and (28) in the proof of Lemma 2, respectively, and applying (39) to (30), we can obtain that for $\epsilon > 0$ and all sufficiently large k

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^2\right) \leq a_{22}^{-1} r_{2\sigma} + \epsilon. \tag{41}$$

In order to show $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) = a_{22}^{-1}r_{2\sigma}$, it is enough to prove that for $\epsilon > 0$ and all sufficiently large k

$$a_{22}^{-1}r_{2\sigma} - \epsilon \le k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right). \tag{42}$$

Suppose that (42) is false, which means that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ satisfying

$$a_{22}^{-1}r_{2\sigma} - \varepsilon_0 > k_m^{-1}\sum_{s=0}^{k_m-1} E\left(z_s^2\right). \tag{43}$$

Then the boundedness of z_k^2 and (30) imply that for all k_m

$$\infty > E\left(\ln z_{k_m}^2\right) > E\left(\ln z_0^2\right) + k_m \mathring{h} a_{22}\varepsilon_0, \tag{44}$$

which is a contradiction. Therefore (42) is true and so the proof is completed due to (41) and (42).
(b) Assume $r_{1\sigma} \ge 0$, which means $\beta_1 = a_{11}^{-1}r_{1\sigma} \ge 0$.
In order to show $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) = \beta_1$, it is enough to prove that for $\epsilon > 0$ and all sufficiently large k

$$\beta_1 - \epsilon \le k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) \tag{45}$$

due to Lemma 2. Suppose that (45) is false, so that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ such that

$$\beta_1 - \varepsilon_0 > k_m^{-1}\sum_{s=0}^{k_m-1} E\left(z_s^1\right). \tag{46}$$

Then the boundedness of z_k^1 and (28) imply that for all k_m

$$\infty > E\left(\ln z_{k_m}^1\right) > E\left(\ln z_0^1\right) + k_m \mathring{h} a_{11}\varepsilon_0, \tag{47}$$

which is a contradiction. Hence (45) is true and, therefore, Lemma 2 with (45) gives

$$\lim_{k\to\infty} \overline{E}(z_k^1) = \beta_1. \tag{48}$$

(b)-(i) Assume that $r_{1\sigma} \ge 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$.
Applying (48) to (30) with both $r_{2\sigma} + a_{21}\beta_1 < 0$ and $z_k^2 > 0$, we have

$$\lim_{k\to\infty} E(\ln z_k^2) = -\infty.$$

Therefore, as (33) implies (34), we can obtain $\lim_{k\to\infty} z_k^2 = 0 \;\; a.s.$
(b)-(ii) Assume that $r_{1\sigma} \ge 0$ and $r_{2\sigma} + a_{21}\beta_1 \ge 0$.
Following the proof of Lemma 2, we can obtain that

$$k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) \le \beta_2 + \epsilon \tag{49}$$

for $\epsilon > 0$ and all sufficiently large k by using (z_k^2, β_2), (29) and (30) instead of (z_k^1, β_1), (27) and (28), respectively, and applying (48) and $\beta_2 = a_{22}^{-1}\left(r_{2\sigma} + a_{21}\beta_1\right) \ge 0$ to (30).
Similarly, following the proof of (45), we can obtain that

$$\beta_2 - \epsilon \le k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) \tag{50}$$

for $\epsilon > 0$ and all sufficiently large k by replacing (z_k^1, β_1) and (28) with (z_k^2, β_2) and (30), respectively, and applying (48) to (30). Therefore (49) and (50) give the desired result. $\square$

Remark 6. The equations (28) and (30) can be written as

$$E(\ln z_k^i) = E(\ln z_0^i) + k\mathring{h}\left\{r_{i\sigma} + \sum\nolimits_{j=1}^{i-1} a_{ij}\overline{E}(z_k^j) - a_{ii}\overline{E}(z_k^i)\right\}. \tag{51}$$

Substituting (26) to (51) yields

$$E(\ln z_k^i) = E(\ln z_0^i) + k\mathring{h}\left[\sum\nolimits_{j=1}^{i-1} a_{ij}\left\{\overline{E}(z_k^j) - \beta_j\right\} - a_{ii}\left\{\overline{E}(z_k^i) - \beta_i\right\}\right]. \tag{52}$$

Applying Lemma 3-(b) and (b)-(ii) to (52) with the notation (20), we have

$$\lim\nolimits_{k\to\infty} k^{-1}E(\ln z_k^i) = 0 \tag{53}$$

under the condition that $\min\{r_{1\sigma}, r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j\} \geq 0$ for $1 \leq i \leq 2$.

Lemma 4. *Let x_k^i and z_k^i be the solutions of (3) and (24), respectively for $i = 1, 2$. Then for $k \geq 0$*

$$0 < x_k^i \leq z_k^i.$$

Proof. Theorem 1 with Remark 2 gives

$$0 < x_k^i. \tag{54}$$

Note that

$$F_{k,y}^1(x) \text{ is nonincreasing in } y \text{ for } x \geq 0 \text{ and } k \geq 0 \tag{55}$$

and

$$F_{k,x}^2(y) \text{ is nondecreasing in } x \text{ for } y \geq 0 \text{ and } k \geq 0 \tag{56}$$

by the definition (8). The proof of this lemma is divided into the following two cases.
Case 1. Let $i = 1$.
Using $\varkappa_0^1 = x_0^2 > 0$ and (55), we have

$$x_1^1 = F_{0,\varkappa_0^1}^1(x_0^1) \leq F_{0,0}^1(x_0^1). \tag{57}$$

It follows from Remark 2, (24), (25), (10) and (13) that

$$0 < x_0^1 \leq z_0^1 < \chi_1 < V_0^1(0,0),$$

with which (9) yields

$$F_{0,0}^1(x_0^1) \leq F_{0,0}^1(z_0^1) = z_1^1. \tag{58}$$

Hence combining (54), (57) and (58) gives

$$0 < x_1^1 \leq z_1^1. \tag{59}$$

Assume that for some positive integer k

$$0 < x_k^1 \leq z_k^1. \tag{60}$$

Using (54), (60), (25), (10) and (13), we have

$$\varkappa_k^1 > 0,\ 0 < x_k^1 \leq z_k^1 < \chi_1 < V_k^1(0,0)$$

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and so

$$x_{k+1}^1 = F_{k,\not{x}_k^1}^1(x_k^1) \leq F_{k,0}^1(x_k^1) \leq F_{k,0}^1(z_k^1) = z_{k+1}^1,$$

where the first inequality is obtained from (55) and the second inequality from (9).
Case 2. Let $i = 2$.
Using $\not{x}_0^2 = x_0^1 \leq z_0^1$ and $0 < x_0^2 \leq z_0^2 < \chi_2 < V_0^2(0,0)$, we have

$$x_1^2 = F_{0,\not{x}_0^2}^2(x_0^2) \leq F_{0,z_0^1}^2(x_0^2) \leq F_{0,z_0^1}^2(z_0^2) = z_1^2 \tag{61}$$

due to (56) and (9). Similarly as in Case 1, using mathematical induction and $z_k^2 \leq \chi_2 < V_k^2(0,0)$ instead of $z_k^1 < \chi_1 < V_k^1(0,0)$ in Case 1, we can obtain the desired result. □

Remark 7. If $\min\{r_{1\sigma}, r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j\} \geq 0$ for $1 \leq i \leq 2$, then Lemma 4 and (53) imply that for $\epsilon > 0$ and all sufficiently large k

$$k^{-1}E(\ln x_k^i) \leq \epsilon, \tag{62}$$

which will be first used in Theorem 4.

5. Extinction and persistence of the discrete solutions

In this section, we present several conditions sufficient for the extinction and persistence (non-extinction) of the solutions x_k^i of (3).

Theorem 2. *Let x_k^i and β_i be the solutions of (3) and (26), respectively for $i = 1, 2$.*

(a) *If $r_{1\sigma} < 0$, then $\lim_{k\to\infty} x_k^1 = 0$ a.s.*
(b) *If $r_{1\sigma} < 0$ and $r_{2\sigma} < 0$, then $\lim_{k\to\infty} x_k^2 = 0$ a.s.*

Proof. The proof is followed by combining Lemma 3-(a) and (a)-(i) with Lemma 4. □

Remark 8. Since $r_{1\sigma} = 0$ gives $\beta_1 = a_{11}^{-1} r_{1\sigma} = 0$, we obtain that

$$\text{if } r_{1\sigma} = 0, \text{ then } \lim_{k\to\infty} k^{-1} \textstyle\sum_{s=0}^{k-1} E\left(x_s^1\right) = 0$$

by combining Lemma 3-(b) with Lemma 4. Similarly, Lemma 3-(b)-(ii) gives

$$\text{if } r_{1\sigma} = r_{2\sigma} = 0, \text{ then } \lim_{k\to\infty} k^{-1} \textstyle\sum_{s=0}^{k-1} E\left(x_s^2\right) = 0$$

since $\beta_2 = a_{22}^{-1}\left(r_{2\sigma} + a_{21}\beta_1\right) = 0$.

Remark 9. By Theorem 2-(a), we find that if $r_1 < \frac{1}{2}\sigma_{1\eta}^2$, then the prey population will be extinct in the future, no matter whether the predator exists. It implies that environmental noise plays a very important role in the biological system.

In order to establish the sufficient condition for the extinction of the predator and the persistence of the prey, we will use the following Lemma 5 as well as Lemma 3-(b).
Using Lemmas 4 and 3-(b) with $\beta_1 = a_{11}^{-1} r_{1\sigma}$ we obtain that

$$\text{if } r_{1\sigma} > 0, \text{ then } \lim_{k\to\infty} k^{-1} \sum_{s=0}^{k-1} E\left(x_s^1\right) \leq a_{11}^{-1} r_{1\sigma}. \tag{63}$$

For finding a lower function of x_k^1, we consider the solution $u_{k,\epsilon}$ of the equation

$$u_{k+1,\epsilon} = u_{k,\epsilon}(1 + \hat{r}_1 - \hat{a}_{11}u_{k,\epsilon} - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}),\ u_{0,\epsilon} = x^1_{N_\epsilon}, \tag{64}$$

in which ϵ satisfies that for some positive integer N_ϵ and all $k \geq N_\epsilon$

$$0 < x_k^2 \leq \epsilon, \tag{65}$$

$$\hat{r}_1 - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\varsigma_* < 1, \tag{66}$$

$$\hat{a}_{12}\epsilon + \tilde{\sigma}_1\varsigma < \tilde{\sigma}_1\varsigma_*, \tag{67}$$

where (65) is possible under the conditions $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$ due to Lemmas 4 and 3-(b)-(i). The inequalities (66) and (67) are possible by (14) and (12), respectively.

Lemma 5. *Assume that $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$. Let ϵ and N_ϵ satisfy (65)–(67). Let x_k^1 and $u_{k,\epsilon}$ be the solutions of (3) and (64), respectively. Then*

(a) $0 < u_{k,\epsilon} < \chi_1$ *for* $k \geq 0$.

(b) $u_{k,\epsilon} \leq x^1_{N_\epsilon+k}$ *for* $k \geq 0$.

(c) *If* $r_{1\sigma} - a_{12}\epsilon > 0$, *then* $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(u_{s,\epsilon}\right) = a_{11}^{-1}\left(r_{1\sigma} - a_{12}\epsilon\right)$.

Proof. (a)We proceed by induction on k.
Since (64) and Theorem 1 with Remark 2 give

$$u_{0,\epsilon} = x^1_{N_\epsilon},\ 0 < x^1_{N_\epsilon} < \chi_1,$$

the statement (a) is true for $k = 0$.
Assume that for a nonnegative integer k

$$0 < u_{k,\epsilon} < \chi_1. \tag{68}$$

Now, in the case of $k + 1$, the proof of (a) is divided into the following two steps.
Step 1. We prove the positivity of $u_{k+1,\epsilon}$.
Denoting

$$\mathcal{U}_k = (2\hat{a}_{11})^{-1}\left(1 + \hat{r}_1 - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}\right)$$

gives that for $k \geq 0$

$$0 < \chi_1 < (2\hat{a}_{11})^{-1}\left(1 + \hat{r}_1 - \tilde{\sigma}_1\varsigma_*\right) < \mathcal{U}_k, \tag{69}$$

where the second inequality is obtained from (13) and the last from (67), (12) and (5).
Letting

$$G_k(x) = x\left(1 + \hat{r}_1 - \hat{a}_{11}x - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}\right),$$

we have

$$G_k(x) \text{ is strictly increasing on } 0 \leq x < \mathcal{U}_k. \tag{70}$$

Applying (68) and (69) to (70), we have the desired positivity.
Step 2. We prove that χ_1 is an upper bound of $u_{k+1,\epsilon}$.
Let $\omega \in \Omega_h$. If $\hat{r}_1 - \hat{a}_{11}u_{k,\epsilon}(\omega) - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}(\omega) \leq 0$, then

$$u_{k+1,\epsilon}(\omega) = G_k(u_{k,\epsilon})(\omega) \leq u_{k,\epsilon}(\omega) < \chi_1,$$

in which (68) gives the last inequality. Otherwise, we have $0 < u_{k,\epsilon}(\omega) < \Delta_k(\omega)$ with

$$\Delta_k = \hat{a}_{11}^{-1}\left(\hat{r}_1 - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}\right).$$

Since $\Delta_k < \mathcal{U}_k$ by (66), we have $0 < u_{k,\epsilon}(\omega) < \Delta_k(\omega) < \mathcal{U}_k(\omega)$ and then (70) gives

$$u_{k+1,\epsilon}(\omega) = G_k(u_{k,\epsilon})(\omega) < G_k(\Delta_k)(\omega) = \Delta_k(\omega) < \chi_1,$$

where the last inequality is obtained from (11), (12) and (5).
(b)We proceed by induction on k.
The statement (b) is true for $k = 0$ due to (64).
Assume that for a nonnegative integer k

$$u_{k,\epsilon} \le x^1_{N_\epsilon+k}. \tag{71}$$

It follows from (a) in this theorem, (71), Theorem 1, Remark 2 and (69) that

$$0 < u_{k,\epsilon} \le x^1_{N_\epsilon+k} < \chi_1 < \mathcal{U}_k$$

and then

$$u_{k+1,\epsilon} = G_k(u_{k,\epsilon}) \le G_k(x^1_{N_\epsilon+k}) = F^1_{N_\epsilon+k,\epsilon}(x^1_{N_\epsilon+k}) \tag{72}$$

due to (70). Combining (55) and (65) also gives

$$F^1_{N_\epsilon+k,\epsilon}(x^1_{N_\epsilon+k}) \le F^1_{N_\epsilon+k,x^2_{N_\epsilon+k}}(x^1_{N_\epsilon+k}) = x^1_{N_\epsilon+k+1}. \tag{73}$$

Therefore, (72) and (73) give the desired result.
(c) Let $\gamma_1 = a_{11}^{-1}(r_{1\sigma} - a_{12}\epsilon)$. Note that

$$\begin{aligned} E(\ln u_{k+1,\epsilon}) &= \ln u_{k,\epsilon} + \mathring{h}\left(r_{1\sigma} - a_{11}u_{k,\epsilon} - a_{12}\epsilon\right), & (74)\\ E(\ln u_{k,\epsilon}) &= E(\ln u_{0,\epsilon}) + k\mathring{h}\left\{r_{1\sigma} - a_{12}\epsilon - a_{11}\overline{E}(u_{k,\epsilon})\right\} \\ &= E(\ln u_{0,\epsilon}) + k\mathring{h}a_{11}\left\{\gamma_1 - k^{-1}\sum\nolimits_{s=0}^{k-1} E(u_{s,\epsilon})\right\} & (75)\end{aligned}$$

as in (27) and (28). Following the proof of Lemma 2, we can obtain that

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E(u_{s,\epsilon}) \le \gamma_1 + \epsilon' \tag{76}$$

for $\epsilon' > 0$ and all sufficiently large k by replacing (27), (28) and $(z^1_k, r_{1\sigma}, \beta_1)$ with (74), (75) and $(u_{k,\epsilon}, r_{1\sigma} - a_{12}\epsilon, \gamma_1)$, respectively.
Similarly, replacing (28) and (z^1_k, β_1) in (45)–(47) with (75) and $(u_{k,\epsilon}, \gamma_1)$, respectively, we can obtain that for $\epsilon' > 0$ and all sufficiently large k

$$\gamma_1 - \epsilon' \le k^{-1}\sum\nolimits_{s=0}^{k-1} E(u_{s,\epsilon}),$$

with which (76) gives the desired result. □

Theorem 3. *Let x^i_k and β_1 be the solutions of (3) and (26), respectively for $i = 1, 2$.*

If $r_{1\sigma} \ge 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$, then $\lim_{k\to\infty} \overline{E}(x^1_k) = \beta_1$ and $\lim_{k\to\infty} x^2_k = 0$ a.s.

Proof. It follows from Lemma 3-(b)-(i), Lemma 4, Theorem 1 and Remark 2 that

$$\lim_{k\to\infty} x_k^2 = 0 \quad a.s.$$

Using Lemma 5-(a) and Lemma 4, we obtain that for $\epsilon > 0$ and all sufficiently large k

$$0 < u_{k,\epsilon} \leq x^1_{N_\epsilon+k} \leq z^1_{N_\epsilon+k}. \tag{77}$$

Lemma 5-(c) and Lemma 3-(b) give

$$\lim_{k\to\infty} \overline{E}\left(u_{k,\epsilon}\right) = a_{11}^{-1}\left(r_{1\sigma} - a_{12}\epsilon\right), \ \lim_{k\to\infty} \overline{E}\left(z_k^1\right) = a_{11}^{-1} r_{1\sigma}, \tag{78}$$

where the first and second equalities are valid under the conditions $r_{1\sigma} - a_{12}\epsilon > 0$ and $r_{1\sigma} \geq 0$, respectively. Therefore using (77), (78) and Remark 8, we obtain the desired result. □

Remark 10. By Theorems 2 and 3, we find that the value $r_{1\sigma}$ is the threshold between the extinction and persistence for the prey population. In addition, although the prey population converges to a non-extinction state in the mean when $r_{1\sigma} > 0$ and $r_{2\sigma}+a_{21}\beta_1 < 0$, the predators dies out when the diffusion coefficient σ_2 is large enough and then

$$-r_{2\sigma} = -r_2 + 0.5\left\{\sigma_2 \cdot (1-\eta_\varsigma)\right\}^2$$

becomes too large.

Remark 11. We can establish one condition for the extinction of the prey and the persistence of the predator as follows. Lemmas 4 and 3-(a)-(ii) yield

$$\text{if } r_{1\sigma} < 0 \text{ and } r_{2\sigma} \geq 0, \text{ then } \lim_{k\to\infty} k^{-1} \sum_{s=0}^{k-1} E\left(x_s^2\right) \leq a_{22}^{-1} r_{2\sigma}. \tag{79}$$

For finding a lower function of x_k^2, we consider the solution $v_{k,\epsilon}$ of the equation

$$v_{k+1,\epsilon} = v_{k,\epsilon}(1 + \hat{r}_2 - \hat{a}_{21}\epsilon - \hat{a}_{22} v_{k,\epsilon} + \tilde{\sigma}_2 \xi^2_{N_\epsilon+k+1}), \ v_{0,\epsilon} = x^2_{N_\epsilon}, \tag{80}$$

in which ϵ satisfies that for some positive integer N_ϵ and all $k \geq N_\epsilon$

$$0 < x_k^1 \leq \epsilon, \tag{81}$$
$$\hat{r}_2 - \hat{a}_{21}\epsilon + \tilde{\sigma}_2\varsigma_* < 1, \tag{82}$$
$$\hat{a}_{21}\epsilon + \tilde{\sigma}_2\varsigma < \tilde{\sigma}_2\varsigma_*. \tag{83}$$

The inequality (81) is possible under the condition $r_{1\sigma} < 0$ due to Lemma 3-(a). Replacing (64)–(67), $r_{1\sigma} > 0$, $r_{2\sigma} + a_{21}\beta_1 < 0$ and $(u_{k,\epsilon}, r_1, a_{11}, a_{12}, \xi^1)$ in the proof of Lemma 5 with (80)–(83), $r_{1\sigma} < 0$, $r_{2\sigma} > 0$ and $(v_{k,\epsilon}, r_2, a_{22}, a_{21}, \xi^2)$, we can obtain that

$$v_{k,\epsilon} \leq x^2_{N_\epsilon+k}, \ \lim_{k\to\infty} k^{-1} \sum_{s=0}^{k-1} E\left(v_{s,\epsilon}\right) = a_{22}^{-1}\left(r_{2\sigma} - a_{21}\epsilon\right), \tag{84}$$

if $r_{2\sigma} - a_{21}\epsilon > 0$. Therefore (79) and (84) give the desired result:

$$\text{if } r_{1\sigma} < 0 \text{ and } r_{2\sigma} > 0, \text{ then } \lim_{k\to\infty}\left(x_k^1, \overline{E}(x_k^2)\right) = \left(0, a_{22}^{-1} r_{2\sigma}\right) \quad a.s.$$

Now, it remains to establish one condition for persistence of the prey and the predator. Define the matrix A and the constants D_i as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \end{pmatrix} = A \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \tag{85}$$

which give

$$|A| = a_{11}a_{22} + a_{12}a_{21} > 0, \ \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = A^{-1} \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \end{pmatrix} = |A|^{-1} \begin{pmatrix} a_{22}r_{1\sigma} - a_{12}r_{2\sigma} \\ a_{11}\left(r_{2\sigma} + a_{21}\beta_1\right) \end{pmatrix} \geq 0 \tag{86}$$

under the conditions $r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma} + a_{21}\beta_1 \geq 0$.
Using (85), the system (23) can be written as the matrix equation

$$\begin{pmatrix} E\left(\ln x_k^1\right) \\ E\left(\ln x_k^2\right) \end{pmatrix} = \begin{pmatrix} E\left(\ln x_0^1\right) \\ E\left(\ln x_0^2\right) \end{pmatrix} + k\mathring{h}A \begin{pmatrix} D_1 - \overline{E}\left(x_k^1\right) \\ D_2 - \overline{E}\left(x_k^2\right) \end{pmatrix} \tag{87}$$

and multiplying the matrix $|A|A^{-1}$ to (87), we have

$$a_{22}E(\ln x_k^1) - a_{12}E(\ln x_k^2) = C_1 + k\mathring{h}|A|\left\{D_1 - \overline{E}\left(x_k^1\right)\right\}, \tag{88}$$

$$a_{21}E(\ln x_k^1) + a_{11}E(\ln x_k^2) = C_2 + k\mathring{h}|A|\left\{D_2 - \overline{E}\left(x_k^2\right)\right\}, \tag{89}$$

where $C_1 = a_{22}E(\ln x_0^1) - a_{12}E(\ln x_0^2)$ and $C_2 = a_{21}E(\ln x_0^1) + a_{11}E(\ln x_0^2)$.

Lemma 6. *Let x_k^1 and β_1 be the solutions of (3) and (26), respectively. If $r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma} + a_{21}\beta_1 \geq 0$, then for $\epsilon > 0$ and all sufficiently large k*

$$\overline{E}(x_k^1) \leq D_1 + \epsilon, \tag{90}$$

where D_1 is defined in (85).

Proof. Suppose that (90) is false, which means that there exist a constant $\epsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ satisfying both for all k_m

$$k_m^{-1}\sum\nolimits_{s=0}^{k_m-1} E\left(x_s^1\right) > D_1 + \epsilon_0, \tag{91}$$

and for all k with $k \neq k_m$

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(x_s^1\right) \leq D_1 + \epsilon_0. \tag{92}$$

Replace (z_k^1, β_1), (31), (32), (28) and (27) in the proof of Lemma 2 with (x_k^1, D_1), (91), (92), (88) and (22), respectively, where we apply (22) with $i = 1$. Then using the boundedness of x_k^1 and following the proof for (37), we can obtain that for all sufficiently large k

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(x_s^1\right) > D_1 + \epsilon_0. \tag{93}$$

Combining (93) and (88) gives

$$a_{22}E(\ln x_k^1) - a_{12}E(\ln x_k^2) < C_1 + k\mathring{h}|A|(-\epsilon_0). \tag{94}$$

Applying Theorem 1 to (22) with $i = 2$, we obtain

$$\sup_{k\geq 0} E(\ln x_k^2) < \infty$$

and then (94) yields

$$\lim_{k\to\infty} E\left(\ln x_k^1\right) = -\infty. \tag{95}$$

Substituting (95) into (22) with $i = 1$ and using the boundedness of x_k^1, we obtain

$$\lim_{k\to\infty} \ln x_k^1 = -\infty \quad a.s.,$$

which implies

$$\lim_{k\to\infty} x_k^1 = 0 \quad a.s.$$

Hence the dominated convergence theorem with Theorem 1 leads to

$$\lim_{k\to\infty} E(x_k^1) = 0,$$

which is contradictory to (93) due to $D_1 + \varepsilon_0 > 0$. This completes the proof. $\square$

Remark 12. The equation (90) with (87) gives that for $\epsilon > 0$ and all sufficiently large k

$$E(\ln x_k^2) \leq E(\ln x_0^2) + k\mathring{h}a_{22}\left\{a_{22}^{-1}a_{21}\epsilon + D_2 - \overline{E}(x_k^2)\right\}. \tag{96}$$

Following the proof of Lemma 6 with (96), we can obtain that

$$\text{if } r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma} \text{ and } r_{2\sigma} + a_{21}\beta_1 \geq 0, \text{ then } \overline{E}(x_k^2) \leq a_{22}^{-1}a_{21}\epsilon + D_2 + \epsilon' \tag{97}$$

for $\epsilon' > 0$ and all sufficiently large k by replacing (x_k^1, D_1) and (88) in the proof of Lemma 6 with $(x_k^2, a_{22}^{-1}a_{21}\epsilon + D_2)$ and (96), respectively.

Theorem 4. *Let x_k^i and β_i be the solutions of (3) and (26), respectively for $i = 1, 2$.*

$$\textit{If } r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma} \textit{ and } r_{2\sigma} + a_{21}\beta_1 \geq 0, \textit{ then } \lim_{k\to\infty} \overline{E}(x_k^i) = D_i,$$

where D_i are defined in (85).

Proof. Substituting (62) into (89) gives that for $\epsilon' > 0$ and all sufficiently large k

$$\epsilon' \geq D_2 - \overline{E}(x_k^2). \tag{98}$$

Combining (98) and (97), we have

$$\lim_{k\to\infty} \overline{E}(x_k^2) = D_2. \tag{99}$$

Applying (99) to (89) with (62) yields

$$\lim_{k\to\infty} k^{-1}E(\ln x_k^1) = \lim_{k\to\infty} k^{-1}E(\ln x_k^2) = 0,$$

with which (88) gives the desired result $\lim\limits_{k\to\infty} \overline{E}(x_k^1) = D_1$. $\square$

Remark 13. Let (x_k, y_k) be the solutions of DDEs (3) with $\sigma_1 = \sigma_2 = 0$ in [35].

(i) If $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 \leq 0$, then $\lim_{k\to\infty}(x_k, y_k) = (a_{11}^{-1}r_1, 0)$.

(ii) If $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 > 0$, then $\lim_{k\to\infty}(x_k, y_k) = (D_x, D_y)$, where (D_x, D_y) is equal to (D_1, D_2) with $\sigma_1 = \sigma_2 = 0$.

Note that the sign of r_2 in the DDE model is fixed to $r_2 < 0$. Adding the noise to the DDEs, we have from Theorems 3 and 4 that

(i)′ If $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} < 0$, then $\lim_{k\to\infty}\left(\overline{E}(x_k^1), x_k^2\right) = \left(a_{11}^{-1}r_{1\sigma}, 0\right)$ $a.s.$

(ii)′ If $r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma}+a_{21}a_{11}^{-1}r_{1\sigma} \geq 0$, then $\lim_{k\to\infty}\left(\overline{E}(x_k^1), \overline{E}(x_k^2)\right) = (D_1, D_2)$.

Hence we demonstrate that the solutions of the DDEs and the DSDEs with small noise have similar asymptotic behavior by comparing (i), (ii) and (i)′, (ii)′, respectively. In addition, when comparing $r_2 + a_{21}a_{11}^{-1}r_1 > 0$ in (ii) and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} < 0$ in (i)′, we understand the effect of strong noise, which changes the behavior of the predator population from non-extinction into extinction. Therefore the main difference between the deterministic and stochastic models is that large stochastic perturbation may result in the extinction of the predator population.

Remark 14. Let (x, y) be the solutions of the SDE model (2), which is a special model in [25] with zero time delays. Note that the sign of r_2 in the SDE model is also negative.

(i) If $r_1 - 0.5\sigma_1^2 < 0$ and $r_2 - 0.5\sigma_2^2 < 0$, then $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$ $a.s.$

(ii) If $r_1 - 0.5\sigma_1^2 > 0$, $r_2 - 0.5\sigma_2^2 < 0$ and $(r_2 - 0.5\sigma_2^2) + a_{21}a_{11}^{-1}(r_1 - 0.5\sigma_1^2) < 0$, then x is stable in the mean and y goes to extinction:

$$\lim_{t\to\infty} t^{-1}\int_0^t x(s)ds = a_{11}^{-1}r_{1\sigma},\ \lim_{t\to\infty} y(t) = 0 \ \ a.s.$$

(iii) If $r_2 - 0.5\sigma_2^2 < 0$ and $(r_2 - 0.5\sigma_2^2) + a_{21}a_{11}^{-1}(r_1 - 0.5\sigma_1^2) > 0$, then both x and y are stable in the mean:

$$\lim_{t\to\infty}\left(t^{-1}\int_0^t x(s)ds,\ t^{-1}\int_0^t y(s)ds\right) = (D_1, D_2) \ \ a.s.$$

Since $r_2 < 0$ in the SDE model (2), the sign of $r_2 - 0.5\sigma_2^2$ in (2) is also negative, which is the reason why the condition $r_2 - 0.5\sigma_2^2 < 0$ is assumed in (i)–(iii). The three results, (i), (ii) and (iii) in this remark, are corresponding to Theorem 2-(b), (i)′ and (ii)′ in Remark 13, respectively. Hence, when replacing the stability of $(x(t), y(t))$ in the mean with the stability of $\left(\overline{E}(x_k^1), \overline{E}(x_k^2)\right)$, we demonstrate that the sufficient conditions for the almost sure global stability of the SDE model (2) also suffice to give the same global stability of the DSDE model (3). In this case, note that there is no constraint on the sign of r_2 in the DSDE model. Therefore we show that the DSDE model (3) is a good discrete model for the corresponding SDE model (2).

6. Numerical examples

In this section, we provide some simulations that illustrate the results in Theorems 1, 2, 3 and 4 with truncation constants $(\varsigma, \varsigma_*) = (19.9, 20)$ in (5) and (12). In this case, we have $0 < \eta_\varsigma < 10^{-85}$, so that we can ignore the effect of the term η_ς when using the values of parameters in the following three examples, where the conditions (12)–(14) are satisfied. In Figures 1, 2 and 3, the DSDE model (3) is simulated 1000 times at each time kh for calculating the expectation values $E\left(x_k\right)$ and $E\left(y_k\right)$, where x_k and y_k denote the

solutions x_k^1 and x_k^2, respectively. We compare our results for the DSDE model (3) with the results for the DDE model in [35], which is the model (3) with $\sigma_1 = \sigma_2 = 0$.

Example 1. Let $h = 0.0001, r_1 = 0.8, r_2 = -0.1, a_{11} = 0.4, a_{12} = 0.001, a_{21} = 0.1, a_{22} = 0.3, \sigma_1^2 = 2.5$ and $\sigma_2^2 = 0.1$. Since $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 > 0$, the solutions x_k and y_k of the DDE model converge to the positive numbers D_x and D_y in Remark 13-(ii), respectively, as displayed in Figure 1-(a). However, since $r_{i\sigma} < 0$ $(i = 1, 2)$, the noises have a large effect on the convergence and, as a result, the solutions of the stochastically perturbed model (3) go to extinction, which are shown in Figures 1-(b) and (c), as in Theorem 2-(a) and (b), respectively. Therefore Figures 1 demonstrates the important role of noise.

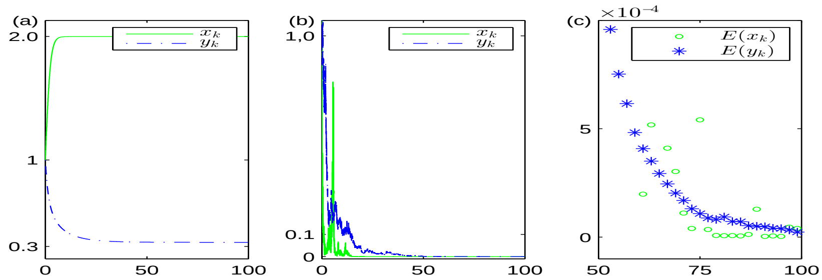

Figure 1: All the x-axes denote time kh. (a) Curves of the solutions of the DDE model. (b) Two realizations of the solutions x_k and y_k of the DSDE model, which converge to zero. (c) Expectation values of the solutions x_k and y_k of the DSDE model, which converge to zero in the mean.

Example 2. Let $h = 0.001, r_1 = 2, r_2 = -2, a_{11} = 1.0, a_{12} = 0.4, a_{21} = a_{22} = 0.3, \sigma_1^2 = 0.2$ and $\sigma_2^2 = 4$. Figure 2-(a) shows that the solutions x_k and y_k of the DDE model converge to $a_{11}^{-1}r_1$ and 0, respectively, as in Remark 13-(i) when $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 \le 0$. The noises satisfy both $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} < 0$, which are the conditions in Theorem 3. Then Figures 2-(b), (c) and (d) show that the stochastically perturbed model (3) behaves similarly to the DDE model in the sense that $k^{-1}\sum_{i=0}^{k-1} E(x_i)$ and y_k converge to $a_{11}^{-1}r_{1\sigma}$ and 0, respectively, which confirms Theorem 3.

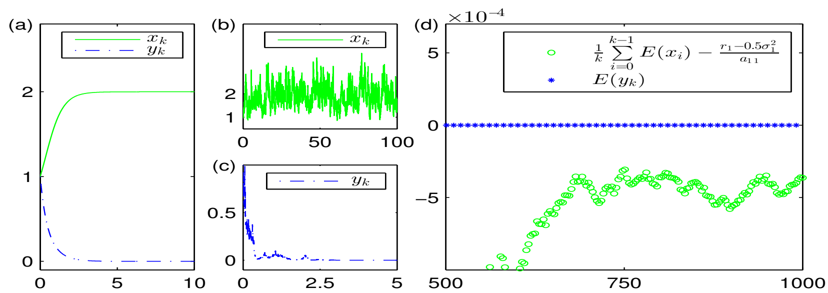

Figure 2: All the x-axes denote time kh. (a) Curves of the solutions of the DDE model. Curves in (b) and (c) are realizations of the solutions x_k and y_k of the DSDE model, respectively. (d) Convergence of average of expectation values of x_k to non-zero and convergence of y_k to zero in the mean.

Example 3. Let $h = 0.001, r_1 = 2.0, r_2 = -0.1, a_{11} = a_{12} = 0.4, a_{21} = 1, a_{22} = 0.3$ and $\sigma_1^2 = \sigma_2^2 = 0.02$, which give that $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 > 0$. Thus Figure 3-(a)

shows that the solutions x_k and y_k of the DDE model converge to D_x and D_y in Remark 13-(ii), respectively, as displayed in Figure 1-(a) in Example 1. However, the condition $r_{1\sigma} > 0$ is different from that in Example 1. Realizations of the solutions of the DSDE model are given in Figures 3-(b) and (c). Since $r_{1\sigma} > a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} > 0$, Figure 3-(d) shows that the DSDE model behaves similarly to the DDE model in the sense that $k^{-1}\sum_{i=0}^{k-1} E(x_i)$ and $k^{-1}\sum_{i=0}^{k-1} E(y_i)$ converge to positive D_1 and D_2, respectively, which demonstrate Theorem 4.

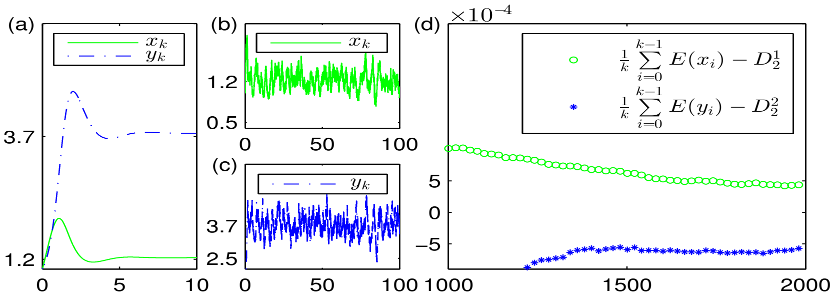

Figure 3: All the x-axes denote time kh. (a) Curves of the solutions of the DDE model. Curves in (b) and (c) are realizations of the solutions x_k and y_k of the DSDE model, respectively. The symbols D_2^1 and D_2^2 in (d) denote D_1 and D_2 defined in (85).

7. Conclusion

In this paper, we have considered a system of discrete-time stochastic difference equations for predator-prey interactions and established sufficient conditions for extinction and non-extinction of the two species. Our results show that if the positive equilibrium point of the deterministic difference system is globally stable, then the stochastic difference model will preserve the nice property in mean provided that the noise is sufficiently small. It is shown, however, that large noise can change the behavior of the predator population from non-extinction into extinction.

Our new discrete Itô formula has played an important role in the two-dimensional DSDE model. In addition we can apply the new formula for the n-dimensional DSDE model

$$x_{k+1}^i = x_k^i \left\{1 + h\left(r_i + \sum_{j=1}^{i-1} a_{ij}x_k^j - \sum_{j=i}^{n} a_{ij}x_k^j\right) + h^{0.5}\sigma_i\xi_{k+1}^i\right\}$$

for $1 \le i \le n$ and $k \ge 0$. Therefore it is a further study to establish sufficient conditions for the extinction and non-extinction of the n species.

Appendix

A.1. The proof of Lemma 1

By Taylor expansion,

$$\varphi(1+x) = \varphi(1) + \varphi'(1)x + 2^{-1}\varphi''(1)x^2 + 6^{-1}\varphi'''(\theta)x^3 \tag{100}$$

with θ lying between 1 and x. Let $x = hf + h^{0.5}g\xi$. Since f, g are $\mathcal{G}$-measurable and ξ is $\mathcal{G}$-independent with $E(\xi) = 0$, we have

$$E\left(x\middle|\,\mathcal{G}\right) = E\left(hf\middle|\,\mathcal{G}\right) + E\left(h^{0.5}g\xi\middle|\,\mathcal{G}\right) = hf + h^{0.5}gE(\xi) = hf \tag{101}$$

and further

$$\begin{aligned} E\left(x^2\middle|\,\mathcal{G}\right) &= E\left((hf)^2\middle|\,\mathcal{G}\right) + E\left(2hfh^{0.5}g\xi\middle|\,\mathcal{G}\right) + E\left(hg^2\xi^2\middle|\,\mathcal{G}\right) \\ &= (hf)^2 + hg^2\cdot(1-\mu) \\ &\leq hfM_5h^{\varepsilon} + hg^2\cdot(1-\mu) \end{aligned} \tag{102}$$

due to $E(\xi^2) = 1-\mu$ and (18). Using Lemma 1-(ii) gives

$$\left|E\left(6^{-1}\varphi'''(\theta)x^3\,\middle|\,\mathcal{G}\right)\right| \leq 6^{-1}M_3E\left(\left|x^3\right|\middle|\,\mathcal{G}\right) \tag{103}$$

and expanding $x^3 = (hf + h^{0.5}g\xi)^3$ yields

$$\begin{aligned} E\left(\left|x^3\right|\middle|\,\mathcal{G}\right) &\leq hf\left\{(hf)^2 + 3hg^2\cdot(1-\mu)\right\} + hg^2M_1h^{0.5}g \\ &\leq hf\left(M_5h^{\varepsilon}\right)^2\left\{1+3(1-\mu)\right\} + hg^2M_1M_4h^{\varepsilon} \end{aligned} \tag{104}$$

because of (18) and (16). Inserting (101)–(104) into (100), we have

$$E\left(\varphi(1+x)\middle|\,\mathcal{G}\right) \tag{105}$$

$$= \varphi(1) + \varphi'(1)hf + 2^{-1}\varphi''(1)hg^2\cdot(1-\mu) + hfO_1\left(h^{\varepsilon}\right) + hg^2O_2\left(h^{\varepsilon}\right), \tag{106}$$

in which the two big O notations denote

$$\begin{aligned} O_1\left(h^{\varepsilon}\right) &= 2^{-1}\varphi''(1)M_5h^{\varepsilon} + 6^{-1}M_3\left(M_5h^{\varepsilon}\right)^2\left\{1+3(1-\mu)\right\}, \\ O_2\left(h^{\varepsilon}\right) &= M_1M_5h^{\varepsilon}. \end{aligned}$$

Now it remains to show

$$E\left(\phi\left(1+hf+h^{0.5}g\xi\right) - \varphi\left(1+hf+h^{0.5}g\xi\right)\ \middle|\,\mathcal{G}\right) = hg^2O\left(h^{\varepsilon}\right).$$

Let $c_1 = 1 + hf$ and $c_2 = h^{0.5}g$. Then the disintegration formula for conditional expectations with respect to $\mathcal{G}$ gives

$$\begin{aligned} &E\left(\phi\left(1+hf+\sqrt{h}g\xi\right) - \varphi\left(1+hf+\sqrt{h}g\xi\right)\middle|\,\mathcal{G}\right) \\ &\qquad = \int_{\mathbb{R}}\left\{\phi\left(c_1+c_2x\right) - \varphi\left(c_1+c_2x\right)\right\}p(x)\ dx \end{aligned} \tag{107}$$

due to Lemma 1-(iii) and the fact that f, g are $\mathcal{G}$-measurable, ξ is $\mathcal{G}$ -independent, ϕ is almost everywhere continuous and φ is also continuous (see Theorem 5.4 in [44] for the disintegration formula). Let $U_\delta = [1-\delta, 1+\delta]$ and $s = c_1 + c_2x$. Then (107) becomes

$$\int_{\mathbb{R}-U_\delta}\left\{\phi\left(s\right) - \varphi\left(s\right)\right\}p\left(\frac{s-c_1}{c_2}\right)\frac{ds}{|c_2|} \tag{108}$$

because of Lemma 1-(i). Here p is the probabilty density function of ξ.

Lemma 1-(iii) gives that

$$\begin{aligned}
&\left|\int_{\mathbb{R}-U_\delta} \{\phi(s)-\varphi(s)\}\, p\left(\frac{s-c_1}{c_2}\right) \frac{ds}{|c_2|}\right| \\
&\leq \left\{\int_{\mathbb{R}-U_\delta} |\phi(s)-\varphi(s)| \frac{ds}{|c_2|}\right\} \sup_{s\notin U_\delta}\left\{p\left(\frac{s-c_1}{c_2}\right) \frac{1}{|c_2|}\right\} \\
&\leq M_4|c_2|^2 \sup_{s\notin U_\delta}\left\{p\left(\frac{s-c_1}{c_2}\right) \frac{1}{|c_2|^3}\right\} \\
&= M_4 h g^2 \sup_{s\notin U_\delta}\left\{p\left(\frac{s-1-hf}{h^{0.5}g}\right) \frac{1}{|h^{0.5}g|^3}\right\}.
\end{aligned}$$

Since there exists some δ_0 such that for $s \notin U_\delta$ and all sufficiently small $h > 0$

$$|s-1-hf| > |s-1| - h|f| > \delta - M_5 h^\epsilon > \delta_0 > 0, \tag{109}$$

letting $y = (s-1-hf)/(h^{0.5}g)$ yields

$$|y| = \frac{|s-1-hf|}{h^{0.5}\,|g|} > \frac{\delta_0}{M_5 h^\varepsilon} \tag{110}$$

and further

$$\sup_{s\notin U_\delta}\left\{p\left(\frac{s-1-hf}{h^{0.5}g}\right) \frac{1}{|h^{0.5}g|^3}\right\} = \sup_{s\notin U_\delta} \frac{p(y)\,|y|^3}{|s-1-hf|^3}.$$

Hence it follows from (17), (109) and (110) that

$$\sup_{s\notin U_\delta} \frac{p(y)\,|y|^3}{|s-1-hf|^3} < M_2 \sup_{s\notin U_\delta} \frac{|y|^{-1}}{|s-1-hf|^3} < M_2 \frac{M_5}{\delta_0^2} h^\varepsilon,$$

which gives

$$\left|\int_{\mathbb{R}-U_\delta} \{\phi(s)-\varphi(s)\}\, p\left(\frac{s-c_1}{c_2}\right) \frac{ds}{|c_2|}\right| < hg^2 \cdot M_4 M_2 \frac{M_5}{\delta_0^2} h^\varepsilon. \tag{111}$$

Therefore using (105), (108) and (111), we obtain the desired result.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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WEIGHTED SUPERPOSITION OPERATORS FROM ZYGMUND SPACES TO μ-BLOCH SPACES

ZHI JIE JIANG, TING WANG, JUAN LIU, TING LUO, TING SONG

ABSTRACT. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Let φ be an entire function on $\mathbb{C}$ and $u \in H(\mathbb{D})$. The boundedness and compactness of the operators $S_{u,\varphi} : f \mapsto u \cdot \varphi \circ f$ from Zygmund spaces to μ-Bloch spaces are characterized.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$ and $H^\infty(\mathbb{D})$ the space of bounded analytic functions. Let φ be a complex-valued function on $\mathbb{C}$ and $u \in H(\mathbb{D})$. We introduce a class of nonlinear operators by

$$S_{u,\varphi} f = u \cdot \varphi \circ f, \ \ f \in H(\mathbb{D}).$$

This operator can be regarded as a generalization of the superposition operator $S_\varphi f = \varphi \circ f$ and the multiplication operator $M_u f = u \cdot f$.

Suppose that X and Y are two metric spaces of analytic functions on $\mathbb{D}$. Note that if X contains the linear functions and S_φ maps X into Y, then φ must be an entire function. In recent years, the following natural questions of the superposition operators are considered.

(a) When does φ induce a superposition operator from X into Y?

(b) When is a superposition operator from X into Y bounded?

(c) When is a superposition operator from X into Y compact?

Although analogous concepts also make sense in the context of real-valued functions and their theory has a long history (see [2]), the study of such natural questions on analytic function spaces has only begun fairly recently. The operators S_φ that map Bergman spaces into area Nevanlinna classes were characterized in [6], which have been extended by other authors to some other analytic function spaces, where it is remarkable the works of Vukotić et. al. in [1], [4] and [5]. It must be mentioned that the authors of [4] gave a very interesting geometric construction of simple connected domain in several analytic function spaces. This technique has been used by many authors; in particular, Xu used it to study the superposition operators from α-Bloch spaces into β-Bloch spaces in [20] and Xiong used it to characterize the superposition operators from Q_p spaces into α-Bloch spaces with $0 < \alpha < 1$ in [18]. It should be noted that quite recently, Castillo et.al. and Ramos Fernández have studied the superposition operators from Bloch-Orlicz spaces into α-Bloch spaces and between weighted Banach spaces of analytic functions in [7] and [14], respectively. In this paper we characterize the boundedness and compactness of the operators $S_{u,\varphi}$ from weighted Zygmund spaces to μ-Bloch spaces. We also consider the superposition operators from weighted Zygmund spaces to weighted Bloch spaces.

Now we present the needed spaces and some facts. The Zygmund space $\mathcal{Z}$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)|f''(z)| < \infty.$$

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1

With the norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|,$$

it is a Banach space. By Zygmund's theorem (see Theorem 5.3 in [9]), we know that $f \in \mathcal{Z}$ if and only if f is continuous on $\overline{D}$ and

$$\sup_{h>0,\theta\in\mathbb{R}} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$

In closed subspaces of $\mathcal{Z}$, the little Zygmund space $\mathcal{Z}_0$ is usually considered, which is defined by

$$\mathcal{Z}_0 = \big\{f \in \mathcal{Z} : \lim_{|z|\to 1}(1-|z|^2)|f''(z)| = 0\big\}.$$

Let $\alpha \in (0,\infty)$. The weighted Zygmund space $\mathcal{Z}_\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f''(z)| < +\infty.$$

With the norm

$$\|f\|_{\mathcal{Z}_\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f''(z)|,$$

$\mathcal{Z}_\alpha$ is also a Banach space. For the weighted Zygmund spaces and the operators from them into some other spaces, see, e.g., [10], [12] and [15].

Suppose that μ is a positive continuous radial function on $\mathbb{D}$ (that is, $\mu(z) = \mu(|z|)$) and decreasing on $[0,1)$ with $\lim_{r\to 1}\mu(r) = 0$. Let μ be a weight. The μ-Bloch space $\mathcal{B}_\mu$ consists of all $f \in H(\mathbb{D})$ such that $\sup_{z\in\mathbb{D}}\mu(z)|f'(z)| < \infty$. With

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{D}}\mu(z)|f'(z)|,$$

$\mathcal{B}_\mu$ is a Banach space. When $\mu(z) = 1-|z|^2$, the space $\mathcal{B}_\mu$ is just Bloch space and denoted by $\mathcal{B}$; while when $\mu(z) = (1-|z|^2)^\alpha$ with $\alpha > 0$, the space $\mathcal{B}_\mu$ becomes the weighted Bloch space $\mathcal{B}_\alpha$. The μ-Bloch spaces appear in the literature in a natural way when one considers properties of some operators in certain spaces of analytic functions; for example, if $\mu(z) = (1-|z|)\log\frac{2}{1-|z|}$, Attele in [3] proved that the Hankel operator on Bergman spaces induced by a function f is bounded if and only if $f \in \mathcal{B}_\mu$. The logarithmic Bloch type space has been defined and studied in [16]. Recently, the Bloch-Orlicz spaces have been introduced by Ramos-Fernandez in [13].

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \simeq b$ means that there is a positive constant C such that $a/C \le b \le Ca$.

2. The operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$

First we enumerate several useful lemmas. The first one below is well-known.

Lemma 2.1 *There is a positive constant C_α depending only on α such that for any $z \in \mathbb{D}$ and $f \in \mathcal{Z}_\alpha$*

(i)

$$\big|f(z)\big| \le \begin{cases} C_\alpha\|f\|_{\mathcal{Z}_\alpha}, & 0<\alpha<2,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}\log\frac{2}{1-|z|^2}, & \alpha = 2,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}(1-|z|^2)^{2-\alpha}, & \alpha>2.\end{cases}$$

(ii)

$$\big|f'(z)\big| \le \begin{cases} C_\alpha\|f\|_{\mathcal{Z}_\alpha}, & 0<\alpha<1,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}\log\frac{2}{1-|z|^2}, & \alpha = 1,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}(1-|z|^2)^{1-\alpha}, & \alpha>1.\end{cases}$$

Let $a \in \mathbb{D}$ and $1/\sqrt{2} < |a| < 1$, define

$$f(z) = (z-1)\Big(\Big(1+\log\frac{1}{1-z}\Big)^2+1\Big)$$

and

$$g_a(z) = \frac{f(\bar{a}z)}{\bar{a}}\Big(\log\frac{1}{1-|a|^2}\Big)^{-1}.$$

The function g_a is called the test function with the following property (see [11]).

Lemma 2.2 *The function g_a belongs to $\mathcal{Z}$ and $\|g_a\|_{\mathcal{Z}} \simeq 1$.*

The following result can be found in [17].

Lemma 2.3 *Let $\alpha \in (0,1]$. Then for every bounded sequence $\{f_n\}$ in $\mathcal{Z}_\alpha$ and $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$, we have*

(i) if $\alpha = 1$, then $\lim\limits_{n\to\infty}\sup\limits_{z\in\mathbb{D}}\big|f_n(z)\big| = 0$.

(ii) if $0 < \alpha < 1$, then $\lim\limits_{n\to\infty}\sup\limits_{z\in\mathbb{D}}\big|f_n'(z)\big| = 0$.

The next result is often used in dealing with the compactness of operators on analytic function spaces. Since the proof is standard (see Proposition 3.11 in [8]), it is omitted .

Lemma 2.4 *Let $u \in H(\mathbb{D})$ and φ an entire function. Then the bounded operator $S_{u,\varphi} : \mathcal{Z}_\alpha \to \mathcal{B}_\mu$ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathcal{Z}_\alpha$ such that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$, it follows that $\lim_{n\to\infty}\|S_{u,\varphi}f_n\|_{\mathcal{B}_\mu} = 0$.*

Now we characterize the boundedness of the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$.

Theorem 2.1 *Let $u \in H(\mathbb{D})$ and φ an entire function with $\varphi'(0) \neq 0$. Then the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded if and only if $u \in \mathcal{B}_\mu$ and*

$$L := \sup_{z\in\mathbb{D}}\mu(z)|u(z)|\log\frac{2}{1-|z|^2} < \infty.$$

Proof. Suppose that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded. By taking f_1 the constant function, we obtain $u \in \mathcal{B}_\mu$. Since operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded, for the function $f_2 = g_a$ there exists a positive constant C such that

$$\begin{aligned}\infty > C\|S_{u,\varphi}\| &\geq \|S_{u,\varphi}f_2\|_{\mathcal{B}_\mu} \geq \mu(a)\big|(S_{u,\varphi}f_2)'(a)\big|\\ &= \mu(a)\big|u'(a)\varphi(f_2(a)) + u(a)\varphi'(f_2(a))f_2'(a)\big|\\ &\geq \mu(a)\big(\big|u(a)\big|\big|\varphi'(f_2(a))\big|\big|f_2'(a)\big| - \big|u'(a)\big|\big|\varphi(f_2(a))\big|\big).\end{aligned}$$

From this, we get

$$\mu(a)\big|u'(a)\big|\big|\varphi(f_2(a))\big| + C\|S_{u,\varphi}\| \geq \mu(a)\big|u(a)\big|\big|\varphi'(f_2(a))\big|\big|f_2'(a)\big|.$$

Set $M = C_\alpha\|f_2\|_{\mathcal{Z}}$ and $M_1 = \max\limits_{|z|=M}|\varphi(z)|$. By Lemma 2.1 (i), we have

$$\begin{aligned}M_1\|u\|_{\mathcal{B}_\mu} + C\|S_{u,\varphi}\| &\geq \mu(a)\big|u'(a)\big|\big|\varphi(f_2(a))\big| + C\|S_{u,\varphi}\|\\ &\geq \mu(a)\big|u(a)\big|\big|\varphi'(f_2(a))\big|\big|f_2'(a)\big|\\ &= \mu(a)\big|u(a)\big|\big|\varphi'(g_a(a))\big|\log\frac{1}{1-|a|^2}\\ &\geq \frac{1}{2}\mu(a)\big|u(a)\big|\big|\varphi'(g_a(a))\big|\log\frac{2}{1-|a|^2},\end{aligned}$$

where we have used that when $|a| > 1/\sqrt{2}$,

$$\log\frac{1}{1-|a|^2} \geq \frac{1}{2}\log\frac{2}{1-|a|^2}.$$

It is easy to see that $g_a(a) \to 0$ as $|a| \to 1$. Therefore from this and the fact that

$$\lim_{|a|\to 1} \big|\varphi'(g_a(a))\big| = |\varphi'(0)| \neq 0,$$

we obtain

$$\sup_{1/2<|z|<1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} < \infty.$$

It is clear that

$$\sup_{|z|\leq 1/2} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} < \infty.$$

Consequently, we obtain $L < \infty$.

Now let $u \in \mathcal{B}_\mu$ and $L < \infty$. Let $f \in \mathcal{Z}$ and $\|f\|_{\mathcal{Z}} \leq M$. Set $M_1 = \max_{|z|=C_\alpha M} |\varphi(z)|$ and $M_2 = \max_{|z|=C_\alpha M} |\varphi'(z)|$. Then by Lemma 2.1, we have

$$\begin{aligned}
\|S_{u,\varphi} f\|_{\mathcal{B}_\mu} &= \big|u(0)\varphi(f(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|(S_{u,\varphi} f)'(z)\big| \\
&= \big|u(0)\varphi(f(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u'(z)\varphi(f(z)) + u(z)\varphi'(f(z))f'(z)\big| \\
&\leq C_\alpha M\|u\|_{\mathcal{B}_\mu} + \sup_{z\in\mathbb{D}} \mu(z)\big|u'(z)\big|\big|\varphi(f(z))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u(z)\big|\big|\varphi'(f(z))\big|\big|f'(z)\big| \\
&\leq C_\alpha M\|u\|_{\mathcal{B}_\mu} + M_1\|u\|_{\mathcal{B}_\mu} + C_\alpha M M_2 \sup_{z\in\mathbb{D}} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \\
&\leq (C_\alpha M + M_1)\|u\|_{\mathcal{B}_\mu} + C_\alpha L M M_2 \\
&< \infty.
\end{aligned}$$

This shows that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded. □

There are a lot of examples satisfying the conditions of Theorem 2.1. Here we take the following two examples. Since the first is clear, its proof is omitted.

Example 2.1 *Let* $u(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *and* $\varphi(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m$, *where* $b_1 \neq 0$. *Then the operator* $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ *is bounded.*

Example 2.2 *Let* $u(z) = \lambda \frac{a-z}{1-\bar{a}z}$ *be the automorphism of* $\mathbb{D}$ *and* $\varphi(z) = e^z$. *Then* $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ *is bounded.*

Proof. Since $\|u\|_\infty \leq 1$ and it is easy to see that

$$|u'(z)| = \frac{1-|a|^2}{|1-\bar{a}z|^2} \leq \frac{2}{1-|a|},$$

we get $u \in \mathcal{B}_\mu$ and $L < \infty$. By Theorem 2.1, the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded. □

From the proof of Theorem 2.1, we can obtain the following sufficient condition of boundedness for the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$.

Theorem 2.2 *Let* $u \in H(\mathbb{D})$ *and* φ *an entire function. If* $u \in \mathcal{B}_\mu$ *and* $L < \infty$, *then* $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ *is bounded.*

We begin to study when the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is compact.

Theorem 2.3 *Let* $u \in H(\mathbb{D})$ *and* φ *an entire function with* $\varphi(0) = 0$ *and* $\varphi'(0) \neq 0$. *Then the operator* $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ *is compact if and only if* $u \in \mathcal{B}_\mu$ *and*

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} = 0.$$

Proof. Suppose that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is compact. Of course, it is bounded, and then $u \in \mathcal{B}_\mu$. Now let us suppose, by the way of contradiction, that

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \neq 0.$$

Then there exists some $\varepsilon_0 > 0$ and a sequence $\{z_n\} \subseteq \mathbb{D}$ such that $|z_n| \to 1$ as $n \to \infty$ and

$$\mu(z_n)\big|u(z_n)\big| \log \frac{2}{1-|z_n|^2} \geq \varepsilon_0.$$

For each $n \in \mathbb{N}$, take the function $f_n = g_{z_n}$. From Lemma 2.2 it follows that $\|f_n\|_{\mathcal{Z}} \leq C$. One can easily check that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$. Thus it follows from Lemma 2.4 that

$$\begin{aligned}
\|S_{u,\varphi} f_n\|_{\mathcal{B}_\mu} &\geq \mu(z_n)\big|(S_{u,\varphi} f_n)'(z_n)\big| \\
&= \mu(z_n)\big|u'(z_n)\varphi(f_n(z_n)) + u(z_n)\varphi'(f_n(z_n))f_n'(z_n)\big| \\
&\geq \mu(z_n)\big(\big|u(z_n)\varphi'(f_n(z_n))f_n'(z_n)\big| - \big|u'(z_n)\varphi(f_n(z_n))\big|\big) \\
&= \mu(z_n)\big|u(z_n)\big|\big|\varphi'(f_n(z_n))\big|\big|f_n'(z_n)\big| - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f_n(z_n))\big| \\
&\geq \mu(z_n)\big|u(z_n)\big|\big|\varphi'(f_n(z_n))\big| \log \frac{1}{1-|z_n|^2} - \|u\|_{\mathcal{B}_\mu}\big|\varphi(f_n(z_n))\big| \\
&\geq \frac{1}{2}\mu(z_n)\big|u(z_n)\big|\big|\varphi'(f_n(z_n))\big| \log \frac{2}{1-|z_n|^2} - \|u\|_{\mathcal{B}_\mu}\big|\varphi(f_n(z_n))\big| \\
&\geq \frac{1}{2}\big|\varphi'(f_n(z_n))\big|\varepsilon_0 - \|u\|_{\mathcal{B}_\mu}\big|\varphi(f_n(z_n))\big|.
\end{aligned}$$

From this and since Lemma 2.3 (i) implies that $|\varphi(f_n(z_n))| = 0$ as $n \to \infty$, we get

$$0 = \lim_{n\to\infty} \|S_{u,\varphi} f_n\|_{\mathcal{B}_\mu} \geq \frac{1}{2}\big|\varphi'(0)\big|\varepsilon_0,$$

which arrives at a contradiction.

Conversely, by the definition of limit we have that for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} < \varepsilon$$

for all $z \in \{z \in \mathbb{D} : \delta < |z| < 1\}$. Let $M_0 > 0$ and $\|f_n\|_{\mathcal{Z}} \leq M_0$ and $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$. By the Cauchy integral formula and an easy calculation, it is clear that $\{f_n'\}$ also uniformly converges to zero on every compact subset of $\mathbb{D}$ as $n \to \infty$. Let $M = \max_{|z|=C_\alpha M_0} |\varphi'(z)|$. By Lemma 2.1 and Lemma 2.3 (i), we have

$$\begin{aligned}
\|S_{u,\varphi} f_n\|_{\mathcal{B}_\mu} &= \big|u(0)\varphi(f_n(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|(S_{u,\varphi} f_n)'(z)\big| \\
&= \big|u(0)\varphi(f_n(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u'(z)\varphi(f_n(z)) + u(z)\varphi'(f_n(z))f_n'(z)\big| \\
&\leq \big|u(0)\varphi(f_n(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u'(z)\big|\big|\varphi(f_n(z))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u(z)\big|\big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\
&\leq \big|u(0)\varphi(f_n(0))\big| + \|u\|_{\mathcal{B}_\mu} \sup_{z\in\mathbb{D}} \big|\varphi(f_n(z))\big| + \sup_{|z|\leq\delta} \mu(z)\big|u(z)\big|\big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\
&\quad + \sup_{\delta<|z|<1} \mu(z)\big|u(z)\big|\big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\
&\leq \big|u(0)\varphi(f_n(0))\big| + \|u\|_{\mathcal{B}_\mu} \sup_{z\in\mathbb{D}} \big|\varphi(f_n(z))\big| + M \max_{|z|\leq\delta} \mu(z)\big|u(z)\big| \max_{|z|\leq\delta} \big|f_n'(z)\big| \\
&\quad + C_\alpha M_0 M \sup_{\delta<|z|<1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2}.
\end{aligned}$$

Taking the limit as $n \to \infty$ in this inequality, we obtain $\lim_{n\to\infty} \|S_{u,\varphi} f_n\|_{\mathcal{B}_\mu} = 0$. By Lemma 2.4, the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is compact. $\square$

Remark 2.1 Considering Theorem 2.3, we have a reason to regard as the limit

$$\lim_{|z|\to 1^-} \mu(z) \log \frac{2}{1-|z|^2}$$

as an important factor for the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ to be compact.

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Theorem 2.4 *Let $u \in H(\mathbb{D})$ and φ an entire function with $\varphi(0) = 0$ and $\varphi'(0) \neq 0$. Then $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is bounded if and only if $u \in \mathcal{B}_{\mu,0}$ and*

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} = 0.$$

Proof. Suppose that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is bounded, then by taking f the constant function we have $u \in \mathcal{B}_{\mu,0}$. Now let us suppose, by the way of contradiction, that

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \neq 0.$$

Then there exist some $\varepsilon_0 > 0$ and a sequence $\{z_n\} \subseteq \mathbb{D}$ with $|z_n| \to 1$ such that

$$\mu(z_n)\big|u(z_n)\big| \log \frac{2}{1-|z_n|^2} \geq \frac{2}{|\varphi'(0)|}\varepsilon_0.$$

Take the function $f = g_{z_n}$. Since $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is bounded, $S_{u,\varphi} f \in \mathcal{B}_{\mu,0}$, that is,

$$\lim_{|z|\to 1} \mu(z)\big|(S_{u,\varphi} f)'(z)\big| = 0;$$

in particular,

$$\lim_{n\to\infty} \mu(z_n)\big|(S_{u,\varphi} f)'(z_n)\big| = 0.$$

Letting $n \to \infty$ in

$$\begin{aligned}
\mu(z_n)\big|(S_{u,\varphi} f)'(z_n)\big| &= \mu(z_n)\big|u'(z_n)\varphi(f(z_n)) + u(z_n)\varphi'(f(z_n))f'(z_n)\big| \\
&\geq \mu(z_n)\big|u(z_n)\big|\big|\varphi'(f(z_n))\big|\big|f'(z_n)\big| - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f(z_n))\big| \\
&\geq \frac{1}{2}\mu(z_n)\big|u(z_n)\big| \log \frac{2}{1-|z_n|^2}\big|\varphi'(f(z_n)\big| - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f(z_n))\big| \\
&\geq \frac{\big|\varphi'(f(z_n))\big|}{|\varphi'(0)|}\varepsilon_0 - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f(z_n))\big|
\end{aligned}$$

arrives at a contradiction.

Conversely, by Theorem 2.1, we know that $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded. It is enough to prove that for any $f \in \mathcal{Z}$, it holds $S_{u,\varphi} f \in \mathcal{B}_{\mu,0}$. Let $f \in \mathcal{Z}$, $M_1 = \max_{|z|=\|f\|_{\mathcal{Z}}} |\varphi(z)|$ and $M_2 = \max_{|z|=\|f\|_{\mathcal{Z}}} |\varphi'(z)|$. Then for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mu(z)|u'(z)| < \frac{\varepsilon}{2M_1}$$

and

$$\mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} < \frac{\varepsilon}{2M_2\|f\|_{\mathcal{Z}}}$$

for all $z \in \{z \in \mathbb{D} : \delta < |z| < 1\}$. So for $z \in \{z \in \mathbb{D} : \delta < |z| < 1\}$, it follows that

$$\begin{aligned}
\mu(z)\big|(S_{u,\varphi} f)'(z)\big| = \mu(z)\big|(S_{u,\varphi} f)'(z)\big| &= \mu(z)\big|u'(z)\varphi(f(z)) + u(z)\varphi'(f(z))f'(z)\big| \\
&\leq M_1\mu(z)\big|u'(z)\big| + M_2\|f\|_{\mathcal{Z}}\mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \\
&< \varepsilon.
\end{aligned}$$

This shows that $S_{u,\varphi} f \in \mathcal{B}_{\mu,0}$. $\square$

Theorem 2.5 *Let $u \in H(\mathbb{D})$ and φ an entire function with $\varphi(0) = 0$ and $\varphi'(0) \neq 0$. Then the bounded operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is compact if and only if $u \in \mathcal{B}_{\mu,0}$ and*

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} = 0.$$

Proof. Similarly as in the proof of Theorem 2.3, this result is true. $\square$

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3. The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$

Although we can obtain some results of the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ from the preceding discussions, we still will individually consider this operator.

Theorem 3.1 *Let $\alpha \in (0,1)$ and φ an entire function. Then the following assertions hold:*

(i) The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded .

(ii) If $\varphi(0) = 0$, then the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.

Proof. We first prove (i). Let $M > 0$, $f \in \mathcal{Z}_\alpha$ and $\|f\|_{\mathcal{Z}_\alpha} \le M$. Set $M_1 = \max_{|z|=C_\alpha M} |\varphi'(z)|$. Then we have

$$(1-|z|^2)^\beta \big|(S_\varphi f)'(z)\big| = (1-|z|^2)^\beta \big|\varphi'(f(z))\big|\big|f'(z)\big| \le C_\alpha M M_1 (1-|z|^2)^\beta < \infty.$$

This means that the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded.

Now we prove (ii). Suppose that $\|f_n\|_{\mathcal{Z}_\alpha} \le M$ and $\{f_n\}$ uniformly converges to zero on every compact subset of $\mathbb{D}$ as $n \to \infty$, then

$$\begin{aligned}\|S_\varphi f_n\|_{\mathcal{B}_\beta} &= |\varphi(f_n(0))| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \big|(S_\varphi f_n)'(z)\big| \\ &= |\varphi(f_n(0))| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\ &\le |\varphi(f_n(0))| + M_1 \sup_{z\in\mathbb{D}} |f_n'(z)|,\end{aligned}$$

where $M_1 = \max_{|z|=C_\alpha M} |\varphi'(z)|$. By $\varphi(0) = 0$ and Lemma 2.3 (ii), we know that $\lim_{n\to\infty} \|S_\varphi f_n\|_{\mathcal{B}_\beta} = 0$. By Lemma 2.4, the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact. $\square$

When $\alpha = 1$, from Theorem 2.1 and Theorem 2.2 we can obtain characterizations of the boundedness and compactness of the operator $S_\varphi : \mathcal{Z} \to \mathcal{B}_\beta$. It is unnecessary to go into details here.

Theorem 3.2 *Let $\alpha \in (1,2)$ and φ an entire function. We have the following assertions:*

(1) If $\alpha \le 1+\beta$, then (i) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded, and
(ii) when $\varphi(0) = 0$, the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.
(2) If $\alpha > 1+\beta$, then the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if φ is a constant function.

Proof. We first prove the assertion (i) of (1). Let $M > 0$, $f \in \mathcal{Z}_\alpha$ and $\|f\|_{\mathcal{Z}_\alpha} \le M$. Set $M_1 = \max_{|z|=C_\alpha M} |\varphi'(z)|$. Then we have

$$(1-|z|^2)^\beta \big|(S_\varphi f)'(z)\big| = (1-|z|^2)^\beta \big|\varphi'(f(z))\big|\big|f'(z)\big| \le C M M_1 (1-|z|^2)^{1-\alpha+\beta} < \infty.$$

This shows that the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded. As the proof of Theorem 3.1 (ii), the assertion (ii) follows.

Note that we have the relation $\mathcal{Z}_\alpha = \mathcal{B}_{\alpha-1}$. By this and Theorem 4 in [5], the assertion (2) is true. $\square$

Theorem 3.3 *Let $\alpha = 2$ and φ an entire function.*

(1) When $\beta > 1$, (i) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if φ is a polynomial of degree $s \le 1$, and
(ii) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.
(2) When $\beta = 1$, (i) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if φ is a linear function, and
(ii) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.
(3) When $0 < \beta < 1$, the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only φ is a constant function.

Proof. By Theorem 7 of [5], the assertions (i) of (1) and (i) of (2) hold. Also from Theorem 4 of [5], the assertion (3) follows. Now we want to prove the assertion (ii) of (1). Let the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ be compact. From the assertion (i) of (1), we know that, if φ is not a constant function, then $\varphi(z) = az + b$ with $a \neq 0$. Therefore, it is enough to show that $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact when $\varphi(z) = az$. At this time, S_φ is just the multiplication operator M_a defined by $M_a f = a \cdot f$. Thus, by Theorem 3.1 of [19], we know that $M_a : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact. Similar to the proof of the assertion (ii) of (1), the assertion (ii) of (2) is right. □

Theorem 3.4 *Let $\alpha > 2$, $\beta > 1$ and φ an entire function.*

(1) The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if

(i) *when $\alpha > \beta$, φ is a constant.*

(ii) *when $\alpha = \beta$, φ is a linear function.*

(iii) *when $\alpha < \beta$, φ is a polynomial of degree $s \leq \frac{\beta-1}{\alpha-2}$.*

(2) The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact if and only if φ is a polynomial of degree $s < \frac{\beta-1}{\alpha-2}$.

Proof. Note that when $\alpha > 2$, it follows that $\mathcal{Z}_\alpha = \mathcal{B}_{\alpha-1} = H_{\alpha-2}$, where $H_{\alpha-2}$ is called the weighted Banach space of analytic functions defined by

$$H_{\alpha-2} = \{f \in H(\mathbb{D}) : (1 - |z|^2)^{\alpha-2}|f(z)| < \infty\}.$$

Then (1) and (2) follow from Theorem 4.2 of [14] and Proposition 3.1 of [4]. □

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Dynamical Analysis Of The Rational Difference Equation $x_{n+1} = \frac{\alpha x_{n-3}}{A+Bx_{n-1}x_{n-3}}$

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ABSTRACT

This article is concerned with the following rational difference equation $x_{n+1} = \frac{\alpha x_{n-3}}{A+Bx_{n-1}x_{n-3}}$ with the initial conditions, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$ are arbitrary real numbers, α, A and B are arbitrary constants. A detailed analytical study of the convergence of the solutions including their dependence on parameters and initial conditions is investigated. The local stability and global attractivity of the difference equation's equilibrium points are discussed. The existence of periodic solutions in the proposed difference equation is also verified analytically. Moreover, numerical simulations are carried out to verify the correctness of the analytical results.

Keywords: Difference equations, Recursive sequences, Analytical study, Infinite products, Convergence, Periodic solution.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Difference equations arise from the study of the evolution of natural phenomena. The applications of difference equations are rapidly increasing to various fields such as economics [1], [12]-[14], mathematical, biology [15]-[16] physics and engineering [7]. Indeed, difference equations represent chief tools of investigating the qualitative behaviors of dynamical systems [33]. Consequently, studying the solutions of difference equations and its qualitative behaviors have become focal topics for research [1]-[36].

In recent years, difference equations have been investigated by many authors. For some results: In [3], Aloqeili found the solution of the difference equation $x_{n+1} = \frac{dx_{n-1}x_{n-k}}{b-cx_{n-s}}$. Cinar [5] obtained the solution of the difference equation $x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}$. In [9], Elabbasy *et al.* discussed the solution and the periodicity character of the difference equations $x_{n+1} = ax_n - \frac{bx_n}{cx_n-dx_{n-1}}$.
In this paper, we study to the following sequence defined recursively by

$$x_{n+1} = \frac{\alpha x_{n-3}}{A + Bx_{n-1}x_{n-3}}, \tag{1}$$

with the initial data: $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$.
Note first that, if $\alpha = 0$, then for all $n \in \mathbb{N}$, $x_n = 0$. Then we will consider that $\alpha \neq 0$. Although we can (by dividing the numerator and denominator by α) obtain a more simply form of such sequences, we will keep them in order to study of the behaviors with respect to α.
Note also that, if one or more of the initial data a, b, c and d is zero, then it will be seen that one or more of the subsequences of $(x_n)_n$ modulo 4 vanish, so that we will suppose that $abcd \neq 0$.
The cases $A = 0$ and $B = 0$ are a trivial, therefore we will assume that $A \neq 0$ and $B \neq 0$. Finally, we will consider the convention: if $(a_p)_p$ is a sequence of complex numbers, and $n > m$, in $\mathbb{Z}$, then $\prod_{p=n}^{m} a_p = 1$.

2. DEFINITIONS AND PRELIMINARIES.

A difference equation of order k is an equation of the form

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-(k-1)}), n = 0, 1, ..., \tag{2}$$

where F is a function that maps on some set I^k into I. A solution of Eq. (2) is a sequence x_n that satisfies Eq. (2) for all $n \geq 0$. With each solution x_n of the Eq. (1), we associate the vector of initial conditions $v_0(x) = (x_0, x_{-1}, ..., x_{-k+1}) \in I^k$.
The norm of the vector $u \in I^k$ will be defined as $\|u\| = \sum_{i=-k+1}^{0} |u_i|$.

Definition 1. (Equilibrium point)
A point $\bar{x} \in \mathbb{R}$ is called an equilibrium point of Eq. (2), if

$$\bar{x} = F(\bar{x}, \bar{x}, ..., \bar{x}).$$

Let $\bar{x} \in \mathbb{R}$ be an equilibrium point of Eq. (2), and denote by $v(\bar{x}) \in I^k$ the vector $v(\bar{x}) = (\bar{x}, \bar{x}, ..., \bar{x})$.
Suppose that the function F is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$.
Consider the linearized equation of Eq. (2) about the equilibrium point $\bar{x}$:

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + ... + q_{k-1} y_{n-(k-1)}, \tag{3}$$

where $q_i = \frac{\partial F}{\partial x_i}(\bar{x}, \bar{x}, ..., \bar{x})$, $i = 0, 1, ..., k-1$, and the characteristic equation of Eq. (3) about $\bar{x}$:

$$\lambda^k - q_0 \lambda^{k-1} - ... - q_{k-2}\lambda - q_{k-1} = 0. \tag{4}$$

Definition 2.

1. When all the roots of Eq. (4) have absolute value less than one, then the equilibrium point of Eq. (2) is locally asymptotically stable.

2. If at least a root of Eq. (4) have absolute value greater than one, then the equilibrium point of Eq. (2) is unstable.

Definition 3.

1. An equilibrium point $\bar{x}$ of Eq. (2) is called hyperbolic if no root of Eq. (4) has absolute value equal one.

2. If there exists a root of Eq. (4) with absolute value equal to one, then the equilibrium point $\bar{x}$ is called nonhyperbolic.

3. An equilibrium point $\bar{x}$ of Eq. (2) is called saddle if there exists a root of Eq. (4) has absolute value less than one. and another root of Eq. (4) greater than one.

4. An equilibrium point $\bar{x}$ of Eq. (2) is called a repeller if all roots of Eq. (4) has absolute value greater than one.

5. A solution x_n of Eq. (2) is called nonoscillatory about $\bar{x}$ or simply nonoscillatory if there exists $N \geq -k$ such that either $x_n \geq \bar{x}$, $\forall n \geq N$ or $x_n \leq \bar{x}$, $\forall n \geq N$. Otherwise, the solution x_n is called oscillatory about $\bar{x}$, or simply oscillatory.

6. A solution x_n of Eq. (2) is called periodic with period p if there exists an integer p, such that

$$x_{n+p} = x_n, \qquad \forall n \geq -k. \tag{5}$$

   A solution is called periodic with prime period p if p is the smallest positive integer for which Eq. (5) holds.

3. ANALYTICAL EXPRESSIONS OF $(X_N)_N$

The following Theorem gives an analytical expression of the sequence $(x_n)_n$.

Theorem 1. *Let $(x_n)_n$ be the sequence given by* (1) *and the initial data that follow, then For all $n \geq 2$*

$$x_{4n-3} = \frac{d\alpha^n \prod_{p=0}^{n-2}\left(A^{2p+2} + Bbd\sum_{i=0}^{2p+1} A^i\alpha^{2p+1-i}\right)}{\prod_{p=0}^{n-1}\left(A^{2p+1} + Bbd\sum_{i=0}^{2p} A^i\alpha^{2p-i}\right)}, \quad x_{4n-2} = \frac{c\alpha^n \prod_{p=0}^{n-2}\left(A^{2p+2} + Bac\sum_{i=0}^{2p+1} A^i\alpha^{2p+1-i}\right)}{\prod_{p=0}^{n-1}\left(A^{2p+1} + Bac\sum_{i=0}^{2p} A^i\alpha^{2p-i}\right)}. \tag{6}$$

$$x_{4n-1} = \frac{b\alpha^n \prod_{p=0}^{n-1}\left(A^{2p+1} + Bbd\sum_{i=0}^{2p} A^i\alpha^{2p-i}\right)}{\prod_{p=0}^{n-1}\left(A^{2p+2} + Bbd\sum_{i=0}^{2p+1} A^i\alpha^{2p+1-i}\right)}, \quad x_{4n} = \frac{a\alpha^n \prod_{p=0}^{n-1}\left(A^{2p+1} + Bac\sum_{i=0}^{2p} A^i\alpha^{2p-i}\right)}{\prod_{p=0}^{n-1}\left(A^{2p+2} + Bac\sum_{i=0}^{2p+1} A^i\alpha^{2p+1-i}\right)}. \tag{7}$$

Proof. By induction, we prove the result for x_{4n-3}. Take $n \geq 2$, and assume that the results hold for the step n, then prove the result for the step $n+1$, we get:

$$\begin{aligned}
x_{4(n+1)-3} &= \frac{\alpha x_{4n-3}}{A + Bx_{4n-1}x_{4n-3}} \\
&= \frac{d\alpha^{n+1} \prod_{p=0}^{n-1}\left(A^{2p+2} + Bbd\sum_{i=0}^{2p+1} A^i\alpha^{2p+1-i}\right)}{\prod_{p=0}^{n-1}\left(A^{2p+1} + Bbd\sum_{i=0}^{2p} A^i\alpha^{2p-i}\right)\left[A\left(A^{2n} + Bbd\sum_{i=0}^{2n-1} A^i\alpha^{2n-1-i}\right) + Bbd\alpha^{2n}\right]} \\
&= \frac{d\alpha^{n+1} \prod_{p=0}^{n-1}\left(A^{2p+2} + Bbd\sum_{i=0}^{2p+1} A^i\alpha^{2p+1-i}\right)}{\prod_{p=0}^{n-1}\left(A^{2p+1} + Bbd\sum_{i=0}^{2p} A^i\alpha^{2p-i}\right)\left(A^{2n+1} + Bbd\left(\sum_{i=1}^{2n} A^i\alpha^{2n-i} + \alpha^{2n}\right)\right)}.
\end{aligned}$$

Hence, we obtain

$$x_{4(n+1)-3} = \frac{d\alpha^{n+1} \prod_{p=0}^{n-1}\left(A^{2p+2} + Bbd\sum_{i=0}^{2p+1} A^i\alpha^{2p+1-i}\right)}{\prod_{p=0}^{n}\left(A^{2p+1} + Bbd\sum_{i=0}^{2p} A^i\alpha^{2p-i}\right)}.$$

Similarly, the expression for x_{4n-2}, x_{4n-1}, x_{4n} can be easily proved.

Notation. If we denote by $(P_n)_n$ the sequence of two variables polynomials defined for every $n \in \mathbb{N}$, x and y as,

$$P_n(x, y) = (A - \alpha + Bxy)A^n - Bxy\alpha^n.$$

The following Corollary gives a simplified analytic expression when $A \neq \alpha$.

Corollary 1. *Consider the sequence $(x_n)_n$ defined by the Eq. (1) for $A \neq \alpha$, the subsequences can be written as:*

$$x_{4n-3} = \frac{d\alpha^n(A-\alpha)\prod_{p=0}^{n-2} P_{2p+2}(b,d)}{\prod_{p=0}^{n-1} P_{2p+1}(b,d)}, \quad x_{4n-2} = \frac{c\alpha^n(A-\alpha)\prod_{p=0}^{n-2} P_{2p+2}(a,c)}{\prod_{p=0}^{n-1} P_{2p+1}(a,c)},$$

$$x_{4n-1} = \frac{b\alpha^n \prod_{p=0}^{n-1} P_{2p+1}(b,d)}{\prod_{p=0}^{n-1} P_{2p+2}(b,d)}, \qquad \textit{and} \qquad x_{4n} = \frac{a\alpha^n \prod_{p=0}^{n-1} P_{2p+1}(a,c)}{\prod_{p=0}^{n-1} P_{2p+2}(a,c)}.$$

Proof. It is sufficient to use the binomial identity $x^{p+1}-y^{p+1} = (x-y)\sum_{k=0}^{p} x^k y^{p-k}$ in the analytical expression of the subsequences defined by Eq. (6) and (7).

Corollary 2. *Consider the sequence $(x_n)_n$ defined by the Eq. (1). For $A=\alpha\neq 0$, the sequence can be expressed in Gamma form as*

$$x_{4n-3} = \frac{A2^{2n-2}\Gamma^2(\frac{A}{2Bbd}+n)\Gamma(\frac{A}{Bbd}+1)}{Bb\Gamma^2(\frac{A}{2Bbd}+1)\Gamma(\frac{A}{Bbd}+2n)}, \qquad x_{4n-2} = \frac{A2^{2n-2}\Gamma^2(\frac{A}{2Bac}+n)\Gamma(\frac{A}{Bac})}{Ba\Gamma^2(\frac{A}{2Bac}+1)\Gamma(\frac{A}{Bac}+2n)},$$

$$x_{4n-1} = \frac{b\Gamma(\frac{A}{Bbd}+2n+1)\Gamma^2(\frac{A}{2Bbd}+1)}{2^{2n}\Gamma(\frac{A}{Bbd}+1)\Gamma^2(\frac{A}{2Bbd}+n+1)}, \qquad x_{4n} = \frac{a\Gamma(\frac{A}{Bac}+2n+1)\Gamma^2(\frac{A}{2Bac}+1)}{2^{2n}\Gamma(\frac{A}{Bac}+1)\Gamma^2(\frac{A}{2Bac}+n+1)},$$

where Γ is the Euler's Gamma function.

Proof. Using Eq. (6) we have:

$$\begin{aligned} x_{4n-3} &= \frac{dA^n \prod_{p=0}^{n-2}\Big(A^{2p+2}+Bbd\sum_{i=0}^{2p+1}A^{2p+1}\Big)}{\prod_{p=0}^{n-1}\Big(A^{2p+1}+Bbd\sum_{i=0}^{2p}A^{2p}\Big)}, \\ &= \frac{dA\prod_{p=0}^{n-2}Bbd\Big(\frac{A}{Bbd}+2p+2\Big)}{\prod_{p=0}^{n-1}Bbd\Big(\frac{A}{Bbd}+2p+1\Big)} = \frac{A\Big[\prod_{p=1}^{n-1}2\Big(\frac{A}{2Bbd}+p\Big)\Big]^2}{Bb\prod_{p=1}^{2n-1}\Big(\frac{A}{Bbd}+p\Big)} \\ &= \frac{A2^{2n-2}\Gamma^2\Big(\frac{A}{2Bbd}+n\Big)\Gamma(\frac{A}{Bbd}+1)}{Bb\,\Gamma\Big(\frac{A}{Bbd}+2n\Big)\Gamma^2\Big(\frac{A}{2Bbd}+1\Big)}. \end{aligned}$$

Similarly, one can prove the other relations. This ended the proof.

Remark 1.

1. A common hypothesis in the study of rational difference equations is the choice of positive coefficients and initial data. Therefore, all the solutions will be automatically well defined. It is, in general a problem of great difficulty to determine the good set of initial conditions without finding the analytical expression of the considered sequence.

2. According to the Corollaries 1 and 2, the good set G of the sequence $(x_n)_n$ is given as

   (a) When $A\neq\alpha$,

$$G = \left\{(a,b,c,d)\in\mathbb{R}^4 \text{ such that } bd,\ ac\in\mathbb{R}-\left\{\frac{-(A-\alpha)A^n}{B(A^n-\alpha^n)},\quad n\in\mathbb{N}\right\}\right\}.$$

(b) When $A=\alpha$, $G=\{(a,b,c,d)\in\mathbb{R}^4$ such that $\frac{A}{Bbd},\frac{A}{Bac}\notin 2\mathbb{Z}_-\}$.

3. If we choose for example $\alpha=A=B$, we obtain the expression of the general term which can be written and in gamma form as

$$x_{4n-3}=\frac{2^{2n-2}\Gamma^2(\frac{1}{2bd}+n)\Gamma(\frac{1}{bd})}{b\Gamma^2(\frac{1}{2bd}+1)\Gamma(\frac{1}{bd}+2n)},\qquad x_{4n-2}=\frac{2^{2n-2}\Gamma^2(\frac{1}{2ac}+n)\Gamma(\frac{1}{ac})}{a\Gamma^2(\frac{1}{2ac}+1)\Gamma(\frac{1}{ac}+2n)},$$

$$x_{4n-1}=\frac{b\Gamma(\frac{1}{bd}+2n+1)\Gamma^2(\frac{1}{2bd}+1)}{2^{2n}\Gamma(\frac{1}{bd}+1)\Gamma^2(\frac{1}{2bd}+n+1)},\qquad x_{4n}=\frac{a\Gamma(\frac{1}{ac}+2n+1)\Gamma^2(\frac{1}{2ac}+1)}{2^{2n}\Gamma(\frac{1}{ac}+1)\Gamma^2(\frac{1}{2ac}+n+1)}.$$

In the following section we will study the convergence of sequence $(x_n)_n$. This will depend evidently on the parameters α, A, B and the initial data.

4. CONVERGENCE OF SOLUTIONS OF *EQ. (1)*

Consider the function F defined on $\mathbb{R}^4$ as: $F(u_0,u_1,u_2,u_3)=\frac{\alpha u_3}{A+Bu_1u_3}$. Using the function F, Eq. (1) can be written as $x_{n+1}=F(x_n,x_{n-1},x_{n-2},x_{n-3})$.

Theorem 2. The following statements are true:

(1) For $B(A-\alpha)\geq 0$, Eq.(1) has a unique equilibrium point $\overline{x}=0$, then

(a) If $A=\alpha$, the equilibrium point is nonhyperbolic.

(b) If $\frac{A}{\alpha}>1$, the equilibrium point is locally asymptotically stable.

(2) For $B(A-\alpha)<0$, then

(a) The Eq. (1) has exactly three equilibrium points which are

$$\overline{x}_1=0,\ \overline{x}_2=\sqrt{\tfrac{\alpha-A}{B}},\ \overline{x}_3=-\sqrt{\tfrac{\alpha-A}{B}}. \tag{8}$$

(b) If $0<A<\alpha$, then

(i) The equilibrium point $\overline{x}_1=0$ is a repeller.

(ii) The equilibrium points $\overline{x}_2$, $\overline{x}_3$ are hyperbolic.

Proof. (1) For $B(A-\alpha)\geq 0$, $\bar{x}$ is an equilibrium point is equivalent to

$$\bar{x}=\frac{\alpha\bar{x}}{A+B\bar{x}^2}\quad\Rightarrow\ B\bar{x}^3+(A-\alpha)\bar{x}=0\quad\Rightarrow\ \bar{x}(B\bar{x}^2+A-\alpha)=0.$$

This shows clearly that if $B(A-\alpha)\geq 0$, $\overline{x}=0$ is the unique equilibrium point of Eq. (1).
$q_i=\frac{\partial F}{\partial u_i}(0,0,0,0)$, then $q_0=q_1=q_2=0$ and $q_3=-\frac{\alpha}{A}$, the characteristic equation of the linearized equation associated with Eq. (1) is then all real roots have absolute value equal to one, so the equilibrium points is nonhyperbolic.$\bar{x}$ is an equilibrium point is equivalent to

$$\lambda^4-\frac{\alpha}{A}=0. \tag{9}$$

(a) Suppose that $A=\alpha$, then all real roots have absolute value equal to one, so the equilibrium points is nonhyperbolic.

(b) Suppose that $\frac{A}{\alpha}>1$, so all the roots of Eq. (9) have absolute value less than one, according the linearized stability Theorem, the equilibrium point $\overline{x}=0$ is locally asymptotically stable.

(2) For $B(A-\alpha)<0$, the equation $\bar{x}(B\bar{x}^2+A-\alpha)=0$ has exactly three solutions which are the equilibrium points in Eqs. (8).

(a) The characteristic equation about $\overline{x}_1 = 0$ is $\lambda^4 - \frac{\alpha}{A} = 0$, since $0 < A < \alpha$ then all roots has absolute value greater than one and $\overline{x}_1 = 0$ is repeller.

(b) The characteristic equation about $\overline{x}_2$ is $\lambda^4 + \frac{\alpha - A}{A}\lambda^2 - \frac{A}{\alpha} = 0$. The real roots of this equation are $\sqrt{\frac{A}{\alpha}}$ and $-\sqrt{\frac{A}{\alpha}}$, they are less than one, so the equilibrium point $\overline{x}_2$ is hyperbolic. The proof for $\overline{x}_3$ can be similarly obtained.

As it is expected, the convergence of $(x_n)_n$ depends on the parameters α, A, B, and the initial data. We will distinguish the following cases:

(i) Case $|\frac{A}{\alpha}| > 1$.

Theorem 3. *Assume that* $|\frac{A}{\alpha}| > 1$

(1) If $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then every solution of Eq. (1) converges toward zero.

(2) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then the solution of Eq. (1) converges iff $a = b = c = d = \pm\sqrt{\frac{\alpha - A}{B}}$.

(3) If $(A - \alpha + Bbd)(A - \alpha + Bac) = 0$ but not both terms of the product are zero, then every solution of Eq. (1).

Proof. (1) Suppose that $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then Corollary 1 implies that

$$
\begin{aligned}
x_{4n-3} &= \frac{d\alpha^n (A-\alpha) \prod_{p=0}^{n-2} \Big(A^{2p+2}(A-\alpha+Bbd) - Bbd\alpha^{2p+2}\Big)}{\prod_{p=0}^{n-1} \Big(A^{2p+1}(A-\alpha+Bbd) - Bbd\alpha^{2p+1}\Big)} \\
&= \frac{d\alpha^n (A-\alpha) A^{n-1} \prod_{p=0}^{n-2} \Big(1 - \frac{Bbd}{A-\alpha+Bbd}(\frac{\alpha}{A})^{2p+2}\Big)}{(A-\alpha+Bbd)A^{2n-1} \prod_{p=0}^{n-2} \Big(1 - \frac{Bbd}{A-\alpha+Bbd}(\frac{\alpha}{A})^{2p+1}\Big)}.
\end{aligned}
$$

Denote by $\beta = \frac{Bbd}{A-\alpha+Bbd}$ and by $(U_p)_p$ the sequence defined as $U_p = \frac{1-\beta(\frac{\alpha}{A})^{2p+2}}{1-\beta(\frac{\alpha}{A})^{2p+1}}$, we get

$$
x_{4n-3} = \frac{d(\frac{\alpha}{A})^n (A-\alpha)}{(A-\alpha+Bbd)\Big(1-\beta(\frac{\alpha}{A})^{2n-1}\Big)} \prod_{p=0}^{n-2} U_p.
$$

We have either: for $p \in \mathbb{N}$ big enough, $U_p > 1$ or for $p \in \mathbb{N}$ big enough, $0 < U_p < 1$.
Using Taylor expansion of the U_p, we obtain

$$
U_p = (1 - \beta(\frac{\alpha}{A})^{2p+2})(1 + \beta(\frac{\alpha}{A})^{2p+1} + o(\frac{\alpha}{A})^{2p+1}) = 1 + \beta(\frac{\alpha}{A})^{2p+1} + o(\frac{\alpha}{A})^{2p+1},
$$

then U_p is equivalent to $1 + \beta(\frac{\alpha}{A})^{2p+1}$ which is the general term of a convergent infinite product.
We can easily deduce that $(x_{4n-3})_n$ converges toward zero. same discussion can be obtained for the other subsequences.

(2) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then by the proof of (1), the subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants: $x_{4n-3} = d$ and $x_{4n-1} = b$, also the subsequences $x_{4n-2} = c$ and $x_{4n} = a$. Thus every solution of Eq. (1) converges to a real number l if and only if $a = b = c = d = l$.

(3) Consider for instance the case $A - \alpha + Bbd = 0$ and $A - \alpha + Bac \neq 0$, by (2), the subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants $x_{4n-3} = d$ and $x_{4n-1} = b$, in other hand and also by the proof of case (1), the subsequences $(x_{4n-2})_n$ and $(x_{4n})_n$ converge to zero, then the sequence $(x_n)_n$ diverges. The proof is completed.

(ii) Case $|\frac{A}{\alpha}| = 1$.

Theorem 4. *Assume that* $|\frac{A}{\alpha}| = 1$. We distinguish two subcases, $A = \alpha$ and $A = -\alpha$.

(1) If $A = \alpha$, and let sequence $(x_n)_n$ be the sequence given by the formula (1), then the sequence $(x_n)_n$ converges toward zero.

(2) If $A = -\alpha$, and let sequence $(x_n)_n$ be the sequence given by the formula (1), then we have $x_{4n-1} = \frac{b}{dx_{4n-3}}$, $x_{4n-2} = \frac{c}{ax_{4n}}$ and the sequence $(x_n)_n$ is divergent.

Proof. (1) For $A = \alpha$, let δ the parameter $\delta = \frac{A}{Bbd}$. In the proof of Corollary 2, we find that

$$x_{4n-3} = \tfrac{A}{Bb(\delta+1)} \prod_{p=1}^{n-1} \Big(\frac{\frac{\delta}{2p}+1}{\frac{\delta+1}{2p}+1} \Big).$$

Denote by $(W_p)_p$ the sequence defined as $W_p = \frac{\frac{\delta}{2p}+1}{\frac{\delta+1}{2p}+1}$, then we get:
For p big enough, we have $0 < W_p < 1$. The Taylor expansion for W_p gives:
$W_p = (1 + \frac{\delta}{2p})(1 - \frac{\delta+1}{2p} + o(\frac{1}{p})) = 1 - \frac{1}{2p} + o(\frac{1}{p})$,
which is a general term of divergent infinite product. Since for p big enough, $0 < W_p < 1$, then $\lim_{n\to\infty} \Pi_{p=1}^{n-1} W_p = 0$. So, we get $\lim_{n\to\infty} x_{4n-3} = 0$. Similarly, one can easily prove that the other subsequences converge to zero, therefore the sequence $(x_n)_n$ converges to zero.

(2) To prove the second part, we replace α by $(-A)$ in the expression of x_{4n-3} of Eq. (6), we obtain

$$x_{4n-3} \quad = \quad \frac{d(-A)^n \prod_{p=0}^{n-2} A^{2p+1}\Big(A+Bbd\sum_{k=0}^{2p+1}(-1)^k\Big)}{\prod_{p=0}^{n-1} A^{2p}\Big(A+Bbd\sum_{k=0}^{2p}(-1)^k\Big)} = \tfrac{d}{(-1-\delta^{-1})^n}, \ .$$

In other hand, If we replace α by $(-A)$ in the first term of Eq. (7), we obtain

$$x_{4n-1} \quad = \quad b(-A)^n \prod_{p=0}^{n-1} \Big(\tfrac{A^{2p}(A+Bbd)}{A^{2p+2}}\Big) = b(-1-\delta^{-1})^n.$$

Thus $x_{4n-1} = \frac{b}{dx_{4n-3}}$, hence

(a) If $|1+\delta^{-1}| > 1$, then the subsequence $(x_{4n-3})_n$ converges to zero, so $(|x_{4n-1}|)_n$ goes to infinity.

(b) If $|1+\delta^{-1}| < 1$, then the subsequence $(|x_{4n-3}|)_n$ goes to infinity.

This completed the proof.

(iii) Case $|\frac{A}{\alpha}| < 1$.

Theorem 5. *Let $(x_n)_n$ be the sequence given by the formula (1), then*
For $|\frac{A}{\alpha}| < 1$, then the subsequences $(x_{4n-3})_n$, $(x_{4n-1})_n$, $(x_{4n-2})_n$ and $(x_{4n})_n$ converge.

Proof. We need to prove that $(x_{4n-3})_n$ converges. Using Corollary (1), we obtain

$$x_{4n-3} \quad = \quad \frac{d\alpha^n(A-\alpha)\prod_{p=0}^{n-2}\Big(A^{2p+2}(A-\alpha+Bbd)-Bbd\alpha^{2p+2}\Big)}{\prod_{p=0}^{n-1}\Big(A^{2p+1}(A-\alpha+Bbd)-Bbd\alpha^{2p+1}\Big)} = \frac{\alpha-A}{Bb(1-\gamma\lambda^{2n-1})}\prod_{p=0}^{n-2} V_p,$$

where $\gamma = \frac{A-\alpha+Bbd}{Bbd}$, $\lambda = \frac{A}{\alpha}$ and $(V_p)_p$ is the sequence defined by $V_p = \frac{1-\gamma\lambda^{2p+2}}{1-\gamma\lambda^{2p+1}}$. For $p \in \mathbb{N}$ big enough, we have two cases; either $V_p > 1$ or $0 < V_p < 1$. Applying the transformation of infinite product of positive terms to infinite series, and assuming p_0 to be big enough, we get

$$x_{4n-3} = \frac{\alpha - A}{Bb(1-\gamma\lambda^{2n-1})}\Big(\prod_{p=0}^{p_0} V_p\Big)\exp\Big(\sum_{p=p_0+1}^{n-2}\ln(V_p)\Big).$$

It is clear that the sequence $\Big(\frac{\alpha-A}{Bb(1-\gamma\lambda^{2n-1})}\Big)_n$ converges toward $\frac{\alpha-A}{Bb}$. The Taylor expansion of V_p to the first order gives

$$V_p = \tfrac{1-\gamma\lambda^{2p+2}}{1-\gamma\lambda^{2p+1}} = 1 + \gamma(1-\lambda)\lambda^{2p+1} + o(\lambda^{2p+1}).$$

So $\ln(V_p)$ is equivalent to $\gamma(1-\lambda)\lambda^{2p+1}$, which is the general term of a convergent infinite series, then the sequence $(x_{4n-3})_n$ is convergent. Similarly, one can prove that the other subsequences are convergent.

Remark 2. (Commentary on the convergence of $(x_n)_n$ in the case $|\frac{A}{\alpha}|<1$).
Suppose that $|\frac{A}{\alpha}|<1$, according to Theorem 5, the subsequences $(x_{4n-3})_n$, $(x_{4n-1})_n$, $(x_{4n-2})_n$ and $(x_{4n})_n$ converge, denote by: l_3, l_2, l_1 and l_0 their limits respectively.
The subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are related by the equations:

$$x_{4(n+1)-3} = \frac{\alpha x_{4n-3}}{A+Bx_{4n-1}x_{4n-3}}, \tag{10}$$

$$x_{4(n+1)-1} = \frac{\alpha x_{4n-1}}{A+Bx_{4(n+1)-3}x_{4n-1}}. \tag{11}$$

Passing to the limit as n goes to infinity in Eq. (10), we obtain $l_3 = \frac{\alpha l_3}{A+Bl_3l_1}$, then $(S_1):\begin{cases} l_3 = 0, \\ or \\ l_3 \neq 0 \text{ and } l_1 = \frac{\alpha-A}{Bl_3}. \end{cases}$

Passing to the limit as n goes to infinity in Eq. (11), we obtain $l_1 = \frac{\alpha l_1}{A+Bl_3l_1}$, then $(S_2):\begin{cases} l_1 = 0, \\ or \\ l_1 \neq 0 \text{ and } l_3 = \frac{\alpha-A}{Bl_1}. \end{cases}$

Combining systems (S_1) and (S_2), since $\alpha \neq A$, we obtain

$$\begin{cases} l_3 = l_1 = 0 \\ or \\ l_1 \neq 0, \quad l_3 \neq 0 \text{ and } (S): \begin{cases} l_3 = \frac{\alpha-A}{Bl_1}, \\ and \\ l_1 = \frac{\alpha-A}{Bl_3}. \end{cases} \end{cases}$$

The proposition $l_3 = l_1 = 0$ contradicts the fact that the infinite product $\prod_{p\geq 0} V_p$ converges, in fact if $\lim_{n\to\infty}\prod_{p=0}^{n} V_p = 0$, then $\lim_{n\to\infty}\sum_{p=p_0}^{n}\ln(V_p) = -\infty$, and this is absurd. Hence the only possibility is that

$$l_1 \neq 0, \quad l_3 \neq 0 \text{ and } (S): \begin{cases} l_3 = \frac{\alpha-A}{Bl_1}, \\ and \\ l_1 = \frac{\alpha-A}{Bl_3}. \end{cases}$$

One can easily see that (S) is equivalent to $l_3 = \frac{\alpha-A}{Bl_1}$. Let f be the function defined on $\mathbb{R}^*$ as $f(x) = \frac{\alpha-A}{Bx}$, we have $fof = Id$ and, l_1 and l_3 are related by $f(l_1) = l_3$.

$$f(x) = x \Leftrightarrow \frac{\alpha-A}{Bx} = x \Leftrightarrow x = \mp\sqrt{\frac{\alpha-A}{B}}.$$

Hence: f has fixed points if and only if $\frac{\alpha-A}{B} > 0$.
The numerical example (Figure 4) given in the end of this paper confirm that even we chose $\frac{\alpha-A}{B} > 0$ and $|\frac{A}{\alpha}|<1$, l_1 and l_3 may be different, which implies the sequence $(x_n)_n$ may converge or diverge.
Finally based on the preview discussion of all preview cases, The following Theorem is now proved.
Theorem 6. (Boundedness of $(x_n)_n$). *The Eq. (1) has an unbounded solutions if and only if $A = -\alpha$.*

5. PERIODICITY CHARACTER OF SOLUTIONS OF *EQ. (1)*

In the sequel, we need the following lemma, which describes sufficient conditions for Eq. (1) to have a periodic solution.
Lemma 1. *Let $(x_n)_{n\geq -3}$ be a solution of Eq. (1) and the initial data that follow. Suppose that there are real numbers l_3, l_2, l_1, l_0 such that $\lim_{n\to\infty} x_{4n-j} = l_j$ for $j = 0, ..., 3$.*

Let $(y_n)_{n\geq -3}$ be the period-4 sequence such that $y_{-j} = l_j$, for all $j = 0, ..., 3$, then the sequence $(y_n)_{n\geq -3}$ is a period-4 solution of Eq. (1). The periodicity results are given by the following Theorem

Theorem 7. *Let $(x_n)_{n\geq -3}$ be a solution of Eq. (1) and the initial data that follow, then*

(1) For $|\frac{A}{\alpha}| > 1$,

(a) If $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then then Eq. (1) has no periodic solutions.

(b) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then the solution of Eq. (1) is a periodic-4 solution.

(c) If either $A - \alpha + Bbd$ or $A - \alpha + Bac$ equals zero but not both of them, then Eq. (1) has a periodic-4 solution.

(2) For $|\frac{A}{\alpha}| = 1$, Eq. (1) has no periodic solutions.

(3) For $|\frac{A}{\alpha}| < 1$, Eq. (1) has periodic-4 solutions.

Proof. (1) Suppose that $|\frac{A}{\alpha}| > 1$,

(a) If $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then by Theorem 3, every solution of Eq. (1) converges to zero, hence, the solutions are not allowed to be periodic (since the solutions are not identically zero).

(b) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then by Theorem 3, the subsequences of $(x_n)_n$ $(x_{4n-j})_n$, $j = 0, .., 3$ are constants: $x_{4n-3} = d$, $x_{4n-2} = c$, $x_{4n-1} = b$ and $x_{4n} = a$, and the sequence $d, c, b, a, d, c, b, a...$ is a periodic-4 solution of Eq. (1).

(c) Consider for instance the case $A - \alpha + Bbd = 0$ and $A - \alpha + Bac \neq 0$, by the proof of Theorem 3, the subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants and equal d and b respectively. Also according to the proof of Theorem 3, the subsequences $(x_{4n-2})_n$ and $(x_{4n})_n$ converge to zero. Applying Lemma 1, the sequence $d, 0, b, 0, d, 0, b, 0, ...$ is a periodic-4 solution of Eq. (1).

(2) The case $A = \alpha$ is similar to (1) (a).
If $A = -\alpha$, then every solution of Eq. (1) is unbounded, so Eq. (1) has no periodic solutions.

(3) If $|\frac{A}{\alpha}| < 1$, then by Theorem 5, there are real numbers l_3, l_2, l_1 and l_0, such that $\lim_{n\to\infty} x_{4n-j} = l_j$ for all $j = 0, .., 3$.
Applying Lemma 1, the sequence $l_3, l_2, l_1, l_0, l_3, l_2, l_1, l_0...$ is a periodic-4 solution of Eq. (1).
This completes the proof.

Remark 3.

(1) Note that if $|\frac{A}{\alpha}| > 1$, $A - \alpha + Bbd = A - \alpha + Bac = 0$, $a = c$, $b = d$, then Eq. (1) has periodic-2 solution $a, b, a, b, ...$

(2) If $|\frac{A}{\alpha}| < 1$, $A - \alpha + Bbd = A - \alpha + Bac = 0$, then, by the proof of Theorem 7, we deduce that the values of the limits of the subsequences are $l_3 = d$, $l_2 = c$, $l_1 = b$ and $l_0 = a$.

6. NUMERICAL SIMULATION

Example 1. Figure (1) illustrates the case $|\frac{A}{\alpha}| > 1$, $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, we choose $a = 2$, $b = -3$, $c = 2$, $d = -2$, $B = 2$, $A = 1.1$ and $\alpha = 1$. We notice that the solution is oscillating about zero with a decreasing amplitude. In fact, according to Theorem 3, the solution has to converge to zero.

Example 2. In order illustrate the case $|\frac{A}{\alpha}| > 1$, $A - \alpha + Bbd = A - \alpha + Bac = 0$, we choose $a = c = 2$, $b = d = -2$, $B = -3$, $A = 13$ and $\alpha = 1$. Figure (2) depicts that the obtained solution is a 2-prime periodic solution. This is coherent with Remark 3.

Example 3. The case $|\frac{A}{\alpha}| > 1$, $A - \alpha + Bbd = 0$ and $A - \alpha + Bac \neq 0$ is illustrated in figure (3), in which we set $a = c = 1$, $b = d = -2$, $B = -2$, $A = 9$ and $\alpha = 1$. The subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants $(x_{4n-3})_n = d$ and $(x_{4n-1})_n = b$, and the subsequences $(x_{4n-2})_n$ and $(x_{4n})_n$ converge to zero. by Lemma 1, the sequence $d, 0, b, 0, d, 0, b, 0, ...$ is a periodic-4 solution of Eq. (1).

Example 4. Figure (4) illustrates the case $|\frac{A}{\alpha}| < 1$, we choose $a = -1$, $b = 0.5$, $c = -0.2$, $d = 0.8$, $B = 1$, $A = 0.5$ and $\alpha = 1$. the subsequences $(x_{4n-3})_n$, $(x_{4n-1})_n$, $(x_{4n-2})_n$ and $(x_{4n})_n$ converge.

Example 5. To illustrate the case $A = \alpha$, we choose $a = 0.1$, $b = 0.2$, $c = 0.3$, $d = -0.4$, $B = 1$, $\alpha = 0.5$ and $A = 0.5$. We notice in the figure (5), that the solution converges to zero (which is coherent to Theorem 4 part (1)), and the Eq. (1) has no periodic solutions (which is coherent to Theorem 7 part (2)).

Example 6. In figure (6) (case $A = -\alpha$), we choose $a = 0.2$, $b = 0.3$, $c = 0.1$, $d = -0.3$, $B = 2$, $\alpha = -0.4$ and $A = 0.4$. We notice that the solution is oscillating about zero with an increasing amplitude and the solution is unbounded, which is coherent to Theorem 4 part (2).

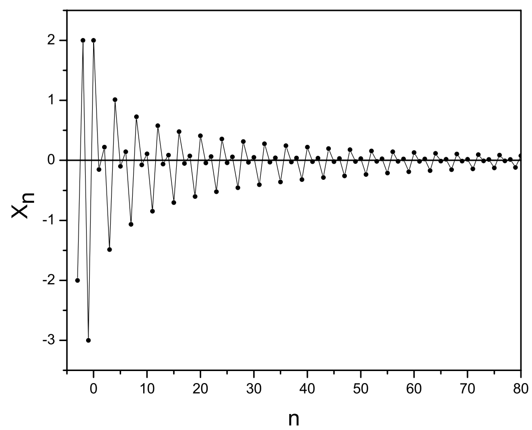

Figure 1.

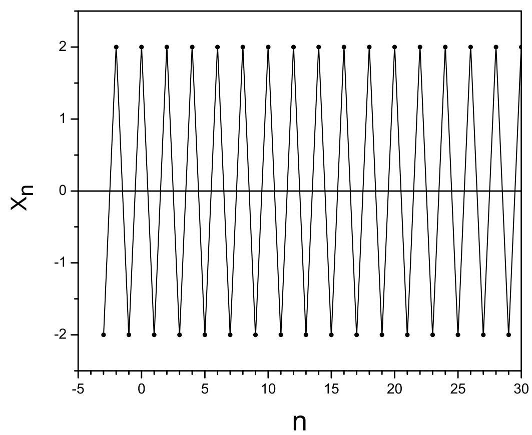

Figure 2.

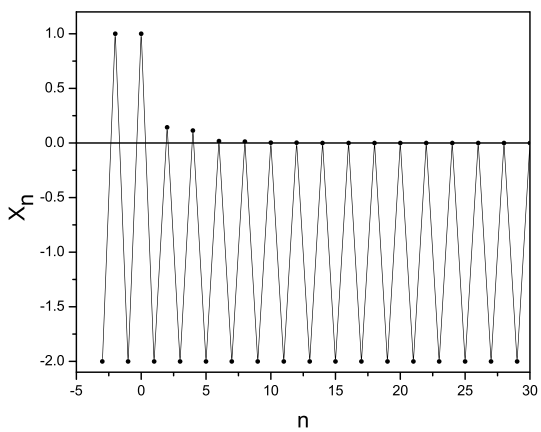

Figure 3.

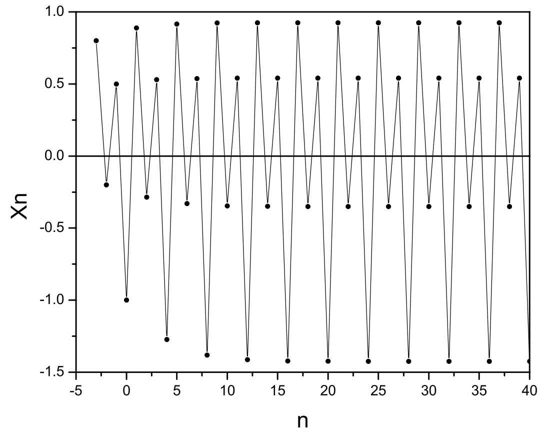

Figure 4.

Conclusion

In this work, some dynamical behaviors of the rational difference equation $x_{n+1} = \frac{\alpha x_{n-3}}{A+Bx_{n-1}x_{n-3}}$ with the initial conditions, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$ are arbitrary real numbers, A and B are arbitrary constants, have been investigated. A detailed analytical study of the convergence of the solutions including their dependence on parameters and initial conditions has been illustrated. The local stability and global attractivity of the difference equation's equilibrium points have been demonstrated. The existence of periodic solutions in the proposed difference equation has also been shown analytically. Finally, numerical simulations have been carried out to match the analytical results.

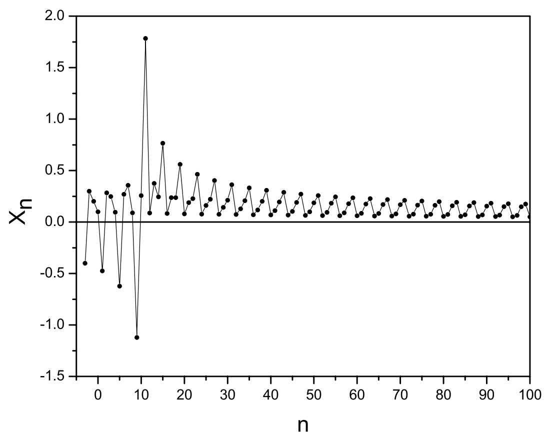

Figure 5.

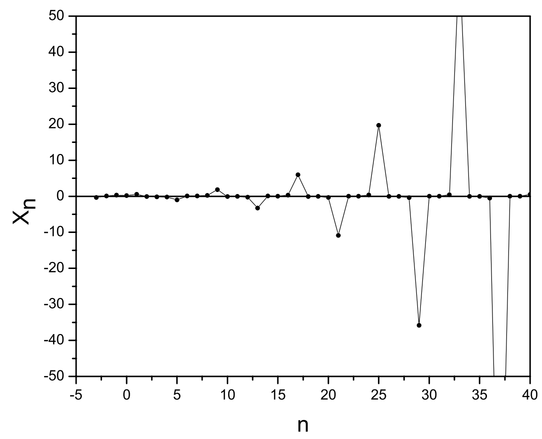

Figure 6.

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QUADRATIC ρ-FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper, we solve the quadratic ρ-functional equations

$$\begin{aligned} & f(x+y)+f(x-y)-2f(x)-2f(y) \\ & \quad = \rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right), \end{aligned} \tag{0.1}$$

where ρ is a fixed non-Archimedean number or a fixed real or complex number with $\rho \neq -1, 2$, and

$$\begin{aligned} & 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y) \\ & \quad = \rho(f(x+y)+f(x-y)-2f(x)-2f(y)), \end{aligned} \tag{0.2}$$

whereρ is a fixed non-Archimedean number or a fixed real or complex number with $\rho \neq -1, \frac{1}{2}$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic ρ-functional equations (0.1) and (0.2) in non-Archimedean Banach spaces and in Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. Gajda [11] following the same approach as in Rassias [22], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [11], as well as by Rassias and Šemrl [21] that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [11], as well as of Rassias and Šemrl [21] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. Găvruta [12], who among others studied the Hyers-Ulam stability of functional equations (cf. the books of Czerwik [8, 9], Hyers, Isac and Th.M. Rassias [14]). The hyperstability of the Cauchy equation was proved by Brzdek [4].

The functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [24] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. See [1, 5, 6, 10, 16, 17, 18, 19, 20, 23] for more

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functional equations. The survey on the Hyers-Ulam stability of functional equations was given by Brillouet-Bulluot, Brzdek and Cieplinski [3].

The functional equation

$$2f\left(\frac{x+y}{2}\right)+2\left(\frac{x-y}{2}\right)=f(x)+f(y)$$

is called a *Jensen type quadratic equation.*

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s|\leq|r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s|\leq\max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0|=0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([15]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|:X\to[0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii) $\|rx\|=|r|\|x\| \qquad (r\in K, x\in X)$;

(iii) the strong triangle inequality

$$\|x+y\|\leq\max\{\|x\|,\|y\|\}, \qquad \forall x,y\in X$$

holds. Then $(X,\|\cdot\|)$ is called a *non-Archimedean normed space.*

In Section 2, we solve the quadratic functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the quadratic functional equation (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the quadratic functional equation (0.2) in vector spaces and prove the Hyers-Ulam stability of the quadratic functional equation (0.2) in non-Archimedean Banach spaces.

In Section 4, we prove the Hyers-Ulam stability of the quadratic functional equation (0.1) in Banach spaces.

In Section 5, we prove the Hyers-Ulam stability of the quadratic functional equation (0.2) in Banach spaces.

2. Quadratic ρ-functional equation (0.1) in non-Archimedean Banach spaces

Throughout Sections 2 and 3, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2|\neq 1$ and let ρ be a fixed non-Archimedean number with $\rho\neq -1,2$.

Lemma 2.1. *Let X and Y be vector spaces. A mapping $f:X\to Y$ satisfies*

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0 \tag{2.1}$$

for all $x, y \in X$ if and only if the mapping $f : X \to Y$ satisfies

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) = 0 \tag{2.2}$$

for all $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $f(0) = 0$.

Letting $y = x$ in (2.1), we get $f(2x) - 4f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$. So $f : X \to Y$ satisfies (2.2).

Assume that $f : X \to Y$ satisfies (2.2).

Letting $x = y = 0$ in (2.2), we get $f(0) = 0$.

Letting $y = 0$ in (2.2), we get $4f\left(\frac{x}{2}\right) = f(x)$ for all $x \in X$. and so $f(2x) = 4f(x)$ for all $x \in X$. So $f : X \to Y$ satisfies (2.1). □

We solve the quadratic ρ-functional equation (0.1) in vector spaces.

Lemma 2.2. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$\begin{aligned} f(x+y) &+ f(x-y) - 2f(x) - 2f(y) \\ &= \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \end{aligned} \tag{2.3}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.3).

Letting $x = y = 0$ in (2.3), we get $-2f(0) = 2\rho f(0)$. So $f(0) = 0$.

Letting $y = x$ in (2.3), we get

$$f(2x) - 4f(x) = 0$$

and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.4}$$

for all $x \in X$.

It follows from (2.3) and (2.4) that

$$\begin{aligned} f(x+y) &+ f(x-y) - 2f(x) - 2f(y) \\ &= \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \\ &= \frac{\rho}{2}(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the quadratic ρ-functional equation (2.3) in non-Archimedean Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\lim_{j\to\infty} |4|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) = 0, \tag{2.5}$$

$$\begin{aligned}&\|f(x+y)+f(x-y)-2f(x)-2f(y)\\&-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\|\le\varphi(x,y)\end{aligned}\tag{2.6}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x)-h(x)\|\le\sup_{j\in\mathbb{N}}\left\{|4|^{j-1}\varphi\left(\frac{x}{2^j},\frac{x}{2^j}\right)\right\}\tag{2.7}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.6), we get

$$\|f(2x)-4f(x)\|\le\varphi(x,x)\tag{2.8}$$

for all $x \in X$. So

$$\left\|f(x)-4f\left(\frac{x}{2}\right)\right\|\le\varphi\left(\frac{x}{2},\frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}&\left\|4^lf\left(\frac{x}{2^l}\right)-4^mf\left(\frac{x}{2^m}\right)\right\|\\&\le\max\left\{\left\|4^lf\left(\frac{x}{2^l}\right)-4^{l+1}f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,\left\|4^{m-1}f\left(\frac{x}{2^{m-1}}\right)-4^mf\left(\frac{x}{2^m}\right)\right\|\right\}\\&\le\max\left\{|4|^l\left\|f\left(\frac{x}{2^l}\right)-4f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,|4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right)-4f\left(\frac{x}{2^m}\right)\right\|\right\}\\&\le\sup_{j\in\{l,l+1,\cdots\}}\left\{|4|^j\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}}\right)\right\}\end{aligned}\tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x):=\lim_{n\to\infty}4^nf(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.7).

It follows from (2.5) and (2.6) that

$$\begin{aligned}&\|h(x+y)+h(x-y)-2h(x)-2h(y)\\&-\rho\left(2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-h(x)-h(y)\right)\|\\&=\lim_{n\to\infty}|4|^n\left\|f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right.\\&\qquad\left.-\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right\|\\&\le\lim_{n\to\infty}|4|^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)=0\end{aligned}$$

for all $x, y \in X$. So

$$h(x+y)+h(x-y)-2h(x)-2h(y)=\rho\left(2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-h(x)-h(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.7). Then we have

$$\begin{aligned}\|h(x) - T(x)\| &= \left\|4^q h\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \max\left\{\left\|4^q h\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\|, \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\|\right\} \\ &\leq \sup_{j\in\mathbb{N}}\left\{|4|^{q+j-1}\varphi\left(\frac{x}{2^{q+j}}, \frac{x}{2^{q+j}}\right)\right\},\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (2.7). □

Corollary 2.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}&\|f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ &-\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\| \leq \theta(\|x\|^r + \|y\|^r)\end{aligned} \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2\theta}{|2|^r}\|x\|^r$$

for all $x \in X$.

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (2.6) and*

$$\lim_{j\to\infty}\left\{\frac{1}{|4|^j}\varphi(2^{j-1}x, 2^{j-1}y)\right\} = 0$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \sup_{j\in\mathbb{N}}\left\{\frac{1}{|4|^j}\varphi(2^{j-1}x, 2^{j-1}x)\right\} \tag{2.11}$$

for all $x \in X$.

Proof. It follows from (2.8) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{|4|}\varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned}&\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| \\ &\leq \max\left\{\left\|\frac{1}{4^l}f\left(2^l x\right) - \frac{1}{4^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{4^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{4^m}f\left(2^m x\right)\right\|\right\} \\ &\leq \max\left\{\frac{1}{|4|^l}\left\|f\left(2^l x\right) - \frac{1}{4}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|4|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{4}f\left(2^m x\right)\right\|\right\} \\ &\leq \sup_{j\in\{l,l+1,\cdots\}}\left\{\frac{1}{|4|^{j+1}}\varphi(2^j x, 2^j x)\right\}\end{aligned} \tag{2.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.12) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.12), we get (2.11).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.10)*. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{2\theta}{|4|}\|x\|^r$$

for all $x \in X$.

3. Quadratic ρ-functional equation (0.2) in non-Archimedean Banach spaces

Let $|2| \neq 1$ and let ρ be a fixed non-Archimedean number with $\rho \neq -1, \frac{1}{2}$.

We solve the quadratic ρ-functional equation (0.2) in vector spaces.

Lemma 3.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$\begin{aligned} 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned} \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $2f(0) = -2\rho f(0)$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$4f\left(\frac{x}{2}\right) - f(x) = 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} &\frac{1}{2}(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \\ &= 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &= \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the quadratic ρ-functional equation (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\lim_{j\to\infty} \left\{ |4|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)\right\} = 0,$$

$$\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \tag{3.3}$$
$$-\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \le \varphi(x,y)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \sup_{j\in\mathbb{N}} \left\{ |4|^{j-1} \varphi\left(\frac{x}{2^{j-1}}, 0\right)\right\} \tag{3.4}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi(x,0) \tag{3.5}$$

for all $x \in X$. So

$$\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \tag{3.6}$$
$$\le \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|\right\}$$
$$\le \max\left\{|4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1} \left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\}$$
$$\le \sup_{j\in\{l,l+1,\cdots\}} \left\{|4|^j \varphi\left(\frac{x}{2^j}, 0\right)\right\}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.3. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \tag{3.7}$$
$$-\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \le \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \theta\|x\|^r$$

for all $x \in X$.

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (3.3) and*

$$\lim_{j\to\infty} \left\{\frac{1}{|4|^j}\varphi(2^j x, 2^j y)\right\} = 0$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \sup_{j\in\mathbb{N}} \left\{ \frac{1}{|4|^j}\varphi(2^j x, 0)\right\} \tag{3.8}$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{|4|}\varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| \qquad (3.9)\\
&\le \max\left\{\left\|\frac{1}{4^l}f\left(2^l x\right) - \frac{1}{4^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{4^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{4^m}f\left(2^m x\right)\right\|\right\}\\
&\le \max\left\{\frac{1}{|4|^l}\left\|f\left(2^l x\right) - \frac{1}{4}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|4|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{4}f\left(2^m x\right)\right\|\right\}\\
&\le \sup_{j\in\{l+1,l+2,\cdots\}}\left\{\frac{1}{|4|^j}\varphi(2^j x, 0)\right\}
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proof of Theorems 2.3. □

Corollary 3.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.7)*. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{|2|^r\theta}{|4|}\|x\|^r$$

for all $x \in X$.

4. Quadratic ρ-functional equation (0.1) in Banach spaces

Throughout Sections 4 and 5, assume that X is a normed space and that Y is a Banach space. Let ρ be a fixed real or complex number with $\rho \neq -1, 2$.

We prove the Hyers-Ulam stability of the quadratic ρ-functional equation (2.3) in Banach spaces.

Theorem 4.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} 4^j\varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{4.1}$$

$$\begin{aligned}
&\|f(x + y) + f(x - y) - 2f(x) - 2f(y) \qquad (4.2)\\
&-\rho\left(2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{x - y}{2}\right) - f(x) - f(y)\right)\| \le \varphi(x, y)
\end{aligned}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{4}\Psi(x, x) \tag{4.3}$$

for all $x \in X$.

Proof. Letting $y = x$ in (4.2), we get

$$\|f(2x) - 4f(x)\| \leq \varphi(x, x) \tag{4.4}$$

for all $x \in X$. So

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\end{aligned} \tag{4.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.5) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.5), we get (4.3).

Now, let $T : X \to Y$ be another quadratic mapping satisfying (4.3). Then we have

$$\begin{aligned}\|Q(x) - T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| \\ &\leq \frac{4^q}{2}\Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q.

It follows from (4.1) and (4.2) that

$$\begin{aligned}&\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) \\ &-\rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\| \\ &= \lim_{n\to\infty} 4^n \left\|f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right. \\ &\qquad \left. -\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\leq \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0\end{aligned}$$

for all $x, y \in X$. So

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic. □

Corollary 4.2. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}&\|f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ &-\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\| \le \theta(\|x\|^r + \|y\|^r)\end{aligned} \tag{4.6}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Theorem 4.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (4.2) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{4}\Psi(x, x) \tag{4.7}$$

for all $x \in X$.

Proof. It follows from (4.4) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{1}{4^{j+1}}\varphi(2^j x, 2^j x)\end{aligned} \tag{4.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.8) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.7).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (4.6)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{4 - 2^r}\|x\|^r$$

for all $x \in X$.

5. Quadratic ρ-functional equation (0.2) in Banach spaces

Let ρ be a fixed real or complex number with $\rho \neq -1, \frac{1}{2}$.

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional equation (3.1) in Banach spaces.

Theorem 5.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty,$$

$$\begin{aligned} &\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &-\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \leq \varphi(x, y) \end{aligned} \tag{5.1}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \Psi(x, 0) \tag{5.2}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (5.1), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| = \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x, 0) \tag{5.3}$$

for all $x \in X$. So

$$\begin{aligned} \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j}, 0\right) \end{aligned} \tag{5.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.4) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.4), we get (5.2).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 5.2. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &-\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \leq \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{5.5}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2^r\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Theorem 5.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (5.1) and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \Psi(x, 0) \tag{5.6}$$

for all $x \in X$.

Proof. It follows from (5.3) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{4}\varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| &\leq \sum_{j=l+1}^{m}\left\|\frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\leq \sum_{j=l+1}^{m}\frac{1}{4^j}\varphi(2^j x, 0) \qquad (5.7)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.7) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.7), we get (5.6).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 5.4. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (5.5). Then there exists a unique quadartic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^r\theta}{4 - 2^r}\|x\|^r$$

for all $x \in X$.

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ON MODIFIED DEGENERATE GENOCCHI POLYNOMIALS AND NUMBERS

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Abstract. In this paper, we consider the modified partially degenerate Genocchi polynomials and investigate some properties of these polynomials. From these properties, we give some new and interesting identities of them.

1. Introduction

The Genocchi polynomials are defined by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} \qquad \text{(see [2, 3, 7, 9, 12, 14, 17, 19, 27, 28])}. \tag{1}$$

When $x=0$, $G_n=G_n(0)$ are called the Genocchi numbers. From (1), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} &= \left(\frac{2t}{e^t+1}\right)e^{xt}\\ &= \left(\sum_{l=0}^{\infty}G_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}x^m\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}G_lx^{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2}$$

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Thus, by comparing the coefficients on both sides of (2), we get

$$G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}. \tag{3}$$

From (1), we can derive the following equation:

$$\begin{aligned} \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} &= -\frac{-2t}{e^{-t}+1}e^{-(1-x)t} \\ &= \sum_{n=0}^{\infty}(-1)^{n-1}G_n(1-x)\frac{t^n}{n!}. \end{aligned} \tag{4}$$

By comparing the coefficients on both sides of (4), we get

$$G_n(x) = (-1)^{n-1}G_n(1-x). \tag{5}$$

The gamma and beta function are defined by the following definite integrals: for $(\alpha > 0, \beta > 0)$,

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt \tag{6}$$

and

$$\begin{aligned} B(\alpha,\beta) &= \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt \\ &= \int_0^{\infty} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}dt \end{aligned} \tag{7}$$

(see [15,23,24]).Thus by (6) and (7), we get

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{8}$$

The classical Genocchi numbers, a sequence of integers introduced by Angelo Genocchi (1817-1889), have been studied in various context in such diverse areas of mathematics and physics as number theory, combinatorics, complex analysis, topology, and quantum physics. In recent years, Genocchi polynomials and numbers have received considerable attention and many researchers have worked on them, their extensions and their connections with some combinatorial counting.

The degenerate Bernoulli polynomials, the rst degenerate version of well-known families of polynomials, were introduced by Carlitz and rediscovered by Ustinov under the name of Korobov polynomials of the second kind. On the other hand, Korobov polynomials (of the rst kind) are the degenerate version of the Bernoulli polynomials of the second kind. Recently, many researchers began to study various kinds of degenerate versions of the familiar polynomials like Bernoulli, Euler, Genocchi, falling factorial and Bell polynomials by using generating functions, umbral calculus, and p-adic integrals.

The goal of this paper is to introduce the modified degenerate Genocchi polynomials and numbers, a degenerate version of the classical Genocchi polynomials and numbers, in order to study their properties and obtain several new and interesting identities involving them. More precisely, we give some properties, explicit formulas, several identities, a connection with Genocchi polynomials, and some integral formulas. Here they were named as the modified degenerate Genocchi polynomials, since there existed what are called the degenerate Genocchi polynomials whose definition is slightly different from ours (see [1, 4-6, 8, 11-16, 18, 20, 21, 22-26, 28]).

2. Modified degenerate Genocchi polynomials

First, we note that

$$e^t = \lim_{\lambda\to 0}(1+\lambda)^{\frac{t}{\lambda}}, \qquad t = loge^t = \lim_{\lambda\to 0} log(1+\lambda)^{\frac{t}{\lambda}} = \lim_{\lambda\to 0}\frac{t}{\lambda}log(1+\lambda). \tag{9}$$

From (1) and (9), we define the modified degenerate Genocchi polynomials as

$$\frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}(1+\lambda)^{\frac{tx}{\lambda}} = \sum_{n=0}^{\infty} g_{n,\lambda}(x)\frac{t^n}{n!} \tag{10}$$

When $x = 0$, $g_{n,\lambda} = g_{n,\lambda}(0)$ are called the modified degenerate Genocchi numbers. From (10), we get

$$\begin{aligned} 2t &= \left((1+\lambda)^{\frac{t}{\lambda}}+1\right)\left(\frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}\right) \\ &= \frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}(1+\lambda)^{\frac{t}{\lambda}} + \frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1} \\ &= \sum_{n=0}^{\infty} g_{n,\lambda}(1)\frac{t^n}{n!} + \sum_{n=0}^{\infty} g_{n,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}(g_{n,\lambda}(1)+g_{n,\lambda})\frac{t^n}{n!}. \end{aligned} \tag{11}$$

By comparing the coefficients on both sides of (11), we get

$$\begin{cases} g_{0,\lambda} = 0 \\ g_{n,\lambda}(1) + g_{n,\lambda} = 2\delta_{1,n}, \end{cases} \tag{12}$$

where $\delta_{1,n}$ is the Kronecker delta. From (10), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} g_{n,\lambda}(x)\frac{t^n}{n!} &= \left(\sum_{m=0}^{\infty} g_{m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}\left(\frac{log(1+\lambda)}{\lambda}\right)^m x^m\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}g_{n-m,\lambda}\left(\frac{log(1+\lambda)}{\lambda}\right)^m x^m\right)\frac{t^n}{n!}. \end{aligned} \tag{13}$$

Thus, by comparing the coefficients on both sides of (13), we obtain the following theorem.

Theorem 2.1. *Let $n \in \mathbb{N}\cup\{0\}$. Then we have*

$$g_{n,\lambda}(x) = \sum_{m=0}^{n}\binom{n}{m}g_{n-m,\lambda}\left(\frac{log(1+\lambda)}{\lambda}\right)^m x^m. \tag{14}$$

From (10), we derive the following equation:

$$\begin{aligned} \sum_{n=0}^{\infty} g_{n,\lambda}(x)\frac{t^n}{n!} &= -\frac{-2t}{(1+\lambda)^{\frac{-t}{\lambda}}+1}(1+\lambda)^{\frac{-(1-x)t}{\lambda}} \\ &= \sum_{n=0}^{\infty}(-1)^{n-1}g_{n,\lambda}(1-x)\frac{t^n}{n!}. \end{aligned} \tag{15}$$

By comparing the coefficients on both sides of (15),

$$g_{n,\lambda}(x) = (-1)^{n-1}g_{n,\lambda}(1-x) \quad (n \geq 0). \tag{16}$$

By (10), we see that

$$\frac{d}{dx}g_{n,\lambda}(x) = g_{n-1,\lambda}(x)\left(\frac{log(1+\lambda)}{\lambda}\right)n \quad (n \geq 1). \tag{17}$$

From (17), we get

$$\begin{aligned} \frac{g_{n+1,\lambda}(1) - g_{n+1,\lambda}}{n+1} &= \int_0^1 \frac{d}{dx}\frac{g_{n+1,\lambda}(x)}{n+1}dx \\ &= \int_0^1 g_{n,\lambda}(x)\left(\frac{log(1+\lambda)}{\lambda}\right)dx. \quad (n \geq 1). \end{aligned} \tag{18}$$

By (18), we obtain the following theorem.

Theorem 2.2. *Let $n \in \mathbb{N} \cup \{0\}$. Then we have*

$$\frac{g_{n+1,\lambda}(1) - g_{n+1,\lambda}}{n+1} = \int_0^1 g_{n,\lambda}(x)\left(\frac{log(1+\lambda)}{\lambda}\right)dx. \tag{19}$$

We note that the Stirling numbers of the first kind are defined as

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \qquad (n \geq 0), \qquad (\text{see } [1, 4-6, 8, 11-16, 18, 20, 21, 22-26, 28]). \tag{20}$$

where $(x)_n = x(x-1)\cdots(x-n+1)(n \geq 1)$, and $(x)_0 = 1$. By (10), we see that

$$\begin{aligned} &\frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}(1+\lambda)^{\frac{tx}{\lambda}} \\ = &\left(\sum_{k=0}^{\infty} g_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{tx}{\lambda}\right)_m \lambda^m\right) \\ = &\left(\sum_{k=0}^{\infty} g_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m} S_1(m,l)\left(\frac{tx}{\lambda}\right)^l\right)\frac{\lambda^m}{m!} \\ = &\left(\sum_{k=0}^{\infty} g_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\left(\sum_{m=l}^{\infty} S_1(m,l)\left(\frac{x}{\lambda}\right)^l\frac{\lambda^m}{m!}l!\right)\frac{t^l}{l!}\right) \\ = &\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=l}^{\infty}\binom{n}{l}g_{n-l,\lambda}S_1(m,l)\left(\frac{x}{\lambda}\right)^l\frac{\lambda^m}{m!}l!\right)\frac{t^n}{n!} \end{aligned} \tag{21}$$

From (21), we obtain the following theorem.

Theorem 2.3. *Let $n \in \mathbb{N} \cup \{0\}$. Then we have*

$$g_{n,\lambda}(x) = \sum_{l=0}^{n}\sum_{m=l}^{\infty}\binom{n}{l}g_{n-l,\lambda}S_1(m,l)\left(\frac{x}{\lambda}\right)^l\frac{\lambda^m}{m!}l!. \tag{22}$$

Let d be an odd integer. Then we see that

$$\begin{aligned} &2t\sum_{l=0}^{d-1}(-1)^l(1+\lambda)^{\frac{lt}{\lambda}} \\ = &\frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}}\left(1-\left(-(1+\lambda)^{\frac{t}{\lambda}}\right)^d\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}}\left(1+(1+\lambda)^{\frac{dt}{\lambda}}\right)\\
&= \frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}} + \frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}}(1+\lambda)^{\frac{dt}{\lambda}}\\
&= \sum_{n=1}^{\infty} g_{n,\lambda}\frac{t^n}{n!} + \sum_{n=1}^{\infty} g_{n,\lambda}(d)\frac{t^n}{n!}\\
&= \sum_{n=1}^{\infty}(g_{n,\lambda}+g_{n,\lambda}(d))\frac{t^n}{n!}\\
&= t\sum_{n=0}^{\infty}\left(\frac{g_{n+1,\lambda}+g_{n+1,\lambda}(d)}{n+1}\right)\frac{t^n}{n!}. \qquad (23)
\end{aligned}
$$

Also, we see that

$$
\begin{aligned}
& 2\sum_{l=0}^{d-1}(-1)^l(1+\lambda)^{\frac{lt}{\lambda}}\\
&= 2\sum_{l=0}^{d-1}\left(\sum_{n=0}^{\infty}(-1)^l\left(\frac{log(1+\lambda)}{\lambda}\right)^n l^n\right)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(2\sum_{l=0}^{d-1}(-1)^l\left(\frac{log(1+\lambda)}{\lambda}\right)^n l^n\right)\frac{t^n}{n!}. \qquad (24)
\end{aligned}
$$

From (23) and (24), we obtain the following theorem.

Theorem 2.4. *Let $n \in \mathbb{N}\cup\{0\}$. Then we have*

$$
2\sum_{l=0}^{d-1}(-1)^l\left(\frac{log(1+\lambda)}{\lambda}\right)^n l^n = \frac{g_{+1n,\lambda}+g_{n+1,\lambda}(d)}{n+1}. \qquad (25)
$$

From (10) and (14), we note that

$$
\begin{aligned}
\int_0^1 y^n g_{n,\lambda}(x+y)dy &= \sum_{m=0}^{n}\binom{n}{m}g_{n-m,\lambda}\left(\frac{log(1+\lambda)}{\lambda}\right)^m\int_0^1 y^{n+m}dy\\
&= \sum_{m=0}^{n}\binom{n}{m}\frac{g_{n-m,\lambda}(x)}{n+m+1}\left(\frac{log(1+\lambda)}{\lambda}\right)^m
\end{aligned} \qquad (26)
$$

By (16), we get

$$
\begin{aligned}
& \int_0^1 y^n g_{n,\lambda}(x+y)dy = (-1)^{n-1}\int_0^1 y^n g_{n,\lambda}(1-(x+y))dy\\
&= (-1)^{n-1}\sum_{m=0}^{n}\binom{n}{m}g_{n-m}(-x)\left(\frac{log(1+\lambda)}{\lambda}\right)^m\int_0^1 y^n(1-y)^m dy\\
&= \sum_{m=0}^{n}\binom{n}{m}(-1)^m g_{n-m,\lambda}(1+x)\left(\frac{log(1+\lambda)}{\lambda}\right)^m B(n+1,m+1)\\
&= \sum_{m=0}^{n}\binom{n}{m}(-1)^m\frac{g_{n-m,\lambda}(1+x)}{n+m+1}\left(\frac{log(1+\lambda)}{\lambda}\right)^m\binom{n+m}{m}^{-1} \qquad (27)
\end{aligned}
$$

By (26) and (27), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} & \sum_{m=0}^{n} \binom{n}{m} \frac{g_{n-m,\lambda}(x)}{n+m+1} \left(\frac{log(1+\lambda)}{\lambda} \right)^m \\ = & \sum_{m=0}^{n} \binom{n}{m} (-1)^m \frac{g_{n-m,\lambda}(1+x)}{n+m+1} \left(\frac{log(1+\lambda)}{\lambda} \right)^m \binom{n+m}{m}^{-1} \end{aligned} \tag{28}$$

From (17), we note that

$$\begin{aligned} & \int_0^1 y^n g_{n,\lambda}(x+y)dy \\ = & \frac{g_{n,\lambda}(x+1)}{n+1} - \frac{n}{n+1} \frac{log(1+\lambda)}{\lambda} \int_0^1 y^{n+1} g_{n-1,\lambda}(x+y)dy \\ = & \frac{g_{n,\lambda}(x+1)}{n+1} - \frac{g_{n-1,\lambda}(x+1)}{n+1} \frac{n}{n+2} \frac{log(1+\lambda)}{\lambda} \\ & +(-1)^2 \frac{n(n-1)}{(n+1)(n+2)} \left(\frac{log(1+\lambda)}{\lambda} \right)^2 \int_0^1 y^{n+2} g_{n-2,\lambda}(x+y)dy \\ = & \frac{g_{n,\lambda}(x+1)}{n+1} - \frac{g_{n-1,\lambda}(x+1)}{n+1} \frac{n}{n+2} \frac{log(1+\lambda)}{\lambda} \\ & +(-1)^2 \frac{g_{n-2,\lambda}(x+1)}{n+1} \frac{n(n-1)}{(n+2)(n+3)} \left(\frac{(log(1+\lambda)}{\lambda} \right)^2 \\ & +(-1)^3 \frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)} \left(\frac{(log(1+\lambda)}{\lambda} \right)^3 \int_0^1 y^{n+3} g_{n-3,\lambda}(x+y)dy \end{aligned} \tag{29}$$

By continuing this process, we have

$$\begin{aligned} \int_0^1 y^n g_{n,\lambda}(x+y)dy = & \frac{g_{n,\lambda}(x+1)}{n+1} \\ & + \sum_{m=1}^{n-1} (-1)^m \frac{n(n-1)\cdots(n-m+1)}{(n+1)(n+2)\cdots(n+m+1)} \left(\frac{(log(1+\lambda)}{\lambda} \right)^m g_{n-m,\lambda}(x+1) \end{aligned} \tag{30}$$

Therefore by (26) and (30), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{N}$, we have*

$$\sum_{m=0}^{n} \binom{n}{m} \frac{g_{n-m,\lambda}(x)}{n+m+1} = \sum_{m=0}^{n-1} (-1)^m \frac{\binom{n}{m}}{\binom{n+m}{m}} \frac{g_{n-m,\lambda}(x+1)}{n+m+1} \left(\frac{(log(1+\lambda)}{\lambda} \right)^m \tag{31}$$

Taking $x = 0$, From (16) and (31), we obtain the following corollary.

Corollary 2.7. *For $n \in \mathbb{N}$, we have*

$$\sum_{m=0}^{n} \binom{n}{m} \frac{g_{n-m,\lambda}}{n+m+1} = \sum_{m=0}^{n-1} (-1)^m \frac{\binom{n}{m}}{\binom{n+m}{m}} \frac{g_{n-m,\lambda}(1)}{n+m+1} \left(\frac{(log(1+\lambda)}{\lambda} \right)^m \tag{32}$$

For $n \in \mathbb{N}$, we observe that

$$\int_0^1 y^n g_{n,\lambda}(x+y)dy$$

$$
\begin{aligned}
&= \frac{\lambda}{log(1+\lambda)}\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{\lambda}{log(1+\lambda)}\frac{n}{n+1}\int_0^1 y^{n-1}g_{n+1,\lambda}(x+y)dy \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n-1}(-1)^n g_{n+1,\lambda}(1-(x+y))dy\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}g_{n+1-l,\lambda}(-x)(-1)^n\int_0^1 y^{n-1}(1-y)^l dy\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}g_{n+1-l,\lambda}(-x)(-1)^n B(n,l+1)\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{1}{n+1}\sum_{l=0}^{n+1}\frac{\binom{n+1}{l}}{\binom{n+l}{l}}(-1)^n g_{n+1-l,\lambda}(-x)\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{1}{n+1}\sum_{l=0}^{n+1}\frac{\binom{n+1}{l}}{\binom{n+l}{l}}(-1)^l g_{n+1-l,\lambda}(1+x)\right)
\end{aligned}
$$

(33)

Therefore, by (30) and (33), we obtain the following theorem.

Theorem 2.8. *For $n \in \mathbb{N}$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n-1}(-1)^l\frac{\binom{n}{l}}{\binom{n+l}{l}}\frac{g_{n-l,\lambda}(1+x)}{n+l+1}\left(\frac{log(1+\lambda)}{\lambda}\right)^{l+1} \\
&= \frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{1}{n+1}\sum_{l=0}^{n+1}(-1)^l\frac{\binom{n+1}{l}}{\binom{n+l}{l}}g_{n+1-l,\lambda}(1+x) \qquad (34)
\end{aligned}
$$

Replacing λ by $e-1$ and t by $(e-1)t$ in (10), we get

$$
\begin{aligned}
\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} &= \frac{2t}{e^t+1}e^{xt} \\
&= \sum_{n=0}^{\infty}g_{n,e-1}(x)(e-1)^{n-1}\frac{t^n}{n!}, \qquad (35)
\end{aligned}
$$

where $G_n(x)$ are the Genocchi polynomials. By comparing both sides of (35), we obtain the following theorem.

Theorem 2.9. *For $n \in \mathbb{N}\cup\{0\}$, we have*

$$
G_n(x) = g_{n,e-1}(x)(e-1)^{n-1}. \qquad (36)
$$

By (12) and (18), we get

$$
\begin{aligned}
\int_0^1 g_{n,\lambda}(x)dx &= \frac{\lambda}{log(1+\lambda)}(n+1)^{-1}\int_0^1 \frac{d}{dx}g_{n+1,\lambda}(x)dx \\
&= \frac{\lambda}{log(1+\lambda)}(n+1)^{-1}(g_{n+1,\lambda}(1) - g_{n+1,\lambda}(0)) \\
&= \frac{(-2)\lambda}{log(1+\lambda)}(n+1)^{-1}g_{n+1,\lambda} \qquad (37)
\end{aligned}
$$

where $n \in \mathbb{N}$. Also, we have

$$\begin{aligned}
&\int_0^1 g_{n,\lambda}(x)g_{m,\lambda}(x)dx \\
= \ & \frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}g_{n+1,\lambda}(x)g_{m,\lambda}(x)\mid_0^1 - \frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}\int_0^1 g_{n+1,\lambda}(x)\frac{d}{dx}g_{m,\lambda}(x)dx \\
= \ & \frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}(g_{n+1,\lambda}(1)g_{m,\lambda}(1) - g_{n+1,\lambda}(0)g_{m,\lambda}(0) \\
& -\frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}\frac{log(1+\lambda)}{\lambda}m\int_0^1 g_{n+1,\lambda}(x)g_{m-1,\lambda}(x)dx \\
= \ & -\frac{m}{n+1}\int_0^1 g_{n+1,\lambda}(x)g_{m-1,\lambda}(x)dx \\
= \ & (-1)^2\frac{m(m-1)}{(n+1)(n+2)}\int_0^1 g_{n+2,\lambda}(x)g_{m-2,\lambda}(x)dx
\end{aligned} \tag{38}$$

By continuing this process, we obtain the following theorem.

Theorem 2.10. *For $m, n \in \mathbb{N}$, we have*

$$\begin{aligned}
&\int_0^1 g_{n,\lambda}(x)g_{m,\lambda}(x)dx \\
= \ & (-1)^{m-2}\frac{m(m-1)\cdots 3}{(n+1)(n+2)\cdots(n+m-2)}\int_0^1 g_{n+m-2,\lambda}(x)g_{2,\lambda}(x)dx.
\end{aligned} \tag{39}$$

Now, we have

$$\begin{aligned}
&\int_0^1 g_{n+m-2,\lambda}(x)g_{2,\lambda}(x)dx \\
= \ & -\frac{2}{n+m-1}\int_0^1 g_{n+m-1,\lambda}(x)g_{1,\lambda}(x)dx \\
= \ & -\frac{2}{n+m-1}\frac{g_{n+m,\lambda}(x)}{n+m}\frac{\lambda}{log(1+\lambda)}\mid_0^1 \\
= \ & -\frac{2}{n+m-1}\frac{\lambda}{log(1+\lambda)}\frac{-2g_{n+m,\lambda}}{n+m}.
\end{aligned} \tag{40}$$

By (41), we obtain the following theorem.

Theorem 2.11. *For $m, n \in \mathbb{N}$, we have*

$$\begin{aligned}
&\int_0^1 g_{n,\lambda}(x)g_{m,\lambda}(x)dx \\
= \ & (-1)^m 2\binom{n+m}{m}^{-1}\frac{\lambda}{log(1+\lambda)}g_{n+m,\lambda}.
\end{aligned} \tag{41}$$

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Hesitant fuzzy implicative filters in BE-algebras

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Abstract. The notion of hesitant fuzzy implicative filter of a BE-algebra is introduced and related properties are investigated. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy implicative filter. Also, as a generalization of hesitant fuzzy implicative filter, we consider the hesitant fuzzy n-fold implicative filter. Characterizations of hesitant fuzzy n-fold implicative filter are discussed.

1. Introduction

In 2007, Kim and Kim [5] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Song et al. [8] considered the fuzzification of ideals in BE-algebras. They introduced the notion of fuzzy ideals in BE-algebras, and investigated related properties. They gave characterizations of a fuzzy ideal in BE-algebras.

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [9] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [11, 12, 13, 14, 15]), and is applied to residuated lattices and MTL-algebras (see [4, 6]).

In this paper, we introduce the notion of hesitant fuzzy implicative filter of a BE-algebra, and investigate some properties of it. We consider characterizations of hesitant fuzzy implicative filter of a BE-algebra. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy implicative filter. Also, as a generalization of hesitant fuzzy implicative filter, we consider the hesitant fuzzy n-fold implicative filter. We discuss characterizations of hesitant fuzzy n-fold implicative filter.

2. Preliminaries

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[0]**Keywords**: filter; implicative filter; hesitant fuzzy (implicative) filter.

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By a *BE-algebra* ([5]) we mean a system $(X;*,1)$ of type $(2,0)$ which the following axioms hold:

(2.1) $(\forall x\in X)\,(x*x=1)$,
(2.2) $(\forall x\in X)\,(x*1=1)$,
(2.3) $(\forall x\in X)\,(1*x=x)$,
(2.4) $(\forall x,y,z\in X)\,(x*(y*z)=y*(x*z)$ (exchange).

We introduce a relation " $\leq$ " on X by $x\leq y$ if and only if $x*y=1$.

A BE-algebra $(X;*,1)$ is said to be *transitive* ([5]) if it satisfies: for any $x,y,z\in X$, $y*z\leq (x*y)*(x*z)$. A BE-algebra $(X;*,1)$ is said to be *self distributive* ([5]) if it satisfies: for any $x,yz\in X$, $x*(y*z)=(x*y)*(x*z)$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general ([5]).

Every self distributive BE-algebra $(X;*,1)$ satisfies the following properties:

(2.5) $(\forall x,y,z\in X)\,(x\leq y\Rightarrow z*x\leq z*y$ and $y*z\leq x*z)$,
(2.6) $(\forall x,y\in X)\,(x*(x*y)=x*y)$,
(2.7) $(\forall x,y,z\in X)\,(x*y\leq (z*x)*(z*y))$,

Definition 2.1.([5]) Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* of X if

(F1) $1\in F$;
(F2) $(\forall x,y\in X)(x*y,x\in F\Rightarrow y\in F)$.

F is an *implicative filter* of X if it satisfies (F1) and

(F3) $(\forall x,y,z\in X)(x*(y*z),x*y\in F\Rightarrow x*z\in F)$.

Definition 2.2.([9]) Let E be a reference set. A *hesitant fuzzy set* on E is defined in terms of a function that when applied to E returns a subset of $[0,1]$, which can be viewed as the following mathematical representation:

$$H_E:=\{(e,h_E(e))|e\in E\}$$

where $h_E:E\to\mathscr{P}([0,1])$.

Definition 2.3. Given a non-empty subset A of X, a hesitant fuzzy set

$$H_X:=\{(x,h_X(x))|x\in X\}$$

on satisfying the following condition:

$$h_X(x)=\emptyset \text{ for all } x\notin A \tag{2.8}$$

is called a *hesitant fuzzy set related to* A (briefly, A-*hesitant fuzzy set*) on X, and is represented by $H_A:=\{(x,h_A(x))\mid x\in X\}$, where h_A is a mapping from X to $\mathscr{P}([0,1])$ with $h_A(x)=\emptyset$ for all $x\notin A$.

For a hesitant set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of a BE-algebra X and a subset γ of $[0,1]$, the hesitant fuzzy γ-inclusive set of H_X, denoted by $H_X(\gamma)$, is defined to be the set

$$H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}.$$

For any hesitant fuzzy set $H_X = \{(x, h_X(x)|x \in X\}$ and $G_X = \{(x, g_X(x))|x \in X\}$, we call H_X a *hesitant fuzzy subset* of G_X, denoted by $H_X \widetilde{\subseteq} G_X$, if $h_X(x) \subseteq g_X(x)$ for all $x \in X$. The *hesitant fuzzy union* of H_X and G_X, denoted by $H_X \widetilde{\cup} G_X$, is defined to be the hesitant fuzzy set $(h_X \widetilde{\cup} g_X)(x) = h_X(x) \cup g_X(x)$ for all $x \in X$. The *hesitant fuzzy intersection* of H_X and G_X, denoted by $H_X \widetilde{\cap} G_X$, is defined to be the hesitant fuzzy set $(h_X \widetilde{\cap} g_X)(x) = h_X(x) \cap g_X(x)$ for all $x \in X$.

3. Hesitant fuzzy implicative filters

Definition 3.1.([3]) Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy filter of X related to A* (briefly, *A-hesitant fuzzy filter* of X) if it satisfies the following condition:

$$(\forall x \in A)\,(h_A(x) \subseteq h_A(1))\,, \tag{3.1}$$

$$(\forall x, y \in A)\,(h_A(x * y) \cap h_A(x) \subseteq h_A(y))\,. \tag{3.2}$$

An A-hesitant fuzzy filter of X with $A = X$ is called a *hesitant fuzzy filter* of X.

Proposition 3.2.([3]) *Let $H_A := \{(x, h_A(x))|x \in X\}$ be an A-hesitant fuzzy filter of X where A is a subalgebra of X. Then the following assertions are valid.*

(i) $(\forall x, y \in A)(x \leq y \Rightarrow h_A(x) \subseteq h_A(y))$,
(ii) $(\forall x, y, z \in A)(h_A(x * (y * z)) \cap h_A(y) \subseteq h_A(x * z))$,
(iii) $(\forall a, x \in A)(h_A(a) \subseteq h_A((a * x) * x)$.

Definition 3.3. Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy implicative filter of X related to A* (briefly, *A-hesitant fuzzy implicative filter* of X) if it satisfies (3.1) and

$$(\forall x, y, z \in A)\,(h_A(x * (y * z)) \cap h_A(x * y) \subseteq h_A(x * z))\,. \tag{3.3}$$

An A-hesitant fuzzy implicative filter of X with $A = X$ is called a *hesitant fuzzy implicative filter* of X.

Example 3.4. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	1	a
d	1	1	a	1	1	a
0	1	1	1	1	1	1

For a subalgebra $A = \{1, a, b\}$ of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X defined by

$$H_A = \left\{(1, [0, 1]), (a, (0, \tfrac{1}{2}]), (b, (0, \tfrac{1}{2})), (c, (0, \tfrac{1}{4})), (d, \emptyset), (0, \emptyset)\right\}$$

It is easy to check that H_A is an A-hesitant fuzzy implicative filter of X.

Proposition 3.5. *Every A-hesitant fuzzy implicative filter of a BE-algebra X is an A-hesitant fuzzy filter of X.*

Proof. Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy implicative filter of X. It follows from (2.4) and (3.3) that

$$\begin{aligned} h_A(y * (x * z)) \cap h_A(x * y) =& h_A(x * (y * z)) \cap h_A(x * y) \\ \subseteq& h_A(x * z) \end{aligned} \tag{3.4}$$

for any $x, y, z \in X$. Setting $x := 1$ in (3.4), we have $h_A(y * z) \cap h_A(y) \subseteq h_A(z)$. Therefore $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy filter of X. □

The converse of Proposition may not be true in general (see Example 3.6).

Example 3.6. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra as in Example 3.4. Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined as follows:

$$h_X : X \to \mathscr{P}([0, 1]),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of $[0, 1]$ with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that H_X is a hesitant fuzzy filter of X. But it is not a hesitant fuzzy implicative filter of X, since $h_X(d*(a*0)) \cap h_X(d*a) = \gamma_2 \nsubseteq \gamma_1 = h_X(d*0)$.

We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy implicative filter.

Proposition 3.7. *Let X be a self distributive BE-algebra. Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy filter of X satisfying*

$$(\forall x, y, z \in X)(h_X(x * (y * (y * z))) \cap h_X(y * x)) \subseteq h_X(y * z). \tag{3.5}$$

Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X.

Proof. Since $x * (y * z) = y * (x * z) \leq (x * y) * (x * (x * z)) = x * (y * (x * z)) = y * (x * (x * z))$ for all $x, y \in X$, we have $h_X(x * (y * z)) \subseteq h_X(y * (x * (x * z)))$ by Proposition 3.2(i). It follows from (3.5) that $h_X(x * z) \supseteq h_X(y * (x * (x * z)) \cap h_X(x * y) \supseteq h_X(x * (y * z)) \cap h_X(x * y)$. Thus $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

Theorem 3.8. *Let X be a transitive BE-algebra. For any hesitant fuzzy filter $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X, the following are equivalent:*

(i) $H_X := \{(x, h_X(x)) \mid x \in X\}$ *is a hesitant fuzzy implicative filter,*
(ii) $(\forall x, y \in X)\,(h_X(x * (x * y)) \subseteq h_X(x * y))$,
(iii) $(\forall x, y, z \in X)\,(h_X(x * (y * z)) \subseteq h_X((x * y) * (x * z)))$.

Proof. (i)⇒(ii) Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. Setting $z := y, y := x$ in (3.3), we get

$$\begin{aligned} h_X(x * y) \supseteq & h_X(x * (x * y)) \cap h_X(x * x) \\ = & h_X(x * (x * y)) \cap h_X(1) \\ = & h_X(x * (x * y)). \end{aligned}$$

Hence (ii) holds.
(ii)⇒(iii) Suppose that (ii) holds. Since $x*(y*z) \leq x*((x*y)*(x*z)) = x*(x*((x*y)*z))$, by Proposition 3.2(i) we have $h_X(x*((x*y)*(x*z))) = h_X(x*(x*((x*y)*z))) \supseteq h_X(x*(y*z))$. It follows from (ii) that

$$\begin{aligned} h_X((x * y) * (x * z)) = & h_X(x * ((x * y) * z)) \\ \supseteq & h_X(x * (x * ((x * y) * z))) \\ \supseteq & h_X(x * (y * z)). \end{aligned}$$

Thus (iii) holds.
(iii)⇒(ii) Assume that (iii) holds. By (3.2) and (iii), we have

$$\begin{aligned} h_X(x * z) \supseteq & h_X((x * y) * (x * z)) \cap h_X(x * y) \\ \supseteq & h_X(x * (y * z)) \cap h_X(x * y). \end{aligned}$$

Therefore $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

Theorem 3.9. *Let X be a self distributive BE-algebra. Then the hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X is a hesitant fuzzy implicative filter of X if and only if it is a hesitant fuzzy filter of X.*

Proof. By Proposition 3.5, every hesitant fuzzy implicative filter of X is a hesitant fuzzy filter of X.

Conversely, assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy filter of X. For any $x, y, z \in X$, by (3.2) we have

$$\begin{aligned} h_X(x * z) \supseteq & h_X((x * y) * (x * z)) \cap h_X(x * y) \\ = & h_X(x * (y * z)) \cap h_X(x * y). \end{aligned}$$

Hence $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

For any element x and y of a BE-algebra X and positive integer n, let $x^n * y$ denote $x * (\cdots * (x * (x * y)) \cdots)$ in which x occurs n times, and $x^0 * y = 1$.

Definition 3.10. Let X be a BE-algebra and let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X. Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is called a *hesitant fuzzy n-fold implicative filter* of X if it satisfies (3.1) and

(3.6) $(\forall x, y, z \in X) \left(h_X(x^n * (y * z)) \cap h_X(x^n * y)\right) \subseteq h_X(x^n * z))$.

Note that a hesitant fuzzy 1-fold implicative filter of X is a hesitant fuzzy implicative filter of X.

Example 3.11. Let $X := \{1, a, b, c, d, 0\}$ is a transitive BE-algebra ([11]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined as follows:

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x \in \{1, b, c\} \\ \gamma_1 & \text{if } x \in \{a, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy n-fold implicative filter of X.

Theorem 3.12. *Every hesitant n-fold fuzzy implicative filter of X is a hesitant fuzzy filter of X.*

Proof. Taking $x := 1$ in (3.6) and (2.3), we have $h_X(z) \supseteq h_X(y * z) \cap h_X(y)$. Hence $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy filter of X. □

The converse of Theorem 3.12 may not be not true in general (see Example 3.13).

Example 3.13. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra as in Example 3.11. Let H_X be a hesitant fuzzy set on X defined as follows:

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that H_X is a hesitant fuzzy filter of X. But it is not a hesitant fuzzy 1-fold implicative filter of X, since $h_X(d * c) = h_X(b) = \gamma_1 \not\supseteq \gamma_2 = h_X(1) = h_X(d * (b * c)) \cap h_X(d * b)$.

Theorem 3.14. *Let X be a transitive BE-algebra. For any hesitant fuzzy filter $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X, the following are equivalent:*

(i) *$H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy n-fold implicative filter,*
(ii) $(\forall x, y \in X)\,(h_X(x^{n+1} * y) \subseteq h_X(x^n * y))$,
(iii) $(\forall x, y, z \in X)\,(h_X(x^n * (y * z)) \subseteq h_X((x^n * y) * (x^n * z)))$.

Proof. (i)⇒(ii) Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy n-fold implicative filter of X. Setting $z := y, y := x$ in (3.6), we have

$$\begin{aligned} h_X(x^n * y) \supseteq& h_X(x^n * (x * y)) \cap h_X(x^n * x) \\ =& h_X(x^{n+1} * y) \cap h_X(1) \\ =& h_X(x^{n+1} * y). \end{aligned}$$

Hence (ii) holds.
(ii)⇒(iii) Suppose that (ii) holds. Since $x^n * (y * z) \leq x^n * ((x^n * y) * (x^n * z))$, we have $h_X(x^n * ((x^n * y) * (x^n * z))) \supseteq h_X(x^n * (y * z))$. Since $x^{n+1} * (x^{n-1} * ((x^n * y) * z)) = x^n * (x^n * ((x^n * y) * z))) = x^n * ((x^n * y)) * (x^n * z))$ and using (ii), we have

$$\begin{aligned} h_X(x^{n+1} * (x^{n-2} * ((x^n * y) * z)) =& h_X(x^n * (x^{n-1} * ((x^n * y) * z)) \\ \supseteq& h_X(x^{n+1} * (x^{n-1} * ((x^n * y) * z))) \\ =& h_X(x^n * ((x^n * y) * (x^n * z))) \\ \supseteq& h_X(x^n * (y * z)). \end{aligned} \tag{3.7}$$

By (ii) and (3.7), we have

$$\begin{aligned} h_X(x^{n+1} * (x^{n-3} * ((x^n * y) * z))) =& h_X(x^n * (x^{n-2} * ((x^n * y) * z))) \\ \supseteq& h_X(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) \\ \supseteq& h_X(x^n * (y * z)). \end{aligned}$$

Continuing this process, we conclude that

$$\begin{aligned} h_X((x^n * y) * (x^n * z)) =& h_X(x^n * ((x^n * y) * z)) \\ \supseteq& h_X(x^n * (y * z)). \end{aligned}$$

(iii)⇒(i) Let $x, y, z \in X$. It follows from (iii) that

$$\begin{aligned} h_X(x^n * z) \supseteq & h_X((x^n * y) * (x^n * z)) \cap h_X(x^n * y) \\ \supseteq & h_X((x^n * (y * z)) \cap h_X(x^n * y). \end{aligned}$$

Hence $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant n-fold fuzzy implicative filter □

Definition 3.15. Let n be a positive integer. A BE-algebra X is said to be *n-fold implicative* if it satisfies the equality $x^{n+1} * y = x^n * y$ for all $x, y \in X$.

Corollary 3.16. *In an n-fold implicative BE-algebra, the notion of hesitant fuzzy filters and hesitant fuzzy n-fold implicative filters coincide.*

Proof. Straightforward. □

Theorem 3.17. *A hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of a BE-algebra X is a hesitant fuzzy implicative filter of X if and only if the hesitant fuzzy γ-inclusive set $H_X(\gamma)$ is an implicative filter of X for all $\gamma \in \mathscr{P}([0,1])$ with $H_X(\gamma) \neq \emptyset$.*

The filter $H_X(\gamma)$ in Theorem 3.17 is called the *γ-inclusive filter* of X.

Proof. Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. Let $x, y, z \in X$ and $\gamma \in \mathscr{P}([0,1])$ be such that $x * (y * z) \in H_X(\gamma)$ and $x * y \in H_X(\gamma)$. Then $\gamma \subseteq h_X(x * (y * z))$ and $\gamma \subseteq h_X(x * y)$. Using (3.1) and (3.3), we have $\gamma \subseteq h_X(1)$ and $\gamma \subseteq h_X(x * (y * z) \cap h_X(x * y) \subseteq h_X(x * z)$ for $x, y, z \in X$. Hence $1 \in H_X(\gamma)$ and $x * z \in H_X(\gamma)$. Thus $H_X(\gamma)$ is an implicative filter of X.

Conversely, suppose that $H_X(\gamma)$ is an implicative filter of X for all $\gamma \in \mathscr{P}([0,1])$ with $H_X(\gamma) \neq \emptyset$. For any $x \in X$, let $h_X(x) = \gamma$. Since $H_X(\gamma)$ is an implicative filter of X, we have $1 \in H_X(\gamma)$ and so $h_X(x) = \gamma \subseteq h_X(1)$. For any $x, y, z \in X$, let $h_X(x*(y*z)) = \gamma_{x*(y*z)}$ and $h_X(x*y) = \gamma_{x*y}$. Take $\gamma = \gamma_{x*(y*z)} \cap \gamma_{x*y}$. Then $x * (y * z) \in H_X(\gamma)$ and $x * y \in H_X(\gamma)$ which imply that $x * z \in H_X(\gamma)$. Hence

$$h_X(x * z) \supseteq \gamma = \gamma_{x*(y*z)} \cap \gamma_{x*y} = h_X(x * (y * z)) \cap h_X(x * y).$$

Thus $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

Theorem 3.18. *Every hesitant fuzzy implicative filter of a BE-algebra can be represented as a hesitant fuzzy γ-inclusive set of a hesitant fuzzy implicative filter.*

Proof. Let F be an implicative filter of a BE-algebra X. For a subset γ of $[0,1]$, define a hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X by

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma & \text{if } x \in F, \\ \emptyset & \text{if } x \notin F. \end{cases}$$

Obviously, $F = H_X(\gamma)$. We now prove that H_X is a hesitant fuzzy implicative filter of X. Since $1 \in F = H_X(\gamma)$, we have $h_X(1) = \gamma \supseteq h_X(x)$ for all $x \in X$. Let $x, y, z \in X$. If $x*(y*z), x*y \in F$, then

$x*z \in F$ because F is an implicative filter of X. Hence $h_X(x*(y*z)) = h_X(x*y) = h_X(x*z) = \gamma$, and so $h_X(x*(y*z)) \cap h_X(x*y) \subseteq h_X(x*z)$. If $x*(y*z) \in F$ and $x*y \notin F$, then $h_X(x*(y*z)) = \gamma$ and $h_X(x*y) = \emptyset$ which imply that

$$h_X(x*(y*z)) \cap h_X(x*y) = \gamma \cap \emptyset = \emptyset \subseteq h_X(x*z).$$

Similarly, if $x*(y*z) \notin F$ and $x*y \in F$, then $h_X(x*(y*z)) \cap h_X(x*y) \subseteq h_X(x*z)$. Obviously, if $x*(y*z) \notin F$ and $x*y \notin F$, then $h_X(x*(y*z)) \cap h_X(x*y) \subseteq h_X(x*z)$. Therefore $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

For two elements a and b of X, consider a hesitant fuzzy set $H_X^{a,b} := \{(x, h_X^{a,b}x)) \mid x \in X\}$ where

$$h_X^{a,b} : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_1 & \text{if } a*(b*x) = 1, \\ \gamma_2 & \text{otherwise,} \end{cases}$$

where γ_1 and γ_2 are subsets of X with $\gamma_2 \subsetneq \gamma_1$.

There exist $a, b \in X$ such that $H_X^{a,b}$ is not a hesitant fuzzy implicative filter of X (see Example 3.19).

Example 3.19. Consider the BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given in Example 3.4. Then $H_X^{1,a}$ is not a hesitant fuzzy implicative filter of X since

$$h_X^{1,a}(1*(a*b)) \cap h_X^{1,a}(1*a) = \gamma_1 \nsubseteq h_X^{1,a}(1*b) = \gamma_2.$$

Now we provide a condition for the hesitant fuzzy set $H_X^{a,b}$ to be a hesitant fuzzy implicative filter of X for all $a, b \in X$.

Theorem 3.20. *If X is a self distributive BE-algebra, then the hesitant fuzzy set $H_X^{a,b}$ is a hesitant fuzzy implicative filter of X for all $a, b \in X$.*

Proof. Let $a, b \in X$. Obviously, $h_X^{a,b}(1) \supseteq h_X^{a,b}(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $a*(b*(x*(y*z))) \neq 1$ or $a*(b*(x*y)) \neq 1$. Then $h_X^{a,b}(x*(y*z)) = \gamma_2$ or $h_X^{a,b}(x*y) = \gamma_2$. Hence

$$h_X^{a,b}(x*(y*z)) \cap h_X^{a,b}(x*y) = \gamma_2 \subseteq h_X^{a,b}(x*z).$$

Assume that $a*(b*(x*(y*z)))=1$ and $a*(b*(x*y))=1$. Then

$$\begin{aligned}1&=a*(b*(x*(y*z)))\\&=a*(b*((x*y)*(x*z)))\\&=a*((b*(x*y))*(b*(x*z)))\\&=(a*(b*(x*y)))*(a*(b*(x*z)))\\&=1*(a*(b*(x*z)))\\&=a*(b*(x*z)),\end{aligned}$$

and so $h_X^{a,b}(x*(y*z))\cap h_X^{a,b}(x*y)=\gamma_1=h_X^{a,b}(x*z)$. Therefore $H_X^{a,b}$ is a hesitant fuzzy implicative filter of X for all $a,b\in X$. □

Theorem 3.21. *If H_X and G_X are hesitant fuzzy implicative filters of a BE-algebra X, then the hesitant fuzzy intersection $H_X\tilde{\cap}G_X$ of H_X and G_X is a hesitant fuzzy implicative filter of X.*

Proof. For any $x\in X$, we have

$$(h_X\tilde{\cap}g_X)(1)=h_X(1)\cap g_X(1)\supseteq h_X(x)\cap g_X(x)=(h_X\tilde{\cap}g_X)(x).$$

Let $x,y,z\in X$. Then

$$\begin{aligned}(h_X\tilde{\cap}g_X)(x*z)&=h_X(x*z)\cap g_X(x*z)\\&\supseteq(h_X(x*(y*z))\cap h_X(x*y))\cap(g_X(x*(y*z))\cap g_X(x*y))\\&=(h_X(x*(y*z))\cap g_X(x*(y*z)))\cap(h_X(x*y)\cap g_X(x*y))\\&=(h_X\tilde{\cap}g_X)(x*(y*z))\cap(h_X\tilde{\cap}g_X)(x*y).\end{aligned}$$

Hence $H_X\tilde{\cap}G_X$ is a hesitant fuzzy implicative filter of X. □

The hesitant fuzzy union of hesitant fuzzy implicative filters of a BE-algebra X may not be a hesitant fuzzy implicative filter of X as the following example.

Example 3.22. Let $X=\{1,a,b,c,d\}$ is a BE-algebra with the following Cayley table ([5]):

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let H_X and G_X be hesitant fuzzy sets of X defined, respectively, as follows:

$$h_X:X\to\mathscr{P}([0,1]),\ x\mapsto\begin{cases}\gamma_3&\text{if }x\in\{1,b\}\\\gamma_1&\text{if }x\in\{a,c,d\}\end{cases}$$

and

$$g_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_4 & \text{if } x \in \{1, a, c\} \\ \gamma_2 & \text{if } x \in \{b, d\} \end{cases}$$

where γ_1, γ_2, γ_3, and γ_4 are subsets of $[0, 1]$ with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3 \subsetneq \gamma_4$. It is easy to check that H_X and G_X are hesitant fuzzy implicative filters of X. But $H_X \tilde{\cup} G_X$ is not a hesitant fuzzy implicative filter of X, since

$$\begin{aligned}(h_X \tilde{\cup} g_X)(1 * (c * d)) \cap (h_X \tilde{\cup} g_X)(1 * c) &= (h_X \tilde{\cup} g_X)(b) \cap (h_X \tilde{\cup} g_X)(c) \\ &= (h_X(b) \cup g_X(b)) \cap (h_X(c) \cup g_X(c)) \\ &= \gamma_3 \cap \gamma_4 = \gamma_3 \nsubseteq \gamma_2 = \gamma_1 \cup \gamma_2 \\ &= h_X(1 * d) \cup g_X(1 * d) = (h_X \tilde{\cup} g_X)(1 * d).\end{aligned}$$

Let H_X be a hesitant fuzzy set set of a BE-algebra X. For any $a, b \in X$ and $k \in \mathbb{N}$, consider the set

$$h_X[a^k; b] := \left\{x \in X \mid h_X\left(a^k * (b * x)\right) = h_X(1)\right\}$$

where $h_X(a^k * x) = h_X(a * (a * (\cdots * (a * (a * x)) \cdots)))$ in which a appears k-times. Note that $a, b, 1 \in h_X[a^k; b]$ for all $a, b \in X$ and $k \in \mathbb{N}$.

Proposition 3.23. *Let H_X be a hesitant fuzzy set of a BE-algebra X such that the condition (3.1) and $h_X(x*y) = h_X(x) \cup h_X(y)$ for all $x, y \in X$. For any $a, b \in X$ and $k \in \mathbb{N}$, if $x \in h_X[a^k; b]$, then $y * x \in h_X[a^k; b]$ for all $y \in X$.*

Proof. Assume that $x \in h_X[a^k; b]$. Then $h_X(a^k * (b * x)) = h_X(1)$, and so

$$\begin{aligned}h_X(a^k * (b * (y * x))) &= h_X(a^k * (y * (b * x))) \\ &= h_X(y * (a^k * (b * x))) \\ &= h_X(y) \cup h_X(a^k * (b * x)) \\ &= h_X(y) \cup h_X(1) = h_X(1)\end{aligned}$$

for all $y \in X$ by the exchange property of the operation $*$. Hence $y*x \in h_X[a^k; b]$ for all $y \in X$. □

Proposition 3.24. *For any hesitant fuzzy set H_X of a BE-algebra X, let $a \in X$ satisfy the following condition $a * x = 1$ for all $x \in X$. Then $h_X[a^k; b] = X = h_X[b^k; a]$ for all $b \in X$ and $k \in \mathbb{N}$.*

Proof. For any $x \in X$, we have

$$\begin{aligned} h_X(a^k * (b * x)) &= h_X(a^{k-1} * (a * (b * x))) \\ &= h_X(a^{k-1} * (b * (a * x))) \\ &= h_X(a^{k-1} * (b * 1)) \\ &= h_X(1), \end{aligned}$$

and so $x \in h_X[a^k; b]$. Similarly, $x \in h_X[b^k; a]$. □

Proposition 3.25. *Let X be a self distributive BE-algebra and let H_X be an order-preserving soft set of X with the property (3.1). If $b \leq c$ in X, then $h_X[a^k; c] \subseteq h_X[a^k; b]$ for all $a \in X$ and $k \in \mathbb{N}$.*

Proof. Let $a, b, c, \in X$ be such that $b \leq c$. For any $k \in \mathbb{N}$, if $x \in h_X[a^k; c]$, then

$$\begin{aligned} h_X(1) = h_X(a^k * (c * x)) &= h_X(c * (a^k * x)) \\ &\subseteq h_X(b * (a^k * x)) = h_X(a^k * (b * x)) \end{aligned}$$

by (2.5), Proposition 3.2(i) and (2.4), and so $h_X(a^k * (b * x)) = h_X(1)$. Thus $x \in h_X[a^k; b]$, which completes the proof. □

The following example shows that there exists a hesitant fuzzy set H_X of X, $a, b \in X$ and $k \in \mathbb{N}$ such that $h_X[a^k; b]$ is not a filter of X.

Example 3.26. Let $X = \{1, a, b, c\}$ is a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let H_X be a hesitant fuzzy set of X U defined as follows:

$$h_X : X \to \mathscr{P}([0,1]), \ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. Then it is a hesitant fuzzy set of X. But $h_X[c; b] = \{x \in X | h_X(c*(b*x)) = h_X(1)\} = \{1, a, b\}$ is not an implicative filter, since $1*(a*c) = a \in h_X[c; b]$, $1 * a = a \in h_X[c; b]$ and $1 * c = c \notin h_X[c; b]$.

We provide conditions for a set $h_X[a^k; b]$ to be an implicative filter.

Theorem 3.27. *Let H_X be a hesitant fuzzy set of a self distributive BE-algebra X. If h_X is injective, then $h_X[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.*

Proof. Assume that X is a self distributive BE-algebra and h_X is injective. Obviously, $1 \in h_X[a^k;b]$. Let $a,b,x,y,z \in X$ and $k \in \mathbb{N}$ be such that $x*(y*z) \in h_X[a^k;b]$ and $x*y \in h_X[a^k;b]$. Then $h_X(a^k*(b*(x*(y*z)))) = h_X(1)$ which implies that $a^k*(b*(x*(y*z))) = 1$ since h_X is injective. Since X is a self distributive BE-algebra, we have

$$\begin{aligned}
h_X(1) &= h_X(a^k*(b*(x*(y*z)))) \\
&= h_X(a^{k-1}*(a*(b*(x*(y*z))))) \\
&= h_X(a^{k-1}*(a*((b*(x*y))*(b*(x*z))))) \\
&= \cdots \\
&= h_X((a^k*(b*(x*y)))*(a^k*(b*(x*z)))) \\
&= h_X(1*(a^k*(b*(x*z)))) \\
&= h_X(a^k*(b*(x*z))),
\end{aligned}$$

which implies that $x*z \in h_X[a^k;b]$. Therefore $h_X[a^k;b]$ is an implicative filter of X for all $a,b \in X$ and $k \in \mathbb{N}$. □

Theorem 3.28. *Let H_X be a hesitant fuzzy set of a self distributive B-algebra X satisfying the condition (3.1) and*

$$(\forall x,y \in X)\,(h_X(x*y) = h_X(x) \cap h_X(y))\,. \tag{3.8}$$

Then $h_X[a^k;b]$ is an implicative filter of X for all $a,b \in X$ and $k \in \mathbb{N}$.

Proof. Let $a,b \in X$ and $k \in \mathbb{N}$. Obviously, $1 \in h_X[a^k;b]$. Let $x,y,z \in X$ be such that $x*(y*z) \in h_X[a^k;b]$ and $x*y \in h_X[a^k;b]$. Then $h_X\left(a^k*(b*(x*(y*z)))\right) = h_X(1)$, which implies from (3.8) and (3.1) that

$$\begin{aligned}
h_X(1) &= h_X(a^k*(b*(x*(y*z)))) \\
&= h_X(a^{k-1}*(a*(b*(x*(y*z))))) \\
&= h_X(a^{k-1}*(a*((b*(x*y))*(b*(x*z))))) \\
&= \cdots \\
&= h_X((a^k*(b*(x*y)))*(a^k*(b*(x*z)))) \\
&= h_X(a^k*(b*(x*y))) \cap h_X(a^k*(b*(x*z))) \\
&= h_X(1) \cap h_X(a^k*(b*(x*z))) \\
&= h_X(a^k*(b*(x*z))).
\end{aligned}$$

Hence $x*z \in h_X[a^k;b]$ and therefore $h_X[a^k;b]$ is an implicative filter of X for all $a,b \in X$ and $k \in \mathbb{N}$. □

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A new quadratic functional equation version and its stability and superstability

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Abstract. Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces. It is shown that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation

$$\begin{aligned} &f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \\ &\qquad = f(x) + f(y) + f(z) \end{aligned} \tag{0.1}$$

if and only if $f : \mathcal{X} \to \mathcal{Y}$ is a quadratic mapping.

Furthermore, we prove the superstability and the Hyers-Ulam stability for the quadratic functional equation (0.1) by using a direct method.

Keywords: Hyers-Ulam stability; quadratic functional equation; fixed point method; quadratic functional inequality; orthogonality space.

1. Introduction and preliminaries

Studying functional equations focusing on their approximate and exact solutions, conduces to one of the most substantial significant study brunches in functional equations, what we would call *"the theory of stability of functional equations"*. This theory specifically analyzes relationships between approximate and exact solutions of functional equations. Actually a functional equation is considered to be *stable*, if one can find an exact solution for any approximate solution of that certain functional equation. Another related and close term in this area is *superstability*, which has a similar nature and concept to the stability problem. As a matter of fact, superstability for a given functional equation occurs when any approximate solution is an exact solution too. In such this situation the functional equation is called *superstable*.

In 1940, the most preliminary form of stability problems was proposed by Ulam [40]. He gave a talk and asked the following: "when and under what conditions does an exact solution of a functional equation near an approximately solution of that exist?"

In 1941, this question that today is considered as the source of the stability theory, was formulated and solved by Hyers [14] for the Cauchy's functional equation in Banach spaces. Then the result of Hyers was generalized by Aoki [1] for additive mappings and by Rassias [32] for linear mappings by considering an unbounded Cauchy difference. In 1994, Găvruţa [13] provided a further generalization of Rassias' theorem in which he replaced the unbounded Cauchy difference by a general control function for the existence of a unique linear mapping. For more epochal information and various aspects about the stability of functional equations theory, we refer the

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reader to the monographs [15, 28, 33, 35], which also include many interesting results concerning the stability of different functional equations in many various spaces.

Consider the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

The function $f(x) = cx^2$ is a solution for the quadratic functional equation and obviously every satisfied function in this equation is said to be a quadratic function. A stability problem for this equation was first proved by Skof [39] and then was generalized by Cholewa [9], Czerwik [7, 8] and others [2, 4, 30, 31, 33, 34]. Moreover, there are some other different types of quadratic functional equations that their stability problems have been investigated by many authors. We refer the readers to the papers [3, 5, 6, 10, 11, 12, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 36, 37, 38, 41].

This paper is organized as follows: In Section 2, we consider the superstability of the quadratic functional equation (0.1) and in Sections 3 and 4, we prove two types of stability problems for the quadratic functional equation (0.1).

2. Superstability of the functional equation (0.1)

To commence proving the superstability of the quadratic functional equation (0.1), we first solve it and then will give a superstability theorem.

Proposition 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces. A mapping $f:\mathcal{X}\to\mathcal{Y}$ satisfies* (0.1) *if and only if the mapping $f:\mathcal{X}\to\mathcal{Y}$ is a quadratic mapping.*

Proof. Sufficiency. Assume that $f:\mathcal{X}\to\mathcal{Y}$ satisfies (0.1).

Letting $x=y=z=0$ in (0.1), we have $4f(0)=3f(0)$. So $f(0)=0$.

Letting $y=z=0$ in (0.1), we get

$$\begin{aligned} 2f\left(\frac{x}{2}\right)+2f\left(-\frac{x}{2}\right)&=f(x), \\ 2f\left(-\frac{x}{2}\right)+2f\left(\frac{x}{2}\right)&=f(-x) \end{aligned} \tag{2.1}$$

for all $x\in\mathcal{X}$, which imply that $f(x)=f(-x)$ for all $x\in\mathcal{X}$.

It follows from (2.1) that $4f\left(\frac{x}{2}\right)=f(x)$ and so $f(2x)=4f(x)$ for all $x\in\mathcal{X}$.

Putting $z=0$ in (0.1), we see that

$$\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)=f(x)+f(y)$$

for all $x,y\in\mathcal{X}$, which means that $f:\mathcal{X}\to\mathcal{Y}$ is a quadratic mapping.

Necessity. Assume that $f:\mathcal{X}\to\mathcal{Y}$ is quadratic.

By (1.1), one can easily get $f(0)=0$, $f(x)=f(-x)$ and $f(2x)=4f(x)$ for all $x\in\mathcal{X}$. So by applying (1.1), we obtain

$$\begin{aligned} &f\Big(\frac{x+y+z}{2}\Big)+f\Big(\frac{x-y-z}{2}\Big)+f\Big(\frac{y-x-z}{2}\Big)+f\Big(\frac{z-x-y}{2}\Big) \\ &=\Big[2f\Big(\frac{x}{2}\Big)+2f\Big(\frac{y+z}{2}\Big)\Big]+\Big[2f\Big(-\frac{x}{2}\Big)+2f\Big(\frac{y-z}{2}\Big)\Big] \\ &=4f\left(\frac{x}{2}\right)+f\Big(\frac{y+z+y-z}{2}\Big)+f\Big(\frac{y+z-y+z}{2}\Big) \\ &=f(x)+f(y)+f(z) \end{aligned}$$

for all $x,y,z\in\mathcal{X}$, which is the functional equation (0.1) and the proof is complete. $\square$

Theorem 2.2. *Let $\mathcal{X}, \mathcal{Y}$ be normed spaces with norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$, respectively. Let δ be a nonnegative real number and $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function with*

$$\varphi(0,0,0) = 0, \qquad \varphi(x, y, 3x + y) = 0$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping such that

$$\begin{aligned} &\left\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) - f(y) - f(z) \right\|_{\mathcal{Y}} \\ &\leq \left\| f(x) - f\Big(\frac{z-x-y}{2}\Big) \right\|_{\mathcal{Y}} + \delta \cdot \varphi(x, y, z) \end{aligned} \tag{2.2}$$

for all $x, y, z \in \mathcal{X}$. Then f is a quadratic mapping.

Proof. Putting $x = y = z = 0$ in (2.2), we get

$$\big\|f(0)\big\|_{\mathcal{Y}} \leq \big\|0\big\|_{\mathcal{Y}} + \delta \cdot \varphi(0,0,0) = 0.$$

So $f(0) = 0$.

Replacing x, y, z by $0, x, x$ in (2.2), respectively, we obtain

$$\big\|f(-x) - f(x)\big\|_{\mathcal{Y}} \leq \big\|0\big\|_{\mathcal{Y}} + \delta \cdot \varphi(0, x, x) = 0.$$

So $f(x) = f(-x)$ for all $x \in \mathcal{X}$.

Replacing x, y and z by $x, -3x$ and 0, and then by $2x, -3x$ and $3x$ in (2.2), respectively, we have

$$\begin{aligned} [f(x) - f(3x)] + 2f(2x) &= 0, \\ 2[f(x) - f(3x)] + f(4x) &= 0, \end{aligned}$$

which result that $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$ for all $x \in \mathcal{X}$.

Letting $x = v - u, y = 2u - v$ and $z = 2v - u$ and then $x = u + v, y = -3v$ and $z = 3u$ in (2.2), respectively, we get the equalities

$$\begin{aligned} f(2u - v) + f(2v - u) &= f(u) + f(v) + f(2u - 2v), \\ f(2u - v) + f(2v - u) &= f(3u) + f(3v) - f(2u + 2v). \end{aligned}$$

Thus

$$f(u) + f(v) + 4f(u - v) = 9f(u) + 9f(v) - 4f(u + v),$$

which is simplified to

$$f(u + v) + f(u - v) = 2f(u) + 2f(v)$$

for all $u, v \in \mathcal{X}$. So f is quadratic. □

Theorem 2.2 covers several other cases for $\varphi : \mathcal{X}^3 \to [0, \infty)$. For example, we can define φ satisfying the mentioned conditions with $\varphi(x, y, z) := \|y\|_{\mathcal{X}} - \|3x - z\|_{\mathcal{X}}$ or $\varphi(x, y, z) := \|3x + y - z\|_{\mathcal{X}}$. In addition, to make a simpler result, one can put $\delta = 0$.

3. Hyers-Ulam stability of the functional equation (0.1): Type A

In this section, we prove the Hyers-Ulam stability of the quadratic functional equation (0.1). We will suppose that $\mathcal{X}$ is a normed space and $\mathcal{Y}$ is a complete normed space with norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$, respectively.

Theorem 3.1. *Let $\varphi : \mathcal{X}^3 \to [0,\infty)$ be a function with $\varphi(0,0,0) = 0$ and the following condition holds:*

$$\text{if} \quad \begin{cases} \|x\|_{\mathcal{X}} \le \|x'\|_{\mathcal{X}}, & \text{or} \\ \|y\|_{\mathcal{X}} \le \|y'\|_{\mathcal{X}}, & \text{or} \\ \|z\|_{\mathcal{X}} \le \|z'\|_{\mathcal{X}}, & \end{cases} \quad \Longrightarrow \quad \varphi(x,y,z) \le \varphi(x',y',z') \tag{3.1}$$

for all $x, y, z, x', y', z' \in \mathcal{X}$. Denote by ϕ a function such that

$$\phi(x,y,z) := \sum_{n=0}^{\infty} 2^{2n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) < \infty \tag{3.2}$$

for all $x, y, z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying

$$\begin{aligned} \left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) - f(y) - f(z) \right\|_{\mathcal{Y}} &\le \left\| f(x) - f\left(\frac{z-x-y}{2}\right) \right\|_{\mathcal{Y}} \\ &+ \varphi(x,y,z) \end{aligned} \tag{3.3}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\| f(x) - \mathcal{Q}(x) \right\|_{\mathcal{Y}} \le 2\phi(x,x,x) \tag{3.4}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x = y = z = 0$ in (3.3), we get

$$\left\| f(0) \right\|_{\mathcal{Y}} \le \left\| 0 \right\|_{\mathcal{Y}} + \varphi(0,0,0) = 0.$$

So $f(0) = 0$.

Replacing x, y, z by $x, x, 4x$ and $x, 0, 3x$ in (3.3), respectively, and then using (3.1), we obtain

$$\begin{aligned} \left\| f(3x) + 2f(2x) - f(x) - f(4x) \right\|_{\mathcal{Y}} &\le \varphi(x,x,4x) \le \varphi(4x,4x,4x), \\ \left\| 2f(2x) + f(x) - f(3x) \right\|_{\mathcal{Y}} &\le \varphi(x,0,3x) \le \varphi(4x,4x,4x) \end{aligned}$$

for all $x \in \mathcal{X}$. These inequalities give

$$\left\| 4f(2x) - f(4x) \right\|_{\mathcal{Y}} \le \left\| f(3x) + 2f(2x) - f(x) - f(4x) \right\|_{\mathcal{Y}} + \left\| 2f(2x) + f(x) - f(3x) \right\|_{\mathcal{Y}} \le 2\varphi(4x,4x,4x).$$

Thus

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\|_{\mathcal{Y}} \le 2\varphi(x,x,x) \tag{3.5}$$

for all $x \in \mathcal{X}$. Using the induction method, we show that

$$\left\| 4^n f\left(\frac{x}{2^n}\right) - f(x) \right\|_{\mathcal{Y}} \le \sum_{s=0}^{n-1} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right) \tag{3.6}$$

for all $n \ge 1$ and all $x \in \mathcal{X}$. The case $n = 1$ is the inequality (3.5). For the case $n+1$, by (3.5) and (3.6), we have

$$\begin{aligned} \left\| 4^{n+1} f\left(\frac{x}{2^{n+1}}\right) - f(x) \right\|_{\mathcal{Y}} &\le 4^n \left\| 4f\left(\frac{1}{2}\left(\frac{x}{2^n}\right)\right) - f\left(\frac{x}{2^n}\right) \right\|_{\mathcal{Y}} + \left\| 4^n f\left(\frac{x}{2^n}\right) - f(x) \right\|_{\mathcal{Y}} \\ &\le 4^n \cdot 2\varphi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right) + \sum_{s=0}^{n-1} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right) = \sum_{s=0}^{n} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right) \end{aligned}$$

for all $x \in \mathcal{X}$, which ends the induction method.

Assume that m, l are positive integers with $m > l$. From (3.6), it follows that

$$\left\| 4^m f\left(\frac{x}{2^m}\right) - 4^l f\left(\frac{x}{2^l}\right) \right\|_{\mathcal{Y}} = 4^l \left\| 4^{m-l} f\left(\frac{1}{2^{m-l}}\left(\frac{x}{2^l}\right)\right) - f\left(\frac{x}{2^l}\right) \right\|_{\mathcal{Y}} \le \sum_{s=l}^{m-1} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right)$$

for all $x \in \mathcal{X}$, in which by (3.2) the right-hand side tends to zero as $m, l \to \infty$. This clarifies that the sequence $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is Cauchy in the complete space $\mathcal{Y}$ and therefore convergent in it. So we can define for all $x \in \mathcal{X}$, the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{Q}(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right).$$

Now passing the limit $n \to \infty$ in (3.6) and then using (3.2), we obtain (3.4).

To end the proof, we show that $\mathcal{Q}$ is a unique quadratic mapping. It follows from (3.3) that

$$\begin{aligned}
&\left\|\mathcal{Q}\Big(\frac{x+y+z}{2}\Big) + \mathcal{Q}\Big(\frac{x-y-z}{2}\Big) + \mathcal{Q}\Big(\frac{y-x-z}{2}\Big) - \mathcal{Q}(y) - \mathcal{Q}(z)\right\|_{\mathcal{Y}} \\
&\quad = \lim_{n\to\infty} 4^n \left\| f\Big(\frac{x+y+z}{2^{n+1}}\Big) + f\Big(\frac{x-y-z}{2^{n+1}}\Big) + f\Big(\frac{y-x-z}{2^{n+1}}\Big) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\|_{\mathcal{Y}} \\
&\quad \le \lim_{n\to\infty} \left\|4^n f\left(\frac{x}{2^n}\right) - 4^n f\Big(\frac{z-x-y}{2^{n+1}}\Big)\right\|_{\mathcal{Y}} + \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$, in which by (3.2), the second term of the right-hand side tends to zero as $n \to \infty$, and therefore we obtain

$$\left\|\mathcal{Q}\Big(\frac{x+y+z}{2}\Big) + \mathcal{Q}\Big(\frac{x-y-z}{2}\Big) + \mathcal{Q}\Big(\frac{y-x-z}{2}\Big) - \mathcal{Q}(y) - \mathcal{Q}(z)\right\|_{\mathcal{Y}} \le \left\|\mathcal{Q}(x) - \mathcal{Q}\Big(\frac{z-x-y}{2}\Big)\right\|_{\mathcal{Y}}$$

for all $x, y, z \in \mathcal{X}$. Now by applying Theorem 2.2 (with $\delta = 0$), we conclude that $\mathcal{Q}$ is a quadratic mapping.

Let $\mathcal{Q}' : \mathcal{X} \to \mathcal{Y}$ be another quadratic mapping satisfying (3.4). Then we have

$$\begin{aligned}
\left\|\mathcal{Q}(x) - \mathcal{Q}'(x)\right\|_{\mathcal{Y}} &\le 4^n \left\|\mathcal{Q}\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_{\mathcal{Y}} + 4^n \left\|\mathcal{Q}'\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_{\mathcal{Y}} \\
&\le 2 \cdot 4^n \cdot 2\phi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right) = 4\sum_{s=n}^{\infty} 2^{2s+1}\varphi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right)
\end{aligned}$$

for all $x \in \mathcal{X}$. By (3.2), the right-hand side tends to zero as $n \to \infty$, and thus $\mathcal{Q}(x) = \mathcal{Q}'(x)$ for all $x \in \mathcal{X}$. This means the uniqueness of $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ and so the proof is complete. □

Theorem 3.2. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function satisfying $\varphi(0,0,0) = 0$ and* (3.1)*. Denote by ϕ a function such that*

$$\phi(x, y, z) := \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \varphi\big(2^n x, 2^n y, 2^n z\big) < \infty \tag{3.7}$$

for all $x, y, z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying (3.3)*. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ satisfying* (3.4)*.*

Proof. As in the proof of Theorem 3.1, we can first get the inequality (3.5), and then by replacing x by $2x$ in (3.5), we obtain

$$\left\|\frac{1}{4} f(2x) - f(x)\right\|_{\mathcal{Y}} \le \frac{1}{2}\varphi(2x, 2x, 2x)$$

for all $x \in \mathcal{X}$.

Using the induction method, we get

$$\left\|\frac{1}{4^n} f(2^n x) - f(x)\right\|_{\mathcal{Y}} \le \sum_{s=1}^{n} \frac{1}{2^{2s-1}} \varphi(2^s x, 2^s x, 2^s x) \tag{3.8}$$

for all $n \ge 1$ and all $x \in \mathcal{X}$.

Now by the same method which was done in the proof of Theorem 3.1, we have the Cauchy sequence $\left\{\frac{1}{4^n} f(2^n x)\right\}$, and then the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ defined by

$$\mathcal{Q}(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in \mathcal{X}$.

And finally we can conclude the inequality (3.4) by (3.7) and (3.8).

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.3. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1, p_2, p_3 > 2$ or $p_1, p_2, p_3 < 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\begin{aligned}\Big\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) - f(y) - f(z)\Big\|_{\mathcal{Y}} \\ \le \Big\| f(x) - f\Big(\frac{z-x-y}{2}\Big)\Big\|_{\mathcal{Y}} + \delta\big(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3}\big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\| f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le \sum_{i=1}^{3} \frac{2^{p_i+1}}{\big|2^{p_i} - 4\big|}\delta\|x\|_{\mathcal{X}}^{p_i}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x, y, z) = \delta\big(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3}\big)$ and applying Theorem 3.1 for the case $p_1, p_2, p_3 > 2$, and Theorem 3.2 for the case $p_1, p_2, p_3 < 2$, we get the desired results. □

Corollary 3.4. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1+p_2+p_3 \neq 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\begin{aligned}\Big\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) - f(y) - f(z)\Big\|_{\mathcal{Y}} \\ \le \Big\| f(x) - f\Big(\frac{z-x-y}{2}\Big)\Big\|_{\mathcal{Y}} + \delta\big(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3}\big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\| f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le \frac{2^{p_1+p_2+p_3+1}}{\big|2^{p_1+p_2+p_3} - 4\big|}\delta\|x\|_{\mathcal{X}}^{p_1+p_2+p_3}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x, y, z) = \delta\big(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3}\big)$ and applying Theorem 3.1 for the case $p_1 + p_2 + p_3 > 2$, and Theorem 3.2 for the case $p_1 + p_2 + p_3 < 2$, we get the desired results. □

4. Hyers-Ulam stability of the functional equation (0.1): Type B

In this section, we bring another type of stability theorems for the quadratic functional equation (0.1) which is more prevalent in considering stability problems rather than the given type in the previous section.

First of all, for convenience, we define for a given mapping $f : \mathcal{X} \to \mathcal{Y}$, the difference operator:

$$\begin{aligned}\mathcal{D}f(x, y, z) =: f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \\ - f(x) - f(y) - f(z)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$.

Theorem 4.1. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function satisfying $\varphi(0, 0, 0) = 0$ and (3.1). Denote by ϕ a function such that*

$$\phi(x, y, z) := \sum_{n=0}^{\infty} \frac{9^n}{4^n}\varphi\left(\frac{2^n}{3^n}x, \frac{2^n}{3^n}y, \frac{2^n}{3^n}z\right) < \infty \tag{4.1}$$

for all $x, y, z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying

$$\big\|\mathcal{D}f(x, y, z)\big\|_{\mathcal{Y}} \le \varphi(x, y, z) \tag{4.2}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq \phi(x, x, x) \tag{4.3}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x = y = z = 0$ in (4.2), we get $f(0) = 0$.

Replacing x, y, z by $0, x, 3x$ and then by $2x, 2x, 2x$ in (4.2), respectively, we obtain

$$\begin{aligned}\big\|2f(2x) + f(x) - f(3x)\big\|_{\mathcal{Y}} &\leq \varphi(0, x, 3x) \leq \varphi(3x, 3x, 3x),\\ \Big\|f(2x) - f(x) - \frac{1}{3}f(3x)\Big\|_{\mathcal{Y}} &\leq \frac{1}{3}\varphi(2x, 2x, 2x) \leq \frac{1}{3}\varphi(3x, 3x, 3x).\end{aligned}$$

Adding the above inequalities, we conclude that $\big\|3f(2x) - \frac{4}{3}f(3x)\big\|_{\mathcal{Y}} \leq \frac{4}{3}\varphi(3x, 3x, 3x)$ and therefore

$$\Big\|\frac{9}{4}f\left(\frac{2}{3}x\right) - f(x)\Big\|_{\mathcal{Y}} \leq \varphi(x, x, x)$$

for all $x \in \mathcal{X}$.

By the induction method, we can show that

$$\Big\|\frac{9^n}{4^n}f\left(\frac{2^n}{3^n}x\right) - f(x)\Big\|_{\mathcal{Y}} \leq \sum_{s=0}^{n-1}\frac{9^s}{4^s}\varphi\left(\frac{2^s}{3^s}x, \frac{2^s}{3^s}x, \frac{2^s}{3^s}x\right) \tag{4.4}$$

for all $x \in \mathcal{X}$.

Now similar to the method in the proof of Theorem 3.1, we have the Cauchy sequence $\left\{\frac{9^n}{4^n}f\left(\frac{2^n}{3^n}x\right)\right\}$, and then the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ defined by

$$\mathcal{Q}(x) := \lim_{n\to\infty}\frac{9^n}{4^n}f\left(\frac{2^n}{3^n}x\right)$$

for all $x \in \mathcal{X}$. This definition and the inequality (4.4) lead us to the inequality (4.3).

It follows from (4.1) and (4.2) that

$$\big\|\mathcal{D}\mathcal{Q}(x, y, z)\big\|_{\mathcal{Y}} \leq \lim_{n\to\infty}\frac{9^n}{4^n}\Big\|\mathcal{D}f\left(\frac{2^n}{3^n}x, \frac{2^n}{3^n}y, \frac{2^n}{3^n}z\right)\Big\|_{\mathcal{Y}} \leq \lim_{n\to\infty}\frac{9^n}{4^n}\varphi\left(\frac{2^n}{3^n}x, \frac{2^n}{3^n}y, \frac{2^n}{3^n}z\right) = 0.$$

Hence $\mathcal{D}\mathcal{Q}(x, y, z) = 0$ for all $x, y, z \in \mathcal{X}$. Now Proposition 2.1 signifies that $\mathcal{Q}$ is a quadratic mapping.

The proof of the uniqueness of $\mathcal{Q}$ is similar to the proof of Theorem 3.1. □

Theorem 4.2. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function satisfying $\varphi(0, 0, 0) = 0$ and (3.1). Denote by ϕ a function such that*

$$\phi(x, y, z) := \sum_{n=0}^{\infty}\frac{4^n}{9^n}\varphi\left(\frac{3^n}{2^n}x, \frac{3^n}{2^n}y, \frac{3^n}{2^n}z\right) < \infty$$

for all $x, y, z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying (4.2). Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ satisfying (4.3).

Proof. The proof is similar to the proof of the previous theorem and thus we omit it. □

Corollary 4.3. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1, p_2, p_3 > 2$ or $p_1, p_2, p_3 < 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\big\|\mathcal{D}f(x, y, z)\big\|_{\mathcal{Y}} \leq \delta(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3})$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq \sum_{i=1}^{3}\frac{2^{p_i-2}}{\left|\frac{2^{p_i}}{9} - \frac{3^{p_i}}{4}\right|}\delta\|x\|_{\mathcal{X}}^{p_i}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x,y,z) = \delta\left(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3}\right)$ and applying Theorem 4.1 for the case $p_1, p_2, p_3 > 2$, and Theorem 4.2 for the case $p_1, p_2, p_3 < 2$, we get the desired results. □

Corollary 4.4. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1+p_2+p_3 \neq 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\left\|\mathcal{D}f(x,y,z)\right\|_{\mathcal{Y}} \leq \delta(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3})$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\|f(x) - \mathcal{Q}(x)\right\|_{\mathcal{Y}} \leq \frac{2^{p_1+p_2+p_3-2}}{\left|\frac{2^{p_1+p_2+p_3}}{9} - \frac{3^{p_1+p_2+p_3}}{4}\right|}\theta\|x\|_{\mathcal{X}}^{p_1+p_2+p_3}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x,y,z) = \delta\left(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3}\right)$ and applying Theorem 4.1 for the case $p_1 + p_2 + p_3 > 2$, and Theorem 4.2 for the case $p_1 + p_2 + p_3 < 2$, we get the desired results. □

This paper is just a start for the quadratic functional equation (0.1). Actually this functional equation and its stability problems can be studied more in various mathematical structures and spaces. Such this studied approach can cause to have a deeper description of this equation's unknown properties which will probably be more interesting and remarkable.

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Some New Results on Preconditioned Generalized Mixed-Type Splitting Iterative Methods

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Abstract

In this paper, we present three preconditioned generalized mixed-type splitting (GMTS) methods for solving the weighted linear least square problem. We compare the spectral radii of the iteration matrices of the preconditioned and the original methods. The comparison results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. Finally, we give two numerical examples to confirm our theoretical results.

Keywords: Preconditioning, GMTS method, linear system, convergence, comparison.

2000 AMS Classification: 65F10.

1. Introduction

We consider the following weighted least squares problem

$$(1.1)\qquad \min_{x\in R^n} (Ax-b)^T W^{-1} (Ax-b),$$

where $A \in R^{n\times n}$ is nonsingular, $b \in R^n$, $W \in R^{n\times n}$ is a symmetric positive definite matrix, see [1,4,9].

In order to solve it, one has to solve a nonsingular linear system as

$$(1.2)\qquad Hy = f,$$

where

$$(1.3)\qquad H = A^T W^{-1} A = \begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix} \in R^{n\times n}$$

is an invertible matrix with

$$B = (b_{ij})_{p\times p},\ C = (c_{ij})_{q\times q},\ L = (l_{ij})_{q\times p},\ U = (u_{ij})_{p\times q},$$

$p + q = n$ and $f = A^T W^{-1} b \in R^n$, see [1,4].

Throughout the paper, we consider the following decomposition for the matrix H,

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$H = \hat{D} - \hat{L} - \hat{U}$, in which

$$\hat{D} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} 0 & 0 \\ -L & 0 \end{pmatrix}, \quad \hat{U} = \begin{pmatrix} B & -U \\ 0 & C \end{pmatrix}. \tag{1.4}$$

In [1], authors established a generalized AOR(GAOR) method to solve systems of linear equations (1.2). In paper [2, 3] , authors studied the preconditioned GAOR methods. In [4], authors presented a generalized mixed-type splitting (GMTS) iterative method which is generalized GAOR method. And they studied the preconditioned generalized mixed-type splitting iterative methods to solve (1.2). They showed that the preconditioned GMTS methods converge faster than the GMTS method, whenever the GMTS method is convergent.

In this paper, we propose three new preconditioners and give the comparison theorems between the preconditioned and original methods. These results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. And we prove that in the case that the GMTS method is convergent, using the third preconditioned GMTS method leads to the better convergence rate than the first and the second preconditioned GMTS methods. In Section 4, we give two examples to confirm our theoretical results. And we know that the preconditioned GMTS methods with preconditioners in this paper have the better converge rate than the preconditioned GMTS method with preconditioner P^*.

2. Preliminaries

2.1 Definition [5] $A \in R^{n\times n}$ is called a Z-matrix if $a_{ij} \le 0$ for $i, j = 1, 2, ..., n$ $(i \ne j)$.

2.2 Definition [5] Let A be a Z-matrix with positive diagonal elements. Then the matrix A is called an M-matrix if A is nonsingular and $A^{-1} \ge 0$.

2.3 Definition [6] The splitting $A = M - N$ is called

(1) a regular splitting of A if $M^{-1} \ge 0$ and $N \ge 0$;
(2) a nonnegative splitting of A if $M^{-1} \ge 0$, $M^{-1}N \ge 0$ and $NM^{-1} \ge 0$;
(3) a weak nonnegative splitting of A if $M^{-1} \ge 0$ and either $M^{-1}N \ge 0$ (the first type) or $NM^{-1} \ge 0$ (the second type);
(4) a convergent splitting of A if $\rho(M^{-1}N) < 1$.

2.1. Lemma. [4] *Let A be a Z-matrix. Moreover, suppose that $A = M - N$ is a weak nonnegative splitting of the first type. Then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.*

2.2. Lemma. [7] *Let $A = M - N$ be a regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and A^{-1} is nonnegative.*

2.3. Lemma. [8] *Let matrix $A = (a_{ij})_{n\times n}$ be given such that*

(1) *$a_{ij} \le 0$ for $i, j = 1, 2, ..., n$ $(i \ne j)$,*
(2) *A is nonsingular,*
(3) *$A^{-1} \ge 0$.*

Then,

(1) *$a_{ii} > 0$ for $i = 1, 2, ..., n$, i.e., A is an M-matrix,*
(2) *$\rho(B) < 1$ where $B = I - D^{-1}A$, where $D = diag\{a_{11}, ..., a_{nn}\}$.*

2.4. Lemma. [6] *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak nonnegative splittings of A, where $A^{-1} \geq 0$, if $M_1^{-1} \geq M_2^{-1}$ then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$.*

3. Comparison results

Consider the linear system (1.2), the generalized mixed-type splitting (GMTS) iterative method is given as follows:

$$(\hat{D} + D_1 + L_1 - \hat{L})y^{(k+1)} = (D_1 + L_1 + \hat{U})y^{(k)} + f \tag{3.1}$$

where $\hat{D}$, $\hat{L}$ and $\hat{U}$ are defined by (1.4), and D_1 is an auxiliary nonnegative block diagonal matrix, L_1 is an auxiliary strictly nonnegative block lower triangular matrix such that $0 \leq D_1 \leq \hat{D}$ and $0 \leq L_1 \leq \hat{L}$. Evidently, the iteration matrix of the GMTS iterative method is given as follow:

$$T = (\hat{D} + D_1 + L_1 - \hat{L})^{-1}(D_1 + L_1 + \hat{U}).$$

In this paper, we propose the new preconditioners as follows,

$$P_i^* = \begin{pmatrix} I + S_i & 0 \\ 0 & I + V_i \end{pmatrix}, \quad i = 1, 2, 3 \tag{3.2}$$

where

$$S_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{p-1,1} & 0 & \cdots & 0 & 0 \\ b_{p1} & 0 & \cdots & 0 & 0 \end{pmatrix}, \ S_2 = \begin{pmatrix} 0 & b_{12} & \cdots & b_{1,p-1} & b_{1p} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & b_{12} & \cdots & b_{1,p-1} & b_{1p} \\ b_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{p-1,1} & 0 & \cdots & 0 & 0 \\ b_{p1} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ c_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{q-1,1} & 0 & \cdots & 0 & 0 \\ c_{q1} & 0 & \cdots & 0 & 0 \end{pmatrix}, \ V_2 = \begin{pmatrix} 0 & c_{12} & \cdots & c_{1,q-1} & c_{1q} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0 & c_{12} & \cdots & c_{1,q-1} & c_{1q} \\ c_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{q-1,1} & 0 & \cdots & 0 & 0 \\ c_{q1} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then $P_i^* H$ can be expressed by

$$P_i^* H = \begin{pmatrix} I - B_i^* & U_i^* \\ L_i^* & I - C_i^* \end{pmatrix},$$

where $B_i^* = B - S_i(I - B)$, $C_i^* = C - V_i(I - C)$, $L_i^* = (I + V_i)L$, $U_i^* = (I + S_i)U$.

Let us consider the corresponding splitting for the preconditioned GMTS method, that is the generalized mixed-type splitting for the $\bar{H}_i = P_i^* H = \bar{M}_i - \bar{N}_i$, where

$$\bar{M}_i = \hat{D}_i^* + \bar{D}_1 + \bar{L}_1 - \hat{L}_i^*, \quad \bar{N}_i = \bar{D}_1 + \bar{L}_1 + \hat{U}_i^*$$

and

$$\hat{D}_i^* = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \ \hat{L}_i^* = \begin{pmatrix} 0 & 0 \\ -L_i^* & 0 \end{pmatrix}, \ \hat{U}_i^* = \begin{pmatrix} B_i^* & -U_i^* \\ 0 & C_i^* \end{pmatrix}, \ i = 1, 2, 3,$$

$\bar{D}_1$ is an auxiliary nonnegative block diagonal matrix with $0 \le \bar{D}_1 \le \hat{D}_i^*$, $\bar{L}_1$ is an auxiliary strictly nonnegative block lower triangular matrix with $0 \le \bar{L}_1 \le \hat{L}_i^*$.

The iteration matrix of the preconditioned GMTS method is

$$T_i^* = (\hat{D}_i^* + \bar{D}_1 + \bar{L}_1 - \hat{L}_i^*)^{-1}(\bar{D}_1 + \bar{L}_1 + \hat{U}_i^*).$$

3.1. Lemma. [4] *Assume that $L \le 0, U \le 0, B \ge 0, C \ge 0$ and H in (1.2) is irreducible. If D_1 is nonsingular, then the iteration matrix of the GMTS method is irreducible.*

3.2. Lemma. [4] *Assume that $L \le 0, U \le 0, B \ge 0, C \ge 0$, then the corresponding splitting of GMTS method is a regular splitting for the matrix H.*

Similar to the proof of Lemma 3.2, we can prove the following lemma.

3.3. Lemma. *Assume that $L \le 0, U \le 0, B \ge 0, C \ge 0$, then the corresponding splitting of PGMTS method is a regular splitting for the matrix $P_i^* H$ $(i = 1, 2, 3)$.*

3.4. Theorem. *Let H be an M-matrix, then $P_i^* H$ $(i = 1, 2, 3)$ is an M-matrix.*

Proof. Consider the following splitting for H, $H = M_1 - N_1$,
where $M_1 = (P_1^*)^{-1}$, $N_1 = (P_1^*)^{-1}(\hat{L}^* + \hat{U}^*)$,
and $\hat{L}^* = \begin{pmatrix} 0 & 0 \\ -L_1^* & 0 \end{pmatrix}, \hat{U}^* = \begin{pmatrix} B_1^* & -U_1^* \\ 0 & C_1^* \end{pmatrix}$.

We can see that $M_1^{-1}N_1 = \hat{L}^* + \hat{U}^*$ and $M_1^{-1} \ge 0$. Then$H = M_1 - N_1$ is a weak nonnegative splitting of the first type. By the assumption H is an M-matrix, hence Lemma 2.1 implies that $\rho(M_1^{-1}N_1) < 1$. Let us assume that $P_1^* H = I - \hat{L}^* - \hat{U}^*$, using the fact that $\rho(\hat{L}^* + \hat{U}^*) = \rho(M_1^{-1}N_1) < 1$, by Lemma 2.2 and Lemma 2.3, it is easy to know that $P_1^* H$ is an M-matrix. The similar results can be gotten when $i = 2, 3$. □

Now, we will show that in the case that the GMTS method converges, the preconditioned GMTS methods converge faster.

3.5. Theorem. *Let T and T_1^* be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively, assume that the matrix H is irreducible, $L \le 0, U \le 0, B \ge 0, C \ge 0, 0 \le D_1 \le \hat{D}$, $0 \le \bar{D}_1 \le \hat{D}_1^*, 0 \le L_1 \le \hat{L}$, $0 \le \bar{L}_1 \le \hat{L}_1^*, b_{i,1} > 0, c_{j,1} > 0$, for some $i \in \{2, 3, ..., p\}, j \in \{2, 3, ..., q\}$. If $\rho(T) < 1, \bar{D}_1 \le D_1$ and $\bar{L}_1 \le L_1$, then $\rho(T_1^*) \le \rho(T)$.*

Proof. As the matrix H is irreducible, so the $P_1^* H$ is irreducible. And by Lemma 3.1, we know that T and T_1^* are irreducible. Consider the GMTS splitting for the matrix $H = M - N$, where $M = \hat{D} + D_1 + L_1 - \hat{L}$, $N = D_1 + L_1 + \hat{U}$.

Obviously, $H = M - N$ is a regular splitting, and by the assumption $\rho(M^{-1}N) < 1$, we can get that H is an M-matrix. From Theorem 3.4, we know that $P_1^* H$ is also an M-matrix. Thus, from Lemma 3.3, we know that $\bar{H}_1 = \bar{M}_1 - \bar{N}_1$ is a regular splitting. Therefore, as H is an M-matrix, we can get $\rho(T_1^*) = \rho(\bar{M}_1^{-1}\bar{N}_1) < 1$.

Now, we define the following splitting for the matrix H, $H = M_1^* - N_1^*$, in which $M_1^* = (I + \bar{S}_1)^{-1}\bar{M}_1$, $N_1^* = (I + \bar{S}_1)^{-1}\bar{N}_1$ and

$$\bar{S}_1 = \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}.$$

Consider the iteration matrix of the GMTS method $T = M^{-1}N$, it is easy to see that

$$M - \bar{M}_1 = \begin{pmatrix} D_{11} - D_{11}^* & 0 \\ L_{21} + L - L_{21}^* - L_1^* & D_{22} - D_{22}^* \end{pmatrix},$$

where

$$D_1 = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix} \le \hat{D},\ \bar{D}_1 = \begin{pmatrix} D_{11}^* & 0 \\ 0 & D_{22}^* \end{pmatrix} \le \hat{D}_1^*,$$

$L_1 = \begin{pmatrix} 0 & 0 \\ L_{21} & 0 \end{pmatrix} \le \hat{L}$ and $\bar{L}_1 = \begin{pmatrix} 0 & 0 \\ L_{21}^* & 0 \end{pmatrix} \le \hat{L}_1$.

It is known that $L_1^* = (I + V_1)L$, hence $L_1^* - L = V_1 L \le 0$.

By computations, we know that $\bar{M}_1 \le M$, so $\bar{M}_1^{-1} \ge M^{-1}$. Consequently,

$$M^{-1} \le \bar{M}_1^{-1} \le \bar{M}_1^{-1}(I + \bar{S}_1) = (M_1^*)^{-1}.$$

From Lemma 2.4, we deduce that

$$\rho(\bar{M}_1^{-1}\bar{N}_1) = \rho((M_1^*)^{-1}N_1^*) \le \rho(M^{-1}N),$$

so $\rho(T_1^*) \le \rho(T)$. □

Similar to the proof of Theorem 3.5, we can get the following two theorems.

3.6. Theorem. *Let T and T_2^* be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively. Assume that the matrix H is irreducible, $L \le 0, U \le 0, B \ge 0, C \ge 0, 0 \le D_2 \le \hat{D}$, $0 \le \bar{D}_2 \le \hat{D}_2^*, 0 \le L_2 \le \hat{L}$, $0 \le \bar{L}_2 \le \hat{L}_2^*, b_{1,i} > 0, c_{1,j} > 0$, for some $i \in \{2, 3, ..., p\}, j \in \{2, 3, ..., q\}$. If $\rho(T) < 1, \bar{D}_2 \le D_2$ and $\bar{L}_2 \le L_2$, then $\rho(T_2^*) \le \rho(T)$.*

3.7. Theorem. *Let T and T_3^* be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively. Assume that the matrix H is irreducible, $L \le 0, U \le 0, B \ge 0, C \ge 0$, $0 \le D_3 \le \hat{D}$, $0 \le \bar{D}_3 \le \hat{D}_3^*$, $0 \le L_3 \le \hat{L}$, $0 \le \bar{L}_3 \le \hat{L}_3^*$, $b_{i,1} > 0, c_{j,1} > 0, b_{1,i} > 0, c_{1,j} > 0$, for some $i \in \{2, 3, ..., p\}, j \in \{2, 3, ..., q\}$. If $\rho(T) < 1, \bar{D}_3 \le D_3$ and $\bar{L}_3 \le L_3$, then $\rho(T_3^*) \le \rho(T)$.*

Now, we prove that in the case that the GMTS method is convergent, using the third preconditioned GMTS method leads to the better convergence rate than the first and the second preconditioned GMTS methods.

3.8. Theorem. *Suppose that the matrix H is irreducible, $L \le 0, U \le 0, B \ge 0, C \ge 0$, $b_{i,1} > 0, c_{j,1} > 0, b_{1,i} > 0, c_{1,j} > 0$, for some $i \in \{2, 3, ..., p\}, j \in$*

$\{2,3,...,q\}$, *the auxiliary block diagonal matrices are chosen as* $\alpha_i I$ *and the auxiliary block lower triangular matrices as* $\beta_i L_i^*$ *for* $i=1,3,\ 0\le\alpha_3\le\alpha_1\le 1, 0\le\beta_1\le\beta_3\le 1$. *Then* $\rho(T_3^*)\le\rho(T_1^*)$ *if* $\rho(T)<1$.

Proof. By the assumption $\rho(T)<1$, and according the Lemma 2.1, H is an M-matrix. Assume that $P_i^*H=\widetilde{M_i}-\widetilde{N}_i, i=1,3$ where

$$\widetilde{M_i}=\begin{pmatrix} I+D_{11}^i & 0\\ L_{21}^i+L_i^* & I+D_{22}^i\end{pmatrix},\qquad \widetilde{N}_i=\begin{pmatrix} B_i^*+D_{11}^i & -U_i^*\\ L_{21}^i & C_i^*+D_{22}^i\end{pmatrix},$$

and $L_{21}^i=-\beta_i L_i^*, D_{11}^i=\alpha_i I_p, D_{22}^i=\alpha_i I_q$ for $i=1,3$.

Now, we define the following splitting for the matrix H, i.e. $H=M_i-N_i (i=1,3)$ such that $M_i=(I+\widetilde{S}_i)^{-1}\widetilde{M_i}$ and $N_i=(I+\widetilde{S}_i)^{-1}\widetilde{N}_i$,

where $\widetilde{S}_i=\begin{pmatrix} S_i & 0\\ 0 & V_i\end{pmatrix}$.

Since

$$L_{21}^1-L_{21}^3=-\beta_1L_1^*+\beta_3L_3^*\ge\beta_1L_3^*-\beta_1L_1^*=-\beta_1(L_1^*-L_3^*),$$

so

$$L_{21}^1-L_{21}^3+L_1^*-L_3^*\ge(1-\beta_1)(L_1^*-L_3^*),$$

then

$$\begin{aligned}\widetilde{M_1}-\widetilde{M_3} &=\begin{pmatrix} D_{11}^1-D_{11}^3 & 0\\ L_{21}^1-L_{21}^3+L_1^*-L_3^* & D_{22}^1-D_{22}^3\end{pmatrix}\\ &\ge\begin{pmatrix} (\alpha_1-\alpha_3)I_p & 0\\ (1-\beta_1)(L_1^*-L_3^*) & (\alpha_1-\alpha_3)I_q\end{pmatrix},\end{aligned}$$

as $L_1^*-L_3^*=(V_1-V_3)L\ge 0$, then $\widetilde{M_1}\ge\widetilde{M_3}$.

Notice that $\widetilde{M_1}^{-1}\ge 0, \widetilde{M_3}^{-1}\ge 0$, hence $\widetilde{M_1}^{-1}\le\widetilde{M_3}^{-1}$ and

$$\begin{aligned}M_1^{-1}&=\widetilde{M_1}^{-1}(I+\widetilde{S}_1)\\ &=\widetilde{M_1}^{-1}+\widetilde{M_1}^{-1}\widetilde{S}_1\\ &\le\widetilde{M_3}^{-1}+\widetilde{M_1}^{-1}(\widetilde{S}_1-\widetilde{S}_3)+\widetilde{M_1}^{-1}\widetilde{S}_3\\ &\le\widetilde{M_3}^{-1}+\widetilde{M_3}^{-1}\widetilde{S}_3\\ &=\widetilde{M_3}^{-1}(I+\widetilde{S}_3)=M_3^{-1}.\end{aligned}$$

Since H is an M-matrix, Lemma 2.4 implies that

$$\rho(M_3^{-1}N_3)\le\rho(M_1^{-1}N_1).$$

According $M_i^{-1}N_i=\widetilde{M_i}^{-1}\widetilde{N}_i$ for $i=1,3$, we can conclude that

$$\rho(T_3^*)\le\rho(T_1^*).$$

□

Similar to the proof of Theorem 3.8, we can get the following theorem.

3.9. Theorem. *Suppose that the matrix* H *is irreducible,* $L\le 0, U\le 0, B\ge 0, C\ge 0,\ b_{i,1}>0, c_{j,1}>0, b_{1,i}>0, c_{1,j}>0$, *for some* $i\in\{2,3,...,p\}, j\in\{2,3,...,q\}$, *the auxiliary block diagonal matrices are chosen as* $\alpha_i I$ *and the auxiliary block lower triangular matrices as* $\beta_i L_i^*$ *for* $i=2,3,\ 0\le\alpha_3\le\alpha_2\le 1, 0\le\beta_2\le\beta_3\le 1$. *Then* $\rho(T_3^*)\le\rho(T_2^*)$ *if* $\rho(T)<1$.

4. Examples

4.1 Example Consider

$$H = \begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix},$$

where $B = (b_{ij})_{p\times p}, C = (c_{ij})_{(n-p)\times(n-p)}, L = (l_{ij})_{(n-p)\times p}$ and $U = (u_{ij})_{p\times(n-p)}$ with

$$b_{ii} = \frac{1}{10\times(i+1)}, \quad i = 1,2,\cdots,p,$$

$$b_{ij} = \frac{1}{30} - \frac{1}{30\times j+i}, \quad i<j, \quad i=1,2,\cdots,p-1, \quad j=2,\cdots,p,$$

$$b_{ij} = \frac{1}{30} - \frac{1}{30\times(i-j+1)+i}, \quad i>j, \quad i=2,\cdots,p,\ j=1,2,\cdots,p-1,$$

$$c_{ii} = \frac{1}{10\times(p+i+1)}, \quad i=1,2,\cdots,n-p,$$

$$c_{ij} = \frac{1}{30} - \frac{1}{30\times(p+j)+p+i},\ i<j,\ i=1,2,\cdots,n-p-1,\ j=2,\cdots,n-p,$$

$$c_{ij} = \frac{1}{30} - \frac{1}{30\times(i-j+1)+p+i},\ i>j,\ i=2,\cdots,n-p,\ j=1,2,\cdots,n-p-1,$$

$$l_{ij} = \frac{1}{30\times(p+i-j+1)+p+i} - \frac{1}{30},\ i=1,2,\cdots,n-p,\ j=1,2,\cdots,p,$$

$$u_{ij} = \frac{1}{30\times(p+j)+i} - \frac{1}{30},\ i=1,2,\cdots,p,\ j=1,2,\cdots,n-p.$$

In the experiments, the auxiliary matrices are chosen such that

$$D_1 = 0.5(\frac{1}{\omega}-1)I,\ \overline{D}_1 = 0.5(\frac{1}{\omega}-1)I,\ L_1 = 0.5(1-\frac{\gamma}{\omega})\widehat{L_i},\ \overline{L}_1 = 0.5(1-\frac{\gamma}{\omega})\widehat{L}_i^*.$$

From Table 1, we see that these results accord with Theorems 3.5 - 3.9.

Table 1. The spectral radii of the GMTS and preconditioned GMTS iteration matrices

n	ω	r	p	$\rho(T)$	$\rho(T_1^*)$	$\rho(T_2^*)$	$\rho(T_3^*)$
10	0.9	0.8	5	0.2352	0.2156	0.2140	0.2048
20	0.8	0.6	5	0.5736	0.5609	0.5605	0.5568
20	0.8	0.6	10	0.5551	0.5413	0.5404	0.5334
25	0.8	0.6	8	0.7164	0.7074	0.7070	0.7033
30	0.9	0.7	10	0.8680	0.8635	0.8633	0.8613
30	0.9	0.7	20	0.8676	0.8630	0.8627	0.8605

In [4], the authors considered the following preconditioner

$$P^* = \begin{pmatrix} I+S & 0 \\ 0 & I+V \end{pmatrix}, \tag{4.1}$$

where

$$S=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \frac{b_{p1}}{\alpha} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$V=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \frac{c_{q1}}{\beta} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Table 2. The spectral radii of the preconditioned GMTS iteration matrices

n	ω	r	p	$\alpha=\beta$	$\rho(T^*)$	$\rho(T_1^*)$	$\rho(T_2^*)$	$\rho(T_3^*)$
10	0.9	0.8	5	3	0.2335	0.2156	0.2140	0.2048
20	0.8	0.6	5	2	0.5729	0.5609	0.5605	0.5568
20	0.8	0.6	10	2	0.5542	0.5413	0.5404	0.5334
25	0.8	0.6	8	3	0.7161	0.7074	0.7070	0.7033
30	0.9	0.7	10	2	0.8678	0.8635	0.8633	0.8613
30	0.9	0.7	20	2	0.8673	0.8630	0.8627	0.8605

Here, T^* is the GMTS iteration matrix for solving $P^*Hy = P^*f$.

From Table 2, we see that the preconditioned GMTS methods with preconditioners in this paper have better converge rates than the preconditioned GMTS method with preconditioner P^*.

4.2 Example The coefficient matrix H in Equation (1.2) is given by

$$H=\begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix},$$

where

$$B=\begin{pmatrix} b_{11} & \frac14 & 0 & 0 \\ \frac14 & 0 & \frac14 & 0 \\ 0 & \frac14 & 0 & \frac14 \\ \frac14 & 0 & \frac14 & 0 \end{pmatrix}, \quad C=\begin{pmatrix} c_{11} & \frac14 & 0 \\ \frac14 & 0 & \frac14 \\ \frac14 & \frac14 & 0 \end{pmatrix},$$

$$L=\begin{pmatrix} -\frac14 & 0 & 0 & -\frac14 \\ 0 & 0 & -\frac14 & 0 \\ 0 & -\frac14 & 0 & 0 \end{pmatrix}, \quad U=\begin{pmatrix} -\frac14 & 0 & 0 \\ 0 & -\frac14 & 0 \\ 0 & 0 & -\frac14 \\ -\frac14 & 0 & 0 \end{pmatrix}.$$

Table 3 displays the spectral radii of the corresponding iteration matrices with $\omega = 0.9, \gamma = 0.8$ and different values of b_{11} and c_{11}.

From Table 3, we can see that $\rho(T_i^*) \le \rho(T)$ for $i = 1, 2, 3$ and $\rho(T_3^*) \le \rho(T_i^*)$ for $i = 1, 2$ when $\rho(T) < 1$. These numerical results are in accordance with the theoretical results given in Theorems 3.5- 3.9.

Table 3. The spectral radii of the GMTS and preconditioned GMTS iteration matrices

b_{11}	c_{11}	$\rho(T)$	$\rho(T_1^*)$	$\rho(T_2^*)$	$\rho(T_3^*)$
0	0	0.6804	0.6303	0.6381	0.6140
0	0.3	0.7657	0.7253	0.7323	0.7071
0.2	0.2	0.7614	0.7186	0.7265	0.6987
0.2	0.5	0.8860	0.8677	0.8713	0.8596
0.5	0.5	0.9553	0.9483	0.9499	0.9453

5. Conclusion

In this paper, we propose three new preconditioners and give comparison theorems between the preconditioned and original methods. These results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. Finally, we give two examples to confirm our theoretical results.

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A Linear Adaptive time-stepping Method for Solving Vibration Problems with Damping Terms

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Abstract

A linear adaptive time-stepping method is devised for linear or nonlinear damping vibration analysis, which has wide applications in civil engineering. In the time direction, the underlying problem is discretized by a linear C^0-continuous discontinuous Galerkin method combined with the technique of linearization. By means of the energy method, some optimal a posteriori error estimates are established for linear vibration problems. Motivated by these estimates, we design an adaptive time-stepping strategy for actual computation. Numerical results are performed to illustrate the efficiency of the adaptive method.

Keywords. Time-stepping method, Vibration, Damping, A posteriori error analysis, Adaptive algorithm

1 Introduction

This paper aims to design and analyze an adaptive time-stepping method for solving the following problem:

For any real number $T > 0$, find $\mathbf{u} : [0, T] \to \mathbb{R}^d$ (with d the spatial dimension) such that

$$\begin{cases} \mathbf{M}\mathbf{u}''(t) + \mathbf{F}\big(t, \mathbf{u}(t), \mathbf{u}'(t)\big) = 0, & 0 < t < T, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{v}_0, \end{cases} \tag{1.1}$$

where $(\cdot)'$ and $(\cdot)''$ denote respectively the first and second order derivatives in time; $\mathbf{M}$ is a given $(d \times d)$ matrix and $\mathbf{F}$ is a given vector-valued function from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{R}^d$; $\mathbf{u}_0$ and $\mathbf{v}_0$ are two given vectors in $\mathbb{R}^d$.

The above problem is frequently encountered in structure analysis of dynamical transient response (cf. [5]). Concretely speaking, the mathematical models for structure analysis are described by a system of second-order linear/nonlinear evolution equations, which give rise to the problem (1.1), after spatial discretization by finite element methods, finite difference methods or spectral methods (cf. [2, 9, 11, 16, 17, 21, 22]).

When the vector-valued function $\mathbf{F}$ is linear with respect to $\mathbf{u}$ and $\mathbf{u}'$, there are various numerical methods for solving the problem (1.1). The most widely used may be classified as modal superposition (cf. [6, 14]) and direct-time integration methods including the Runge-Kutta, central difference, Houbolt, Newmark-β and Wilson-θ methods (see [11] and the references therein for details). The space-time finite element method (cf. [7, 12, 13]) is another widely developed approach for solving second order time evolution equations. One typical way is using the time-discontinuous Galerkin (TDG) method (cf. [7, 15]) in the time direction for the displacement and velocity fields together, but it has the disadvantage that an ill-conditioned (4×4) block system must be solved at each time step, which is time consuming. To overcome this difficulty, some linear C^0-continuous time-stepping methods were used in [18], where only the primal variables are involved and only a (1×1) block system

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should be solved at each time step. Moreover, an adaptive method was proposed in [18] for solving second order abstract evolution equations, where the optimal a posteriori error estimates are established, which, in conjunction with the error equidistribution strategy and some ideas implied in the Runge-Kutta-Felberg method, leads to an adaptive time-stepping method.

In this paper, we intend to use some ideas in [18] to develop an adaptive time-stepping method for solving the problem (1.1). In the time direction, the problem (1.1) is discretized by a linear C^0-continuous discontinuous Galerkin method combined with the technique of linearization (including three linearization methods). Then, by means of the energy method, some optimal a posteriori error estimates are established for linear vibration problems via some ideas in [18]. It deserves to emphasize that the mathematical argument developed here is greatly simplified by using the Lagrange basis functions instead of the Legendre polynomials. Motivated by these estimates, we construct a posteriori error estimates for nonlinear problems, based on which we design an adaptive time-stepping strategy for actual computation. Numerical results are performed to illustrate the efficiency of the adaptive method.

The rest of this paper is organized as follows. In Section 2, we present a time-stepping finite element method for the problem (1.1), and the detailed implementation of the previous method is also developed for actual computation. In Section 3, a posteriori error analysis is established in detail for linear vibration problems. In Section 4, we propose an adaptive algorithm based on some a posteriori error estimates. A series of numerical results are performed in the final section.

2 A linear time-stepping finite element method

2.1 The formulation of a linear time-stepping finite element method

Throughout this paper, we assume that Problem (1.1) has a unique solution and the matrix $\mathbf{M}$ is symmetric positive definite. We use a standard time-stepping method to discretize Problem (1.1) (cf. [10, 18, 19]). To this end, we first partition the time interval $I := (0, T)$ with the nodes

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

to get the following subintervals:

$$J_n = (t_{n-1}, t_n], \quad k_n = t_n - t_{n-1}, \quad 1 \le n \le N.$$

Define

$$\mathcal{V}_1 = \left\{ \mathbf{v} : \bar{I} \to \mathbb{R}^d;\ \mathbf{v} \in C(\bar{I}),\ \mathbf{v}|_{J_n}(t) = \sum_{j=0}^{1} t^j \mathbf{w}_j,\ \mathbf{w}_j \in \mathbb{R}^d,\ 1 \le n \le N \right\},$$

$$\mathcal{W}_2 = \left\{ \mathbf{v} : \bar{I} \to \mathbb{R}^d;\ \mathbf{v} \in C^1(\bar{I}),\ \mathbf{v}|_{J_n}(t) = \sum_{j=0}^{2} t^j \mathbf{w}_j,\ \mathbf{w}_j \in \mathbb{R}^d,\ 1 \le n \le N \right\},$$

$$\mathcal{H}_q = \left\{ \mathbf{v} : \bar{I} \to L^2(I);\ \mathbf{v}|_{J_n}(t) = \sum_{j=0}^{q} t^j \mathbf{w}_j,\ \mathbf{w}_j \in \mathbb{R}^d,\ 1 \le n \le N \right\}, \quad q = 0, 1.$$

Let $\mathcal{V}_1(J_n)$ and $\mathcal{W}_2(J_n)$ be the restrictions of $\mathcal{V}_1$ and $\mathcal{W}_2$ to J_n, respectively. Similarly, denote by $\mathcal{H}_q(J_n)$ the restriction of $\mathcal{H}_q$ to J_n. Thus, our time-stepping method for (1.1) is

to find $\mathbf{U} \in \mathcal{V}_1$ such that

$$\begin{cases} \int_{J_n} \left(\langle \mathbf{U}'', \ \mathbf{w}' \rangle_{\mathbf{M}} + \langle \mathbf{F}(t, \mathbf{U}, \mathbf{U}'), \ \mathbf{w}' \rangle\right) \mathrm{d}t + \left\langle \dot{\mathbf{U}}_+^{n-1} - \dot{\mathbf{U}}_-^{n-1}, \ \dot{\mathbf{w}}_+^{n-1} \right\rangle_{\mathbf{M}} = 0, \\ \mathbf{U}^0 = \mathbf{u}_0, \quad \dot{\mathbf{U}}_-^0 = \mathbf{v}_0, \qquad\qquad \mathbf{w} \in \mathcal{V}_1(J_n), \quad 1 \leq n \leq N, \end{cases} \tag{2.1}$$

where

$$\langle \mathbf{a}, \mathbf{b} \rangle := \mathbf{b}^{\mathrm{T}}\mathbf{a}, \quad \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{A}} := \mathbf{b}^{\mathrm{T}}\mathbf{A}\mathbf{a}, \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^d, \ \mathbf{A} \in \mathbb{R}^{d \times d}, \tag{2.2}$$

$$\dot{\mathbf{w}}_\pm^{n-1} := \lim_{s \to 0^+} \mathbf{w}'(t_{n-1} \pm s), \quad \mathbf{w}^{n-1} := \mathbf{w}(t_{n-1}).$$

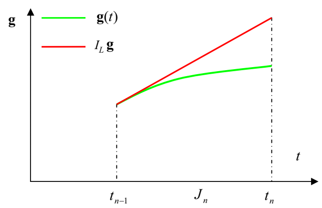
2.2 Implementation of the time-stepping method

Since $\mathbf{U} \in \mathcal{V}_1$, we have by a direct manipulation that, for any $t \in J_n$,

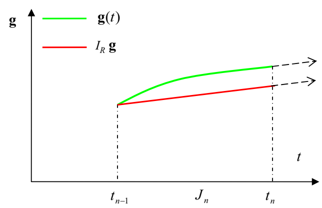
$$\mathbf{U}(t) = \mathbf{U}^{n-1} + (t - t_{n-1})\dot{\mathbf{U}}_-^n, \qquad \mathbf{U}'(t) = \dot{\mathbf{U}}_-^n, \qquad \mathbf{U}''(t) = \mathbf{0}. \tag{2.3}$$

To implement the method (2.1) in actual computation, we require to linearize the nonlinear function $\mathbf{F}(\mathbf{t}, \mathbf{U}, \mathbf{U}')$ with respect to $\mathbf{U}$. As shown in Figure 1, for a given function $g(t)$, its linearization over J_n are usually the interpolants given by

$$\mathcal{I}_L\mathbf{g}(t) = \mathbf{g}(t_{n-1}) + (t - t_{n-1})\mathbf{g}'(t_{n-1}) \quad \text{or} \quad \mathcal{I}_R\mathbf{g}(t) = \mathbf{g}(t_{n-1}) + (t - t_{n-1})\mathbf{g}'(t_n), \qquad t \in J_n.$$



(a) $\mathcal{I}_L$



(b) $\mathcal{I}_R$

Figure 1: Diagrams of the (local) interpolate operators $\mathcal{I}_L$ and $\mathcal{I}_R$.

Note that the function $\mathbf{F} = \mathbf{F}(\mathbf{t}, \mathbf{U}, \mathbf{U}')$ is discontinuous at the interior node t_n. Recalling the expression (2.3), we have by the direct computation that the right limit of $\mathbf{F}$ at $t = t_{n-1}$ can be expressed as

$$\mathbf{F}_+^{n-1} = \mathbf{F}(t_{n-1}, \mathbf{U}^{n-1}, \dot{\mathbf{U}}_+^{n-1}) = \mathbf{F}(t_{n-1}, \mathbf{U}^{n-1}, \dot{\mathbf{U}}_-^n). \tag{2.4}$$

Using the chain rule for differentiation and (2.3), we find that, at $t = t_n$, the left limit of the full derivative of $\mathbf{F}(t, \mathbf{U}, \mathbf{U}')$ with respect to t is given as follows:

$$\begin{aligned} \dot{\mathbf{F}}_-^n &= \frac{\partial \mathbf{F}}{\partial t}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) + \frac{\partial \mathbf{F}}{\partial \mathbf{U}}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n)\dot{\mathbf{U}}_-^n + \frac{\partial \mathbf{F}}{\partial \mathbf{U}'}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n)\mathbf{0} \\ &= \frac{\partial \mathbf{F}}{\partial t}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) + \frac{\partial \mathbf{F}}{\partial \mathbf{U}}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n)\dot{\mathbf{U}}_-^n \\ &=: \left.\frac{\partial \mathbf{F}}{\partial t}\right|_{t_-^n} + \left.\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\right|_{t_-^n} \dot{\mathbf{U}}_-^n. \end{aligned}$$

Similarly, we have

$$\dot{\mathbf{F}}_+^{n-1} := \left.\frac{\partial \mathbf{F}}{\partial t}\right|_{t_+^{n-1}} + \left.\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\right|_{t_+^{n-1}} \dot{\mathbf{U}}_-^n.$$

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With these results in mind, we have by the definitions of the interpolation operators $\mathcal{I}_L$ and $\mathcal{I}_R$ that

$$\text{Left side Scheme}: \ \mathbf{F}\big(t, \mathbf{U}(t), \mathbf{U}'(t)\big) \ \approx \mathcal{I}_L\mathbf{F} = \mathbf{F}_+^{n-1} + (t-t_{n-1})\dot{\mathbf{F}}_+^{n-1}, \tag{2.5}$$

$$\text{Right side Scheme}: \ \mathbf{F}\big(t, \mathbf{U}(t), \mathbf{U}'(t)\big) \ \approx \mathcal{I}_R\mathbf{F} = \mathbf{F}_+^{n-1} + (t-t_{n-1})\dot{\mathbf{F}}_-^{n}. \tag{2.6}$$

Now, inserting (2.3) and (2.6) into the first equation of (2.1) and taking $\dot{\mathbf{w}}$ to be $\mathbf{w}^*$ or $(t-t_{n-1})\mathbf{w}^*$, where $\mathbf{w}^*$ is any constant vector in $\mathbb{R}^d$, we find that the method (2.1) is equivalent to finding $\{\dot{\mathbf{U}}_-^n\}_{n=0}^N$ such that

$$\left(\mathbf{M} + \frac{k_n^2}{2}\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\Big|_{t_-^n}\right)\dot{\mathbf{U}}_-^n + \frac{1}{2}k_n^2\frac{\partial \mathbf{F}}{\partial t}\Big|_{t_-^n} + k_n\mathbf{F}_+^{n-1} = \mathbf{M}\dot{\mathbf{U}}_-^{n-1}, \quad 1 \le n \le N. \tag{2.7}$$

Note that the quantities $\frac{\partial \mathbf{F}}{\partial t}\big|_{t_-^n}$, $\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\big|_{t_-^n}$ and $\mathbf{F}_+^{n-1}$ are all the functions of the unknown vector $\dot{\mathbf{U}}_-^n$, so the above scheme is implicit. However, if we use the linearization formulation (2.5) instead of (2.6), then the system (2.1) reduces to

$$\left(\mathbf{M} + \frac{k_n^2}{2}\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\Big|_{t_+^{n-1}}\right)\dot{\mathbf{U}}_-^n + \frac{1}{2}k_n^2\frac{\partial \mathbf{F}}{\partial t}\Big|_{t_+^{n-1}} + k_n\mathbf{F}_+^{n-1} = \mathbf{M}\dot{\mathbf{U}}_-^{n-1}, \quad 1 \le n \le N. \tag{2.8}$$

It is noted that in most vibration problems, it suffices for us to deal with the linear damping case, indicating that the function $\mathbf{F}$ is linear with respect to the independent variable $\mathbf{u}'$. In this case, since the quantities $\frac{\partial \mathbf{F}}{\partial t}\big|_{t_+^{n-1}}$ and $\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\big|_{t_+^{n-1}}$ in (2.8) do not depend on $\dot{\mathbf{U}}_-^n$, the system (2.8) is essentially a linear system of the unknown vector $\dot{\mathbf{U}}_-^n$. Hence, we can work out $\dot{\mathbf{U}}_-^n$ with much less computational cost, compared to the method (2.7).

In order to balance the efficiency and stability of the time-stepping method, it is very natural to split the nonlinear term $\mathbf{F}$ into two parts $\mathbf{F}_L$ and $\mathbf{F}_R$, which correspond to the non-stiff and the stiff terms of the original system (1.1), respectively. Then, it is better for us to use $\mathcal{I}_L\mathbf{F}_L + \mathcal{I}_R\mathbf{F}_R$ to approximate $\mathbf{F}$ in (2.1). In other words, we have

$$\text{Semi} - \text{side Scheme}: \ \mathbf{F} \ \approx \mathcal{I}_L\mathbf{F}_L + \mathcal{I}_R\mathbf{F}_R = \mathbf{F}_+^{n-1} + (t-t_{n-1})\big(\dot{\mathbf{F}}_{L+}^{n-1} + \dot{\mathbf{F}}_{R-}^{n}\big). \tag{2.9}$$

It is noted that for the linear damping system, the semi-side scheme also yields a linear system for getting the unknown vector $\dot{\mathbf{U}}_-^n$.

Now, let us present the solution process of the method (1.1) in detail. Once we obtain $\mathbf{U}$ in J_{n-1}, we can get $\dot{\mathbf{U}}_-^n$ by solving the system (2.7) or (2.8). Then the function $\mathbf{U}$ over J_n is completely determined using the formulation $\mathbf{U}(t) = \mathbf{U}^{n-1} + (t-t_{n-1})\dot{\mathbf{U}}_-^n$ for all $t \in J_n$. On implementing this computation recursively, we can thereby determine the function $\mathbf{U}$ completely.

In the last part of this subsection, we give the solution process explicitly for the vibration analysis related to linear transient dynamic response. At this moment, we can reformulate the problem (1.1) as follows.

For any real number $T > 0$, find $\mathbf{u} : [0, T] \to \mathbb{R}^d$ such that

$$\begin{cases} \mathbf{M}\mathbf{u}'' + \mathbf{C}\mathbf{u}' + \mathbf{K}\mathbf{u} = \mathbf{f}, & 0 < t < T, \\ \mathbf{u}(0) = \mathbf{u}_0, \\ \mathbf{u}'(0) = \mathbf{v}_0, \end{cases} \tag{2.10}$$

where $\mathbf{C}$ and $\mathbf{K}$ are the $(d \times d)$ damping and stiffness matrices of the dynamic system, respectively. We assume that $\mathbf{C}$ and $\mathbf{K}$ are symmetric and semi-definite. Observing that

$$\mathbf{F}\big(t, \mathbf{u}(t), \mathbf{u}'(t)\big) = \mathbf{C}\mathbf{u}' + \mathbf{K}\mathbf{u} - \mathbf{f},$$

we have from the variational formulation (2.1) that

$$\left(\frac{k_n^2}{2}\mathbf{K}+k_n\mathbf{C}+\mathbf{M}\right)\dot{\mathbf{U}}_-^n=\mathbf{M}\dot{\mathbf{U}}_-^{n-1}-k_n\mathbf{K}\mathbf{U}^{n-1}+\mathbf{f}^n,\quad 1\le n\le N, \tag{2.11}$$

where $\mathbf{f}^n := \int_{J_n}\mathbf{f}\mathrm{d}t$.

3 A posteriori error analysis for linear problems

For the numerical method (2.1) for the linear vibration problem (2.10), following the similar arguments leading to Theorem 2.5 in [18], we can derive some stability estimates to the numerical solution $\mathbf{U}$ and then establish the required a priori error estimates. Another way to derive such estimates is to use the mathematical argument due to [24]. Since the objective of this article is to develop efficient adaptive time stepping method for the linear vibration problem (2.10) and the generalized problem (1.1), we will focus on in this section a posteriori error analysis for the problem (2.10) discretized by the method (2.1). Motivated by such an analysis, we will heuristically mention in the next section some error estimators for the nonlinear problem (1.1) and then devise the corresponding adaptive time stepping method.

3.1 Reconstruction

As shown in [18], in order to get efficient a posteriori error estimates for the method (2.1), we require to construct a higher order reconstruction $\widetilde{\mathbf{U}}$ from the approximate solution $\mathbf{U}$. So let us first recall such a reconstruction given in [18]. Introduce an invertible linear operator $\widetilde{I}_2:\ \mathcal{V}_1\to\mathcal{W}_2$ as follows. With any $\mathbf{w}\in\mathcal{V}_1$ we associate an element $\widetilde{\mathbf{w}}:=\ \widetilde{I}_2\mathbf{w}\in\mathcal{W}_2$ defined by locally interpolating $\mathbf{w}$ in each subinterval $J_n(1\le n\le N)$, i.e., $\widetilde{\mathbf{w}}|_{J_n}\in\mathcal{W}_2(J_n)$ is uniquely determined by

$$\widetilde{\mathbf{w}}(t)=\widetilde{\mathbf{w}}(t_{n-1})+k_n\dot{\mathbf{w}}_-^{n-1}\Phi_0(\frac{t-t_{n-1}}{k_n})+k_n\dot{\mathbf{w}}_-^n\Phi_1(\frac{t-t_{n-1}}{k_n}),\qquad 1\le n\le N, \tag{3.1}$$

and the initial values $\widetilde{\mathbf{w}}(0)=\mathbf{w}(0)$, $\widetilde{\mathbf{w}}'(0)=\mathbf{w}'(0)$. In (3.1), the definition of Φ_0, Φ_1 are given as

$$\Phi_0(\xi)=-\frac{1}{2}\xi^2+\xi,\qquad\qquad \Phi_1(\xi)=\frac{1}{2}\xi^2. \tag{3.2}$$

We call $\widetilde{\mathbf{w}}$ a time reconstruction of $\mathbf{w}$, as shown in Figure 2. It is easy to check by the

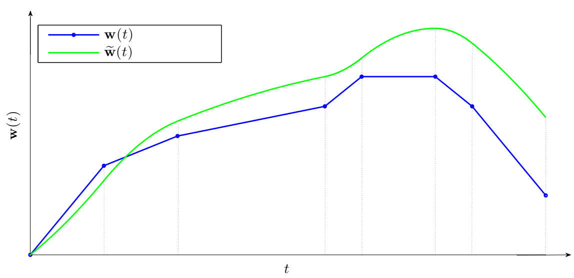

Figure 2: Diagram of $\widetilde{I}_2 w$.

above construction that

$$\widetilde{\mathbf{w}}'(t_n)=\dot{\mathbf{w}}_-^n,\quad 1\le n\le N. \tag{3.3}$$

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Thus, for an approximate solution U, the reconstructed function we hope to find is $\widetilde{\mathbf{U}} \in \mathcal{W}_2$, defined by

$$\widetilde{\mathbf{U}}(t) = \widetilde{\mathbf{U}}(t_{n-1}) + k_n \dot{\mathbf{U}}_-^{n-1} \Phi_0(\frac{t - t_{n-1}}{k_n}) + k_n \dot{\mathbf{U}}_-^n \Phi_1(\frac{t - t_{n-1}}{k_n}), \qquad 1 \le n \le N. \tag{3.4}$$

By a direct computation we have

$$\widetilde{\mathbf{U}}''(t) = \frac{1}{k_n}(\dot{\mathbf{U}}_-^n - \dot{\mathbf{U}}_-^{n-1}), \qquad 1 \le n \le N. \tag{3.5}$$

Observing that the function $\mathbf{U}(t)$ can be rewritten as

$$\mathbf{U}(t) = \mathbf{U}(t_{n-1}) + k_n \dot{\mathbf{U}}_+^{n-1} \Phi_0(\frac{t - t_{n-1}}{k_n}) + k_n \dot{\mathbf{U}}_-^n \Phi_1(\frac{t - t_{n-1}}{k_n}), \quad t \in J_n,$$

subtracting which from (3.4) we know

$$\mathbf{U}(t) - \widetilde{\mathbf{U}}(t) = \mathbf{U}^{n-1} - \widetilde{\mathbf{U}}^{n-1} + k_n(\dot{\mathbf{U}}_+^{n-1} - \dot{\mathbf{U}}_-^{n-1})\Phi_0(\frac{t - t_{n-1}}{k_n}), \quad t \in J_n. \tag{3.6}$$

Hence,

$$\mathbf{U}^n - \widetilde{\mathbf{U}}^n = \mathbf{U}^{n-1} - \widetilde{\mathbf{U}}^{n-1} + \frac{1}{2}k_n^2 \widetilde{\mathbf{U}}'', \quad t \in J_n,$$

i.e.,

$$\mathbf{U}^n - \widetilde{\mathbf{U}}^n = \frac{1}{2}\sum_{m=1}^{n} k_m^2 \widetilde{\mathbf{U}}''|_{J_m}, \quad t \in J_n. \tag{3.7}$$

Moreover, by integration by parts and (3.3), it follows that

$$\int_{J_n} \langle \widetilde{\mathbf{U}}'', \ \mathbf{w}' \rangle_{\mathbf{M}} \mathrm{d}t = \int_{J_n} \langle \mathbf{U}'', \ \mathbf{w}' \rangle_{\mathbf{M}} \mathrm{d}t + \langle \dot{\mathbf{U}}_+^{n-1} - \dot{\mathbf{U}}_-^{n-1}, \dot{\mathbf{w}}_+^{n-1} \rangle_{\mathbf{M}}, \quad \mathbf{w} \in \mathcal{V}_1(J_n),$$

and use the variational equation in (2.1) we further have

$$\int_{J_n} (\langle \widetilde{\mathbf{U}}'', \ \mathbf{w}' \rangle_{\mathbf{M}} + \langle \mathbf{C}\mathbf{U}' + \mathbf{K}\mathbf{U} - \mathbf{f}, \ \mathbf{w}' \rangle) \ \mathrm{d}t = 0, \quad \mathbf{w} \in \mathcal{V}_1 \quad 1 \le n \le N,$$

i.e.,

$$\mathbf{M}\widetilde{\mathbf{U}}'' + P_0(\mathbf{C}\mathbf{U}' + \mathbf{K}\mathbf{U} - \mathbf{f}) = 0, \qquad t \in J_n, \tag{3.8}$$

where P_q $(q = 0, 1)$ stands for the (local) L^2 orthogonal projection operator on to $\mathcal{H}_q(J_n)$ (cf. [1]), defined by

$$\int_{J_n} \langle P_q \mathbf{v} - \mathbf{v}, \ \mathbf{w} \rangle \mathrm{d}t = 0, \qquad \mathbf{w} \in \mathcal{H}_q(J_n). \tag{3.9}$$

3.2 Error estimates

Let $\|\cdot\|$, $\|\cdot\|_{\mathbf{M}}$, $\|\cdot\|_{\mathbf{C}}$ and $\|\cdot\|_{\mathbf{K}}$ be the norms (or seminorms) over $\mathbb{R}^d$, defined by the inner products (2.2), respectively. We further define

$$\|\mathbf{v}\|_{L^\infty_{\mathbf{M}}(G)} = \operatorname*{ess\,sup}_{t \in G} \|\mathbf{v}(t)\|_{\mathbf{M}}, \qquad \|\mathbf{v}\|_{L^\infty_{\mathbf{M}^{-1}}(G)} = \operatorname*{ess\,sup}_{t \in G} \|\mathbf{v}(t)\|_{\mathbf{M}^{-1}}, \tag{3.10}$$

where $\mathbf{M}^{-1}$ is the inverse of the matrix $\mathbf{M}$. We assume that for the given function f, the linear problem (2.10) has a unique solution satisfying that

$$\mathbf{u} \in C([0, T]; \ \mathbb{R}^d) \cap C^1([0, T]; \ \mathbb{R}^d).$$

Let $\widetilde{\mathbf{e}} := \mathbf{u} - \widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{R}}$ be the residual of $\widetilde{\mathbf{U}}$ given by

$$\widetilde{\mathbf{R}}(t) := \mathbf{M}^{-1}(\mathbf{M}\widetilde{\mathbf{U}}''(t) + \mathbf{C}\widetilde{\mathbf{U}}'(t) + \mathbf{K}\widetilde{\mathbf{U}}(t) - \mathbf{f}(t)), \qquad t \in J_n, \ 1 \le n \le N. \tag{3.11}$$

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Theorem 3.1 *Let* $\mathbf{u}$ *and* $\mathbf{U}$ *be the solution of* (2.10) *and* (2.1), *respectively. Let* $\widetilde{\mathbf{U}}$ *be the reconstruction of* $\mathbf{U}$ *by* (3.1). *Then for any* $t \in [0, T]$, *there holds*

$$\max_{0 \le \tau \le t} \|(\mathbf{u} - \widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} \le 2 \int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}} \, \mathrm{d}s, \tag{3.12}$$

where $\widetilde{\mathbf{R}}$ *is given by* (3.11).

Proof. Subtracting (3.11) from (2.10) gives

$$\mathbf{M}\widetilde{\mathbf{e}}''(t) + \mathbf{C}\widetilde{\mathbf{e}}'(t) + \mathbf{K}\widetilde{\mathbf{e}}(t) = -\mathbf{M}\widetilde{\mathbf{R}}(t). \tag{3.13}$$

Then, we test (3.13) by $\widetilde{\mathbf{e}}'$ and integrate over $t \in [0,\ \tau]$ to get

$$\begin{aligned} &\int_0^\tau \left(\langle \widetilde{\mathbf{e}}''(s),\ \widetilde{\mathbf{e}}'(s) \rangle_{\mathbf{M}} + \langle \widetilde{\mathbf{e}}'(s),\ \widetilde{\mathbf{e}}'(s) \rangle_{\mathbf{C}} + \langle \widetilde{\mathbf{e}}(s),\ \widetilde{\mathbf{e}}'(s) \rangle_{\mathbf{K}} \right) \mathrm{d}s \\ &\quad = \int_0^\tau \langle -\widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s) \rangle_{\mathbf{M}} \, \mathrm{d}s. \end{aligned} \tag{3.14}$$

Moreover, using integration by parts and noting that $\widetilde{\mathbf{e}}(0) = \widetilde{\mathbf{e}}'(0) = 0$, we arrive at

$$\frac{1}{2}\|\widetilde{\mathbf{e}}'(\tau)\|^2_{\mathbf{M}} + \int_0^\tau \|\widetilde{\mathbf{e}}'(s)\|^2_{\mathbf{C}} \, \mathrm{d}s + \frac{1}{2}\|\widetilde{\mathbf{e}}(\tau)\|^2_{\mathbf{K}} = \int_0^\tau \langle -\widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s) \rangle_{\mathbf{M}} \, \mathrm{d}s,\ \tau \in [0, t]. \tag{3.15}$$

Hence, it follows from (3.15) and the Cauchy-Schwarz inequality that

$$\begin{aligned} \frac{1}{2}(\max_{0 \le \tau \le t} \|\widetilde{\mathbf{e}}'(\tau)\|_{\mathbf{M}})^2 &\le \max_{0 \le \tau \le t} \int_0^\tau |\langle \widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s) \rangle_{\mathbf{M}}| \, \mathrm{d}s \\ &\le \int_0^t |\langle \widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s) \rangle_{\mathbf{M}}| \, \mathrm{d}s \ \le \ \max_{0 \le \tau \le t} \|\widetilde{\mathbf{e}}'(\tau)\|_{\mathbf{M}} \int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}} \, \mathrm{d}s, \end{aligned}$$

which readily yields

$$\max_{0 \le \tau \le t} \|\widetilde{\mathbf{e}}'(\tau)\|_{\mathbf{M}} \le 2 \int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}} \, \mathrm{d}s, \tag{3.16}$$

as required. ■

Now, we proceed with the efficiency of the above a posteriori error estimates.

Lemma 3.1 *For* $t \in J_n$, $1 \le n \le N$,

$$\mathbf{U}(t) - P_0\mathbf{U}(t) = (t - t_{n-1} - \frac{1}{2}k_n)\dot{\mathbf{U}}^{\mathbf{n}}_{-}. \tag{3.17}$$

Moreover, for $1 \le n \le N$,

$$\|(\mathbf{U} - \widetilde{\mathbf{U}})'\|_{L^\infty_{\mathbf{M}}(J_n)} = k_n \|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)}. \tag{3.18}$$

Furthermore, there holds

$$\begin{aligned} 2\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}} \, \mathrm{d}s \le \sum_{m-1}^{n} \bigg(&\frac{2}{3}k_m^3 \|\mathbf{K}\widetilde{\mathbf{U}}^{(3)}\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)} + t k_m^2 \|\mathbf{K}\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)} \\ &+ \frac{1}{2}k_m^2 \|\mathbf{K}\mathbf{U}'\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)} + k_m^2 \|\mathbf{C}\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)} \\ &+ 2\int_{J_m} \|\mathbf{f}(s) - P_0\mathbf{f}(s)\|_{\mathbf{M}^{-1}} \, \mathrm{d}s \bigg). \end{aligned} \tag{3.19}$$

Proof. First of all, recalling the definition of (Local) L^2 projection (3.9), we can deduce that

$$\begin{aligned} P_0\mathbf{U}(t) &= \frac{1}{k_n}\int_{J_n}\mathbf{U}(s)\mathrm{d}s = \frac{1}{k_n}\int_{J_n}\big(\mathbf{U}^{n-1}+(s-t_{n-1})\dot{\mathbf{U}}_-^n\big)\mathrm{d}s \\ &= \mathbf{U}^{n-1}+\frac{1}{2}k_n\dot{\mathbf{U}}_-^n, \qquad\qquad t\in J_n, \end{aligned}$$

so

$$\mathbf{U}(t)-P_0\mathbf{U}(t) = (t-t_{n-1}-\frac{1}{2}k_n)\dot{\mathbf{U}}_-^n, \qquad t\in J_n. \tag{3.20}$$

On the other hand, differentiating (3.6) with respect to the variable t directly yields

$$(\mathbf{U}-\widetilde{\mathbf{U}})'(t) = -(t-t_{n-1})\widetilde{\mathbf{U}}'', \qquad t\in J_n, \tag{3.21}$$

which implies (3.18).

Moreover, we have by (3.8) and (3.11) that

$$\mathbf{M}\widetilde{\mathbf{R}} = \mathbf{K}(\widetilde{\mathbf{U}}-P_0\mathbf{U})+\mathbf{C}\big(\widetilde{\mathbf{U}}'-P_0(\mathbf{U}')\big)-(\mathbf{f}-P_0\mathbf{f}). \tag{3.22}$$

Write

$$\mathbf{K}(\widetilde{\mathbf{U}}-P_0\mathbf{U}) = \mathbf{K}(\widetilde{\mathbf{U}}-\mathbf{U})+\mathbf{K}(\mathbf{U}-P_0\mathbf{U}),$$

and owing to the fact that$P_0(\mathbf{U}')=\mathbf{U}'$ we know

$$\mathbf{C}\big(\widetilde{\mathbf{U}}'-P_0(\mathbf{U}')\big) = \mathbf{C}(\widetilde{\mathbf{U}}-\mathbf{U})'.$$

Hence, the equation (3.22) can be reformulated as

$$\mathbf{M}\widetilde{\mathbf{R}}(s) = \mathbf{K}(\widetilde{\mathbf{U}}-\mathbf{U})(s)+\mathbf{K}(\mathbf{U}-P_0\mathbf{U})(s)+\mathbf{C}(\widetilde{\mathbf{U}}-\mathbf{U})'(s)-(\mathbf{f}-P_0\mathbf{f})(s),$$

which, in conjunction with (3.6), (3.20) and (3.21), yields the estimate (3.19). ■

Now, let us continue to discuss the lower and upper a posteriori error bound for the method (2.1).

Theorem 3.2 (lower and upper bounds) *Let* $\mathbf{u}$ *and* $\mathbf{U}$ *be the solution of* (2.10) *and* (2.1), *respectively. Let* $\widetilde{\mathbf{U}}$ *be the reconstruction of* $\mathbf{U}$ *by* (3.1). *Then for* $t\in[0,T]$, $1\le n\le N$,

$$\begin{aligned} \max_{1\le m\le n} k_m^2\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} &\le \|(\mathbf{u}-\mathbf{U})'\|_{L^\infty_{\mathbf{M}}(0,t)}+\max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} \\ &\le \max_{1\le m\le n} k_m\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)}+4\int_0^t\|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s, \end{aligned} \tag{3.23}$$

where the a posteriori term $\widetilde{\mathbf{R}}$ *is given by* (3.11).

Proof. Using the triangle inequality and (3.18), we obtain

$$\begin{aligned} \max_{1\le m\le n} k_m^2\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} &= \|(\mathbf{U}-\widetilde{\mathbf{U}})'\|_{L^\infty_{\mathbf{M}}(0,t)} \\ &\le \|(\mathbf{u}-\mathbf{U})'\|_{L^\infty_{\mathbf{M}}(0,t)}+\max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}}, \end{aligned} \tag{3.24}$$

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which implies the left side estimate of (3.23). Again, by the triangle inequality, (3.18) and (3.16), we have

$$\begin{aligned}\|(\mathbf{u}-\mathbf{U})'\|_{L^\infty_{\mathbf{M}}(0,t)} &\le \max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} + \|(\mathbf{U}-\widetilde{\mathbf{U}})'(\tau)\|_{L^\infty_{\mathbf{M}}(0,t)} \\ &\le \max_{1\le m\le n} k_m\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} + 2\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s. \end{aligned} \tag{3.25}$$

This together with (3.16) and (3.24) yields

$$\begin{aligned}&\max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} + \|(\mathbf{u}-\mathbf{U})'(\tau)\|_{L^\infty_{\mathbf{M}}(0,t)} \\ &\le \max_{1\le m\le n} k_m\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} + 4\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s,\end{aligned}$$

which leads to the right side estimate of (3.23). ■

4 An adaptive algorithm

Motivated by Theorem 3.2 (cf. the estimate (3.25)), we are tempted to introduce a posteriori error estimator of the time-stepping method (2.1) for solving even a nonlinear problem (1.1) heuristically. That means, let

$$\eta := \max_{1\le n\le N} k_n\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)} + 2\int_0^T \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\mathrm{d}s, \tag{4.1}$$

where $\widetilde{\mathbf{R}}$ is the residual of a nonlinear problem, defined by

$$\widetilde{\mathbf{R}}(t) = \mathbf{M}^{-1}\Big(\mathbf{M}\widetilde{\mathbf{U}}''(t) + \mathbf{F}\big(t,\widetilde{\mathbf{U}}(t),\widetilde{\mathbf{U}}'(t)\big)\Big), \qquad t\in J_n,\ 1\le n\le N.$$

Then the quantity η may be viewed as a posteriori error estimator for the method (2.1). Until now, it is beyond our power to develop reliability and efficiency estimates for such an estimator.

Based on the above error estimator, using the error equidistribution strategy as used in [4, 20], we can construct the error indicator corresponding to the subinterval J_n as

$$\Theta := 2\ \max\{\ \Theta_1,\ \ \Theta_2\ \}, \tag{4.2}$$

where

$$\Theta_1 := k_n\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)}, \qquad \Theta_2 := 2\frac{T}{k_n}\int_{J_n}\|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\mathrm{d}s.$$

The magnitude of Θ affects the choice of k_n, the length of the subinterval J_n.

Next, let us study how to compute the quantities Θ_1 and Θ_2 after we get $\dot{\mathbf{U}}^n_-$ at each time step by (2.1). First of all, from (3.5) and the definition of Θ_1, we have

$$\Theta_1 = \|\dot{\mathbf{U}}^n_- - \dot{\mathbf{U}}^{n-1}_-\|_{\mathbf{M}}\ . \tag{4.3}$$

For deriving Θ_2, we should obtain $\widetilde{\mathbf{R}}(t)$ in advance. It follows from (3.4) that

$$\begin{aligned}&\widetilde{\mathbf{U}}(t) = \widetilde{\mathbf{U}}(t_{n-1}) + k_n\dot{\mathbf{U}}^{n-1}_-\Phi_0(\xi) + k_n\dot{\mathbf{U}}^n_-\Phi_1(\xi),\\ &\widetilde{\mathbf{U}}'(t) = \dot{\mathbf{U}}^{n-1}_-(1-\xi) + \dot{\mathbf{U}}^n_-\xi, \qquad \widetilde{\mathbf{U}}''(t) = \frac{1}{k_n}(-\dot{\mathbf{U}}^{n-1}_- + \dot{\mathbf{U}}^n_-),\end{aligned} \tag{4.4}$$

where $\xi = (t - t_{n-1})/k_n$ and $\Phi_0,\ \Phi_1$ are defined as in (3.2).

Furthermore, in actual computation, we will use the Gaussian quadrature formula (cf. [23]) to evaluate Θ_2 numerically. In other words, for $t \in J_n$, $1 \le n \le N$,

$$\int_{J_n} \|\widetilde{\mathbf{R}}(t)\|_{\mathbf{M}} \mathrm{d}t \ \approx \ \sum_{j=1}^{N_g} k_n \omega_j \|\widetilde{\mathbf{R}}(t_{n-1} + k_n \zeta_j)\|_{\mathbf{M}}, \tag{4.5}$$

where ζ_j and ω_j $(1 \le j \le N_g)$ are the Gaussian quadrature points and weights on reference interval $[0,1]$, respectively.

Remark 4.1 *Let us discuss the cost of computing Θ_2 briefly. It is evident that the cost is taken in numerical integration by Gaussian quadrature formula* (4.5)*. Since the quadrature method is highly accurate, very few nodes are enough for actual computation (with the number ≤ 10). Next, we have to evaluate $\|\widetilde{\mathbf{R}}(\cdot)\|_{\mathbf{M}}$ at the quadrature nodes, the main cost of which corresponds to numerical solution of a linear system with $\mathbf{M}$ as a coefficient matrix. Generally speaking, the mass matrix $\mathbf{M}$ is a well-conditioned symmetric positive definite matrix, so the linear system can be solved by the conjugate gradient method very efficiently. According to the above analysis, we find that the cost for computing Θ_2 is inexpensive.*

With the help of the previous preparations and using some ideas implied in the Runge-Kutta-Felberg method (cf. [23]), we are ready to present the following Algorithm 1 to compute the numerical solution of the problem (1.1) by using the adaptive time-stepping strategy.

Algorithm 1 Adaptive Time Stepping Method

Given a tolerance ϵ, a parameter $\delta \in (0,1)$, and the max (min) time step size $k_{\max}$ ($k_{\min}$) by user

· **Step 0:** Initialize $n = 1$, $t_0 = 0$, $k_1 = k_{\max}$, $\mathbf{U}^0 = \mathbf{u}_0$, $\dot{\mathbf{U}}_-^0 = \mathbf{v}_0$
· **WHILE** $t_{n-1} < T$
· **Step 1:** Given t_{n-1}, k_n, $\mathbf{U}^{n-1}$, $\dot{\mathbf{U}}_-^{n-1}$
· **1(a):** Get the numerical solution $\mathbf{U}^n$, $\dot{\mathbf{U}}_-^n$ by (2.7)
· **1(b):** Get the approximation $\widetilde{\mathbf{U}}^n$ by (3.4)
· **1(c):** Evaluate Θ_1 by (4.3)
· **1(d):** Get $\widetilde{\mathbf{R}}(t)$ at Gaussian quadrature points by (4.4) and (3.11)
· **1(e):** Summation to get the value of Θ_2 by (4.5)
· **1(f):** Get Θ by (4.2)
· **Step 2:** If $\delta\epsilon \le \Theta \le \epsilon$, $k_{n+1} = k_n$, go to **Step 5**
· **Step 3:** If $\Theta < \delta\epsilon$, $k_{n+1} = \min\{2k_n,\ k_{\max}\}$, go to **Step 5**
· **Step 4:** If $\Theta > \epsilon$, $k_n = \max\{k_n/2,\ k_{\min}\}$, go to **Step 1**
· **Step 5:** Let $t_n = t_{n-1} + k_n$, $n = n + 1$, go to loop condition judgment
· **END WHILE**

Remark 4.2 *Similar to the Runge-Kutta-Felberg method (cf. [23]), the parameter $\delta \in (0,1)$ in Algorithm 1 is used to determine how to enlarge the step size during the computation process (see Step 3 in Algorithm 1). The choice of δ is very technical. If δ is chosen too small, the over-refined meshes would be used in time, deteriorating the efficiency of Algorithm 1. If it is chosen too large, the algorithm would enlarge the step size more frequently, increasing the extra computational cost remarkably. From our numerical experience, it's better to choose δ such that $1/32 \le \delta \le 1/2$.*

5 Numerical experiments

5.1 Efficiency of the estimators

Example 5.1 (Nonlinear lumped mass system) *For illustrating the effectiveness of the a posteriori error estimates developed in the previous sections, we first study the vibration of a multi-structure model, a similar one as given in [3]. As shown in Figure 3, the structure consists of two rigid elements (vehicles) with lumped masses equal to m_1 and m_2, respectively; these elements are connected with each other by soften, classical and harden springs with linear damping. And the restoring force of these springs are given as follows:*

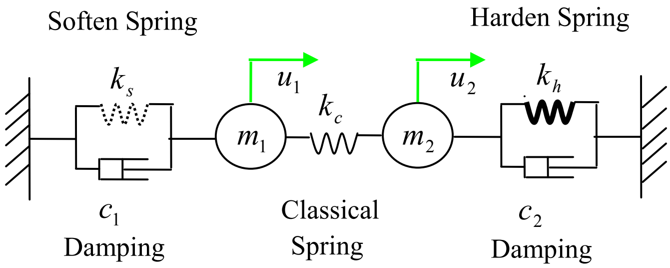

Figure 3: Example: 5.1: The nonlinear dynamic system.

$$\text{Classical Spring } (k_c): \quad f_c = -\kappa_1 u, \tag{5.1}$$
$$\text{Softening Spring } (k_s): \quad f_s = -\kappa_2 \tanh(u), \tag{5.2}$$
$$\text{Hardening Spring } (k_h): \quad f_h = -\kappa_3 u(1+\kappa_4 u^2). \tag{5.3}$$

In our actual computation, we choose $m_1 = m_2 = 1$, and choose the spring stiffness as $\kappa_1 = \kappa_2 = \kappa_3 = 1$. The damping coefficients are taken as $c_1 = c_2 = 1$. Hence by d Alembert's principle, we can get the following system of nonlinear dynamic equations,

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}'' = \begin{pmatrix} c_1 u_1'(t) + f_s\big(u_1(t)\big) + f_c\big(u_1(t)\big) - f_c\big(u_2(t)\big) - f_1(t) \\ c_2 u_2'(t) + f_h\big(u_2(t)\big) + f_c\big(u_2(t)\big) - f_c\big(u_1(t)\big) - f_2(t) \end{pmatrix}, \tag{5.4}$$

where f_1 and f_2 are the external forces. We choose $T = 1$ and the exact solution to be $\mathbf{u}(t) = (u_1, u_2)^{\mathrm{T}} = (\sin(\pi t), \sin(2\pi t))^{\mathrm{T}}$, so the force term $\mathbf{f}$ can be computed by the equations (5.4). We solve the solution of the dynamical system by the method (2.1) combined with the right side scheme (2.7).

In our numerical computation, for a given natural number N, we adopt the uniform partition in time with the mesh size $k = T/N$, $1 \le n \le N$. To show the computational performance of our method, define

$$\mathrm{Ed} = \max_{0\le\tau\le T} \|(\mathbf{u} - \mathbf{U})'(\tau)\|_{\mathbf{M}} , \qquad \mathrm{Et} = \max_{0\le\tau\le T} \|(\mathbf{u} - \widetilde{\mathbf{U}})(\tau)\|_{\mathbf{M}} ,$$
$$\mathrm{Etd} = \max_{0\le\tau\le T} \|(\mathbf{u} - \widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} , \qquad \varepsilon_1 = 2\int_0^T \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\, \mathrm{d}s ,$$
$$\varepsilon_2 = \max_{0\le n\le N} k_n \|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)} , \qquad \varepsilon_3 = \eta = 2\varepsilon_1 + \varepsilon_2 .$$
$$\mathrm{Effld} = \frac{\varepsilon_2}{\mathrm{Ed} + \mathrm{Etd}}, \qquad \mathrm{Effud} = \frac{\varepsilon_3}{\mathrm{Ed} + \mathrm{Etd}}.$$

In Figure 4(a) we present the values of Et *and* ε_1 *as well as their orders (which are* 1*). In Figure 4(b) we give the estimates of the reconstruction solution Et and Etd as well as*

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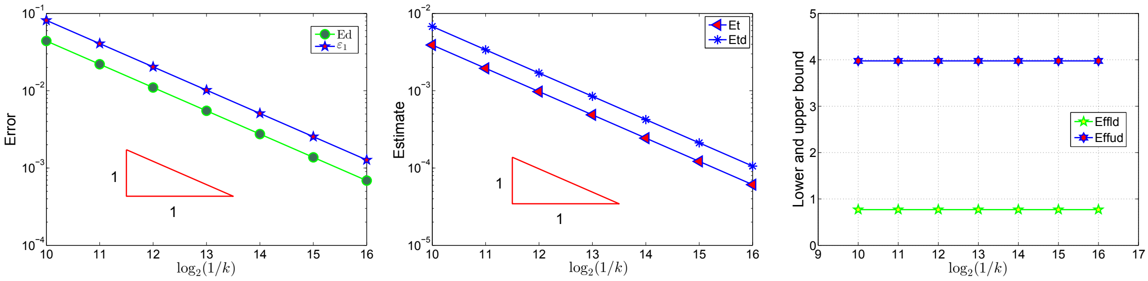

(a) Order of Ed and ε_1 (b) Order of Et and Etd (c) Lower and Upper bound

Figure 4: Example 5.1. Numerical results corresponding to estimators in Theorem 3.12 and Theorem 3.2.

their orders. Moreover, we present the values of these effectivity indices in Figure 4(c), from which we can observe that $0.77 \approx \text{Effld} < 1 < \text{Effud} \approx 3.98$*. Therefore, our a posteriori error estimator* (4.1) *is rather efficient.*

5.2 Efficiency of the adaptive algorithm

Example 5.2 (Nonlinear Klein-Gordon equation) *In order to test the effectiveness of our adaptive Algorithm 1, we consider the nonlinear Klein-Gordon equations (cf. [8]),*

$$u_{tt}(\mathbf{x},t) - \Delta u(\mathbf{x},t) + \beta u_t(\mathbf{x},t) + u^2(\mathbf{x},t) = f(\mathbf{x},t),$$

equipped with the homogeneous Dirichlet boundary condition and the initial conditions. After the discretization by P_1 *conforming element in the space direction, we obtain the following system of nonlinear ODEs,*

$$\begin{cases} \mathbf{M}\mathbf{u}''(t) + \mathbf{C}\mathbf{u}'(t) + \mathbf{K}\mathbf{u}(t) + \mathbf{M}\mathbf{u}^2(t) = \mathbf{f}(t), & 0 < t < T, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{v}_0, \end{cases} \tag{5.5}$$

where $\mathbf{u}$ *is the vector representation of the finite element solution* u_h *in terms of the shape basis functions* $\{\varphi_i\}$*, i.e.,* $u_h(\mathbf{x},t) = \sum_{i=1}^{M}\{\mathbf{u}(t)\}_i\varphi_i(\mathbf{x})$*. The mass matrix* $\mathbf{M}$*, the stiff matrix* $\mathbf{K}$*, the damping matrix* $\mathbf{C}$ *and the force* $\mathbf{F}$ *are defined respectively by* $[\mathbf{M}]_{ij} = \int_\Omega \varphi_j\varphi_i \mathrm{d}\Omega$*,* $[\mathbf{K}]_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \mathrm{d}\Omega$*,* $[\mathbf{C}]_{ij} = \beta\int_\Omega \varphi_j\varphi_i \mathrm{d}\Omega$ *and* $\{\mathbf{f}\}_i = \int_\Omega f(t)\varphi_i \mathrm{d}\Omega$*. In the numerical computation, we choose the damping coefficient* $\beta = 0.05$ *and the terminal time* $T = 1.0$*. Consider the* 1*-dim case of the above problem with the force* f *given such that the exact solution is*

$$u(x,t) = \mathrm{e}^{-t/2}x(1-x)\sin((1.5\pi + \mathrm{actan}(500(2t-1))x), \qquad 0 < x < 1,$$

which varies rapidly around $t = 0.5$*. After the discretization in space direction with a fine uniform mesh* $h = 1/5000$*, we solve the semi-discrete problem by using Algorithm 1 combined with the semi-side scheme* (2.9) *with* $\mathbf{F}$ *split into* $\mathbf{F}^R := \mathbf{C}\mathbf{u}' + \mathbf{K}\mathbf{u} - \mathbf{f}$ *and* $\mathbf{F}^L := \mathbf{M}\mathbf{u}^2$*, so that we only require to solve a linear system at each time subinterval. When implementing Algorithm 1 in this example, we set the related parameters by* $\epsilon = 2.5e-1$*,* $\delta = 1/2$*,* $k_{\max} = 1e-1$ *and* $k_{\min} = 2e-4$*.*

To show the efficiency of Algorithm 1, we also carry out the numerical simulation using the uniform time stepping method with the same number of subintervals as for the adaptive method. The numerical solution obtained by the uniform time stepping method with $k = k_{\min}/100$ *is used as a reference solution.*

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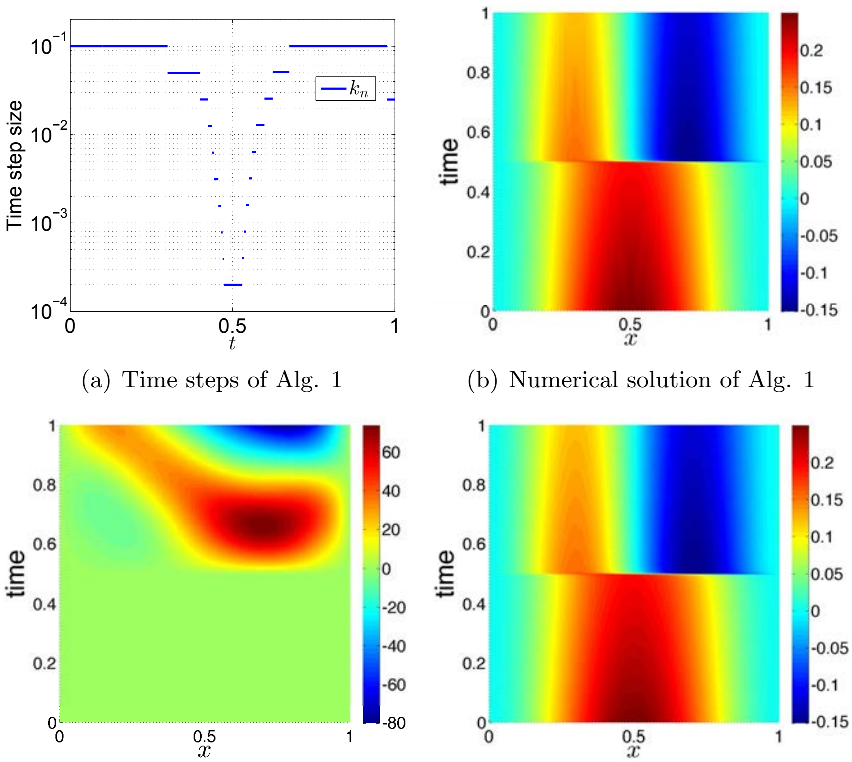

(a) Time steps of Alg. 1

(b) Numerical solution of Alg. 1

(c) Numerical solution with uniform stepsize

(d) Reference solution

Figure 5: Example 5.2. Comparison of numerical results.

From Figure 5(a) we can see the time step size becomes extremely small around $t = 0.5$ *in order to capture the rapid change of the solution, and the step size will become large automatically when the solution varies slowly, which illustrates the efficiency of Algorithm 1. The numerical results with Algorithm 1 and the uniform time stepping method, and the reference solution are shown in Figures 5(b), 5(c) and 5(d), respectively, from which we may find that the adaptive method can approximate the exact solution very well even if it varies rapidly, but the uniform time stepping method fails. We mention further that for the adaptive method in this example, the total CPU time used is approximately* 147.1 s, *while the one for computing* Θ *is only* 7.4 s, *only covers a very small amount of the total time.*

Acknowledgments

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A fractional Means inequality

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Abstract

Here we produce an interesting fractional means scalar inequality.

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We make

Remark 1 *Let $\nu > 0$, $n := \lceil \nu \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), $f(\cdot, y) \in AC^n([a,b])$, $\forall\, y \in [c,d]$ (it means $\frac{\partial^{n-1} f(\cdot,y)}{\partial x^{n-1}} \in AC([a,b])$, $\forall\, y \in [c,d]$). Then the left Caputo partial fractional derivative with respect to x, is given by (see [1], p. 270)*

$$\frac{\partial_{*a}^{\nu} f(x,y)}{\partial x^{\nu}} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \frac{\partial^n f(t,y)}{\partial x^n} dt, \tag{1}$$

$\forall\, y \in [c,d]$, and it exists almost everywhere for x in $[a,b]$, Γ denotes the gamma function.

Then, we get the left Caputo fractional Taylor formula ([2], p. 54)

$$f(x,y) = \sum_{k=0}^{n-1} \frac{\partial^k f(a,y)}{\partial x^k} (x-a)^k + \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} \frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}} dt, \tag{2}$$

$\forall\, x \in [a,b]$, for each $y \in [c,d]$.

Above $\left(\int_a^x (x-t)^{\nu-1} \frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}} dt\right) \in AC^n([a,b])$, $\forall\, y \in [c,d]$.

Let now $f(x,\cdot) \in AC^n([c,d])$, $\forall\, x \in [a,b]$ (it means $\frac{\partial^{n-1} f(x,\cdot)}{\partial y^{n-1}} \in AC([c,d])$, $\forall\, x \in [a,b]$). Then the left Caputo partial fractional derivative with respect to y, is given by

$$\frac{\partial_{*c}^{\nu} f(x,y)}{\partial y^{\nu}} = \frac{1}{\Gamma(n-\nu)} \int_c^y (y-s)^{n-\nu-1} \frac{\partial^n f(x,s)}{\partial y^n} ds, \tag{3}$$

$\forall\, x \in [a,b]$, and it exists almost everywhere for y in $[c,d]$.

Then, we get the left Caputo fractional Taylor formula

$$f(x,y)=\sum_{k=0}^{n-1}\frac{\partial^{k}f(x,c)}{\partial y^{k}}(y-c)^{k}+\frac{1}{\Gamma(\nu)}\int_{c}^{y}(y-s)^{\nu-1}\frac{\partial_{*c}^{\nu}f(x,s)}{\partial y^{\nu}}ds, \tag{4}$$

$\forall\ y\in[c,d]$, *for each* $x\in[a,b]$.

Above $\left(\int_{c}^{y}(y-s)^{\nu-1}\frac{\partial_{*c}^{\nu}f(x,s)}{\partial y^{\nu}}ds\right)\in AC^{n}([c,d])$, $\forall\ x\in[a,b]$.

Assume

$$\frac{\partial^{k}f(a,y)}{\partial x^{k}}=0,\ \ for\ k=1,...,n-1,\ \forall\ y\in[c,d], \tag{5}$$

we get

$$f(x,y)-f(a,y)=\frac{1}{\Gamma(\nu)}\int_{a}^{x}(x-t)^{\nu-1}\frac{\partial_{*a}^{\nu}f(t,y)}{\partial x^{\nu}}dt. \tag{6}$$

Additionally assume $f(a,y)=0$, $\forall\ y\in[c,d]$, *then*

$$f(x,y)=\frac{1}{\Gamma(\nu)}\int_{a}^{x}(x-t)^{\nu-1}\frac{\partial_{*a}^{\nu}f(t,y)}{\partial x^{\nu}}dt, \tag{7}$$

$\forall\ y\in[c,d]$, $\forall\ x\in[a,b]$.

Assume

$$\frac{\partial^{k}f(x,c)}{\partial y^{k}}=0,\ \ for\ k=1,...,n-1,\ \forall\ x\in[a,b], \tag{8}$$

we get

$$f(x,y)-f(x,c)=\frac{1}{\Gamma(\nu)}\int_{c}^{y}(y-s)^{\nu-1}\frac{\partial_{*c}^{\nu}f(x,s)}{\partial y^{\nu}}ds, \tag{9}$$

$\forall\ y\in[c,d]$, $\forall\ x\in[a,b]$.

Additionally assume that $f(x,c)=0$, $\forall\ x\in[a,b]$, *then*

$$f(x,y)=\frac{1}{\Gamma(\nu)}\int_{c}^{y}(y-s)^{\nu-1}\frac{\partial_{*c}^{\nu}f(x,s)}{\partial y^{\nu}}ds, \tag{10}$$

$\forall\ y\in[c,d]$, $\forall\ x\in[a,b]$.

Assuming (5) and (8), we get

$$2f(x,y)-f(a,y)-f(x,c)=$$
$$\frac{1}{\Gamma(\nu)}\left\{\int_{a}^{x}(x-t)^{\nu-1}\frac{\partial_{*a}^{\nu}f(t,y)}{\partial x^{\nu}}dt+\int_{c}^{y}(y-s)^{\nu-1}\frac{\partial_{*c}^{\nu}f(x,s)}{\partial y^{\nu}}ds\right\}, \tag{11}$$

$\forall\ x\in[a,b]$, $\forall\ y\in[c,d]$.

Additionally assume that $f(a,y)=0$, $\forall\ y\in[c,d]$, *and* $f(x,c)=0$, $\forall\ x\in[a,b]$, *we obtain*

$$f(x,y)=\frac{1}{2\Gamma(\nu)}\left\{\int_{a}^{x}(x-t)^{\nu-1}\frac{\partial_{*a}^{\nu}f(t,y)}{\partial x^{\nu}}dt+\int_{c}^{y}(y-s)^{\nu-1}\frac{\partial_{*c}^{\nu}f(x,s)}{\partial y^{\nu}}ds\right\}, \tag{12}$$

2

$\forall\ x \in [a,b]\,,\ \forall\ y \in [c,d]\,.$

We can rewrite (11) as follows:

$$f(x,y) - \left(\frac{f(a,y)+f(x,c)}{2}\right) =$$

$$\frac{1}{2\Gamma(\nu)}\left\{\int_a^x (x-t)^{\nu-1}\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}dt + \int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds\right\}, \qquad (13)$$

$\forall\ x \in [a,b]\,,\ \forall\ y \in [c,d]\,.$

If $0<\nu<1$, then $n=1$, and (13) is valid without (5) and (8), which in this case are void conditions.

Call

$$\Delta f(x,y) := f(x,y) - \left(\frac{f(a,y)+f(x,c)}{2}\right). \qquad (14)$$

Assume $f \in C([a,b]\times[c,d])$, then

$$\int_a^b\int_c^d \Delta f(x,y)\,dxdy = \int_a^b\int_c^d f(x,y)\,dxdy -$$

$$\left(\frac{(b-a)\int_c^d f(a,y)\,dy + (d-c)\int_a^b f(x,c)\,dx}{2}\right). \qquad (15)$$

Hence it holds

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \Delta f(x,y)\,dxdy = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy -$$

$$\left(\frac{\frac{1}{(d-c)}\int_c^d f(a,y)\,dy + \frac{1}{(b-a)}\int_a^b f(x,c)\,dx}{2}\right). \qquad (16)$$

Assume now that

$$\frac{\partial_{*a}^{\nu} f(x,y)}{\partial x^{\nu}}, \frac{\partial_{*c}^{\nu} f(x,y)}{\partial y^{\nu}} \in C([a,b]\times[c,d]) \qquad (17)$$

Clearly, it holds

$$|\Delta f(x,y)| \leq$$

$$\frac{1}{2\Gamma(\nu)}\left\{\int_a^x (x-t)^{\nu-1}\left|\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}\right|dt + \int_c^y (y-s)^{\nu-1}\left|\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}\right|ds\right\} \leq$$

$$\frac{1}{2\Gamma(\nu)}\left\{\frac{(x-a)^{\nu}}{\nu}\left\|\frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}}\right\|_{\infty} + \frac{(y-c)^{\nu}}{\nu}\left\|\frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}}\right\|_{\infty}\right\} \leq \qquad (18)$$

$$\frac{1}{2\Gamma(\nu+1)}\left\{(b-a)^{\nu}\left\|\frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}}\right\|_{\infty} + (d-c)^{\nu}\left\|\frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}}\right\|_{\infty}\right\}.$$

That is

$$|\Delta f(x,y)| \leq \frac{1}{2\Gamma(\nu+1)} \left\{ (b-a)^{\nu} \left\| \frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}} \right\|_{\infty} + (d-c)^{\nu} \left\| \frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}} \right\|_{\infty} \right\} =: \lambda. \quad (19)$$

Hence

$$\frac{1}{(b-a)(d-c)} \left| \int_a^b \int_c^d \Delta f(x,y)\, dxdy \right| \leq$$

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d |\Delta f(x,y)|\, dxdy \leq \lambda.$$

We have derived:

Theorem 2 *Let $\nu > 0$, $n := \lceil \nu \rceil$, $f(\cdot, y) \in AC^n([a,b])$, $\forall\ y \in [c,d]$; and $f(x,\cdot) \in AC^n([c,d])$, $\forall\ x \in [a,b]$. Assume $\frac{\partial^k f(a,y)}{\partial x^k} = 0$, for $k = 1,...,n-1$, $\forall\ y \in [c,d]$; and $\frac{\partial^k f(x,c)}{\partial y^k} = 0$, for $k = 1,...,n-1$, $\forall\ x \in [a,b]$. Furthermore, assume $f \in C([a,b] \times [c,d])$ and $\frac{\partial_{*a}^{\nu} f(x,y)}{\partial x^{\nu}}, \frac{\partial_{*c}^{\nu} f(x,y)}{\partial y^{\nu}} \in C([a,b] \times [c,d])$. Then*

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dxdy - \left(\frac{\frac{1}{(b-a)} \int_a^b f(x,c)\, dx + \frac{1}{(d-c)} \int_c^d f(a,y)\, dy}{2} \right) \right|$$

$$\leq \frac{1}{2\Gamma(\nu+1)} \left\{ (b-a)^{\nu} \left\| \frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}} \right\|_{\infty} + (d-c)^{\nu} \left\| \frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}} \right\|_{\infty} \right\}. \quad (20)$$

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